Étale Algebraic Series

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Setting :

$x = (x_1, ..., x_n)$ variables,

$x' = (x_1, ..., x_{n-1})$,

y a single variable,

K a field, $\mathrm{char}(K) = 0$,

$K[[x]] = K[[x_1, ..., x_n]]$ formal power series ring,

$K[[x']] = K[[x_1, ..., x_{n-1}]]$,

$K\langle x\rangle = K\langle x_1, ..., x_n\rangle$ algebraic power series ring,

$K\langle x'\rangle = K\langle x_1, ..., x_{n-1}\rangle$,

$K[x, y] = K[x_1, ..., x_n, y]$ polynomial ring.

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Algebraic series :

$h \in K[[x_1, ..., x_n]]$,

$P(x, h(x)) = 0$ for some (irreducible) polynomial $P(x, y) \in K[x, y]$:

$$p_d(x)h^d + p_{d-1}(x)h^{d-1} + \ldots + p_1(x)h + p_0(x) = 0.$$

Convention:

$$h(0) = 0.$$

Examples :

$$h(x) = \frac{x}{1+x}, \qquad \sqrt{1+x} - 1, \qquad x\sqrt{1+x}, \qquad \sqrt{1+x} - \sqrt[3]{1+x},$$

$$P(x, y) = (1+x)y - x = 0,$$

$$(y+1)^2 - (1+x) = 0,$$

$$y^2 - (x^2 + x^3) = 0,$$

$$[y^3 + (1+x)(1+3y)]^2 - (1+x)(1+x-3y^2)^2 = 0.$$

Étale algebraic series :

$P(x, h(x)) = 0$ with $P(x, y)$ satisfying

$$\partial_y P(0,0) \neq 0,$$

say,

$$P(x, y) = y + Q(x, y),$$

$$\operatorname{ord} Q(0, y) \geq 2.$$

We also say: h is étale algebraic over $K[x]$.

Examples :

$h(x) = \sqrt{1+x} - 1$	étale:	$y^2 + 2y - x = 0,$
$h(x) = x\sqrt{1+x}$	not étale:	$y^2 - (x^2 + x^3) = 0.$

Similarly, for a ring $K[x] \subset N \subset K[[x]]$:

h étale algebraic over N, if $P(h) = 0$ for some $P \in N[y]$ with $\partial_y P(0,0) \neq 0$,

say,

$$P(x, y) = y + Q(y),$$

with $Q \in N[y]$ and $\operatorname{ord} Q(0, y) \geq 2$.

Inverse Function Theorem :

$$f_1(z_1, ..., z_n) = z_1 + \widetilde{f}_1(z_1, ..., z_n),$$
$$\dots$$
$$\dots$$
$$f_n(z_1, ..., z_n) = z_n + \widetilde{f}_n(z_1, ..., z_n),$$

$f = (f_1, ..., f_n)$ polynomials, or algebraic, convergent or formal series, $\mathrm{ord}\, \widetilde{f}_i(z) \geq 2$.

Define

$$g^1(z) := z,$$

$$g^{k+1}(z) := z - \widetilde{f}(g^k(z)),$$

$$g := \lim g^k.$$

Then:

$$g = f^{-1},$$

$g = (g_1, ..., g_n)$, with g_i algebraic, convergent, or formal series, respectively.

Note: g^k is given by substitution and hence polynomial if f is polynomial.

Univariate Étalization Lemma :

h an algebraic series in one variable x.

There exists an $e \geq 1$ such that

$$h(x) = k(x) + a(x) \cdot x^e$$

with k a polynomial of degree $\leq e$ and a an étale algebraic series, $a(0) = 0$.

Proof :

$P(x, y)$ minimal polynomial of h, say, $P(x, h(x)) = 0$; then $\partial_y P(x, h(x)) \neq 0$.

$$e := \operatorname{ord} \partial_y P(x, h(x)).$$

Decompose

$$h = k + a \cdot x^e,$$

k polynomial of degree $\leq e$, a series with $a(0) = 0$.

