Enumeration of diagonally symmetric alternating sign matrices

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(joint work with Roger Behrend and Ilse Fischer)

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Alternating Sign Matrix (ASM)

Definition:

- quadratic matrix $(n \times n)$
- entries are 0, 1, or -1
- 1's and -1's alternate along rows and along columns
- all row sums and all column sums are 1

Example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A bit of history

- ASMs arose during the study of Dodgson condensation for determinants (Mills, Robbins, Rumsey, 1982), conjectures
- Proof: Zeilberger (1996), Kuperberg (1996)

$$\mathrm{ASM}(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Submitted: May 1, 1995; Accepted: July 25, 1995

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

A bit of history

- ASMs arose during the study of Dodgson condensation for determinants (Mills, Robbins, Rumsey, 1982), conjectures
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$$\mathrm{ASM}(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

- Various symmetry classes: ASM, VSASM, VHSASM, HTSASM, QTSASM, DSASM, DASASM, TSASM
- Behrend/Fischer/Konvalinka (2016):

$$\mathrm{DASASM}(2n+1) = \prod_{i=0}^{n} \frac{(3i)!}{(n+i)!}$$

DSASMs

Diagonally symmetric alternating sign matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

DSASMs

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Fundamental domain

DSASMs

Diagonally symmetric alternating sign matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

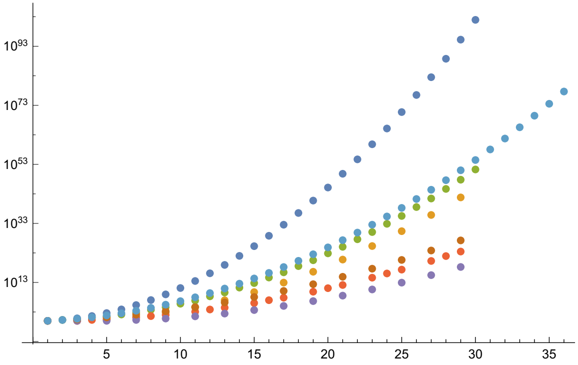
Fundamental domain

Let r denote the ratio between the size of a fundamental domain and the total number of entries, e.g., $r_{\mathrm{DSASM}} = \frac{1}{2}$.

Growth

Conjecture. For an ASM symmetry class $C \in \{\mathrm{ASM}, \mathrm{VSASM}, \mathrm{VHSASM}, \mathrm{HTSASM}, \mathrm{QTSASM}, \mathrm{DSASM}, \mathrm{DASASM}, \mathrm{TSASM}\}$, let $C(n)$ denote the number of $n \times n$ ASMs in C. Then

$$\limsup_{n\to\infty} C(n)^{1/(r_C \cdot n^2)} = \frac{3\sqrt{3}}{4} \approx 1.29904.$$



DSASMs for $n = 1, 2, 3, 4$

$$\begin{pmatrix} 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix} \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} \begin{pmatrix} 0&1&0\\1&-1&1\\0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&-1&1\\0&0&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&-1&1\\0&0&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0\\0&1&-1&1\\1&-1&1&0\\0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}$$

Number of DSASMs

A005163	Number of alternating sign n X n matrices that are symmetric about a diagonal. (Formerly M1500)	+20 0

1, 2, 5, 16, 67, 368, 2630, 24376, 293770, 4610624, 94080653, 2492747656, 85827875506, 3842929319936, 223624506056156, 16901839470598576, 1659776507866213636, 211853506422044996288, 35137231473111223912310, 7569998079873075147860464 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS Robbins's paper does not give a formula for this sequence. On the contrary he states: "Apparently these numbers do not factor into small primes, so a simple product formula seems unlikely. Of course this does not rule out other very simple formulas, but these would be more difficult to discover (let alone prove)." As far as I know no formula is currently known. - Herman Jamke (hermanjamke(AT)fastmail.fm), Feb 23 2008

REFERENCES

Bousquet-Mélou, Mireille; and Habsieger, Laurent; Sur les matrices a signes alternants, [On alternating-sign matrices] in Formal power series and algebraic combinatorics (Montreal, PQ, 1992). Discrete Math. 139 (1995), 57-72.

N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

R. P. Stanley, A baker's dozen of conjectures concerning plane partitions, pp. 285-293 of "Combinatoire Enumerative (Montreal 1985)", Lect. Notes Math. 1234, 1986.

Number of DSASMs