Taylor:

$$\partial_y P(x, k) = \partial_y P(x, h - a \cdot x^e)$$

$$= \partial_y P(x, h) - \partial_y^2 P(x, h) \cdot a \cdot x^e + T(x, a \cdot x^e).$$

Comparison of orders:

$$\operatorname{ord} \partial_y P(x, k) = \operatorname{ord} \partial_y P(x, h) = e.$$

Taylor:

$$
\begin{aligned}
0 &= P(x, h) \\
&= P(x, k + a \cdot x^e) \\
&= P(x, k) + \partial_y P(x, k) \cdot a \cdot x^e + S(x, a \cdot x^e).
\end{aligned}
$$

Comparison of orders:

$$\operatorname{ord} P(x, k) > 2e.$$

Set

$$P(x, k) =: x^{2e} \cdot R(x),$$

$$S(x, y \cdot x^e) =: x^{2e} \cdot Q(x, y).$$

Divide

$$P(x, k) + \partial_y P(x, k) \cdot a \cdot x^e + S(x, a \cdot x^e) = 0$$

by x^{2e}:

$$R(x) + a + Q(x, a) = 0.$$

$\operatorname{ord} Q(0, y) \geq 2$, hence:

a is étale algebraic.

Multivariate Étalization Lemma :

h an algebraic series in $x = (x_1, ..., x_n)$.

There exists a polynomial

$$k(x) = \sum_{i=0}^{e} k_i(x')x_n^i,$$

in x_n with $k_i \in K\langle x' \rangle$, and a polynomial $Q(x, y) \in K[x, y]$ such that

$$h(x) = k(x) + a(x) \cdot Q(x, k),$$

where $a \in K\langle x \rangle$ with $a(0) = 0$ is an étale algebraic series over

$$K(x, k_0, ..., k_e) \cap K\langle x \rangle.$$

Proof :

$P(x, h) = 0$, with $\partial_y P(x, h) \neq 0$ of order e. Linear coordinate change: $\partial_y P(x, h)$ is x_n-regular of order e,

$$\partial_y P(x, h) = x_n^e + \text{ terms in } x_1, ..., x_n.$$

Set $g := \partial_y P(x, h)$.

Weierstrass:

$$h(x) = k(x) + c(x) \cdot g(x),$$

with $k \in K\langle x' \rangle[x_n]$ a polynomial in x_n of degree $\leq e$ with algebraic series coefficients in x' and $c \in K\langle x \rangle$ an algebraic series.

Taylor:

$$\partial_y P(x, k) = \partial_y P(x, h - c \cdot g)$$

$$= g - \partial_y^2 P(x, h) \cdot c \cdot g + T(x, c \cdot g)$$

$$= g \cdot [1 - \partial_y^2 P(x, h) \cdot c + T(x, c \cdot g) \cdot g^{-1}].$$

Comparison of orders:

$$\partial_y P(x, k) = \partial_y P(x, h) \cdot u(x),$$

$u \in K\langle x \rangle^*$ a unit.

Set $f := \partial_y P(x, k)$ and $a := c \cdot u^{-1}$, so that $a \cdot f = c \cdot g$ and hence

$$h = k + a \cdot f.$$

Taylor:

$$0 = P(x,h) = P(x, k + a \cdot f)$$

$$= P(x,k) + \partial_y P(x,k) \cdot a \cdot f + S(x, a \cdot f)$$

$$= P(x,k) + f^2 \cdot a + S(x, a \cdot f)$$

Get:

$$0 = P(x,k) + f^2 \cdot [a + S(x, a \cdot f) \cdot f^{-2}].$$

In particular:

$$R(x,k) := P(x,k) \cdot f^{-2} = \frac{P(x,k)}{\partial_y P(x,k)^2} \in K(x, k_0, ..., k_e) \cap K\langle x \rangle$$

is an algebraic series.

Set $U(x,z,y) := S(x,yz) \cdot z^{-2}$, a polynomial of order ≥ 2 in y.

From

$$P(x,k) + f^2 \cdot [a + S(x, a \cdot f) \cdot f^{-2}] = 0$$

follows

$$R(x,k) + a + U(x, \partial_y P(x,k), a) = 0,$$

Proven:

a is étale algebraic over $K(x, k_0, ..., k_e) \cap K\langle x \rangle$.

Persistence Theorem :

If a certain property

$$\mathcal{P} : K[[x]] \to \{0, 1\}$$

holds for polynomials as well as for étale algebraic series,

$$\mathcal{P}(K[x]) = \mathcal{P}(K_{et}\langle x\rangle) = 1,$$

and is closed under addition, multiplication and division, it holds for all algebraic series,

$$\mathcal{P}(K\langle x\rangle) = 1.$$

Applications :

Recursion.

Convergence.

Asymptotics.

D-finiteness.

Eisenstein.

Diagonals.

Finite support.

Artin Approximation :

$P(x,y)$ convergent series, $h(x)$ formal solution, $P(x,h(x))=0$, and $c\geq 1$. There exists a convergent solution $\widetilde{h}(x)$ with $\widetilde{h}\equiv h$ modulo x^c.

Note : Same for several equations $P_1=...=P_k=0$ in variables $x_1,...,x_n,y_1,...,y_m$ and solution vectors $h=(h_1,...,h_m)$. If P polynomial or algebraic, then $\widetilde{h}$ algebraic.

Idea of proof :

If $\partial_y P(x,h(x))=0$, add $\partial_y P(x,y)$ to your equation, get system, and work with suitable minors of the Jacobian matrix instead of $\partial_y P(x,y)$.

If $\partial_y P(x,h(x))\neq 0$, proceed as in the theorem above: Write $h=k+a\cdot f$, where $f=\partial_y P(x,k)$ is x_n-regular of order e, and k polynomial in x_n.

Weierstrass:

$$P(x,k)=B\cdot f^2+\sum_{i=0}^{2e-1}R_i(x',k_0,...,k_e)\cdot x_n^i,$$

with B and R_i convergent.

Know from the above:

$$P(x,k)\in f^2\cdot K\langle x\rangle.$$

Uniqueness of remainder:

$$R_i(x',k_0,...,k_e)=0.$$

System of convergent equations in one variable less, $x'=(x_1,...,x_n)$, with formal solutions $k_0,...,k_e$. Apply induction.

Artin-Mazur Lemma :

h algebraic, $P(x,h) = 0$, minimal polynomial,

$$R := K[x,y]/\langle P(x,y)\rangle,$$

$$\widetilde{R} := \text{integral closure of } R,$$

$$\text{know: } \widetilde{R} = K[x, z_1, ..., z_k]/\langle P_1, ..., P_k\rangle.$$

Lift $(x, h(x))$ from $X = \text{Spec}(R)$ to $\widetilde{X} = \text{Spec}(\widetilde{R})$ (universal property of normalization). Get solution $(h_1, ..., h_k)$ of $P_1 = \ldots = P_k = 0$ at, say, $0 \in \widetilde{X}$, and

$$\pi : \widetilde{X} \to X,\ (x, z_1, ..., z_k) \to (x, z_1).$$

But $\widetilde{X}$ is graph and by Zariski's theorem analytically irreducible, hence smooth,

$$(\partial_{z_i} P_j)(0,0) \text{ invertible.}$$

Then:

$$R \subset \widetilde{R} = K[x, h_1, ..., h_k] \text{ étale extension,}$$

and

$$h_1 = h.$$

Conclusion :

Every algebraic series is a component of a vector of algebraic series satisfying an étale system of polynomial equations.

Standard étale extensions: (Chevalley)

Every étale extension is locally standard étale:

$h_1, ..., h_k$ solutions of $P_1(x,y) = \ldots = P_k(x,y) = 0$,

$y = (y_1, ..., y_k)$, P_i polynomials,

h_i algebraic series, $h_i(0) = 0$

$(\partial_{y_i} P_j)(0,0)$ invertible.

There exists an étale algebraic series $a(x)$ such that

$$h = \frac{s_0(x) + s_1(x) \cdot a + \ldots + s_d(x) \cdot a^d}{t_0(x) + t_1(x) \cdot a + \ldots + t_e(x) \cdot a^e},$$

s_i, t_j polynomials, $t_0(0) \neq 0$.

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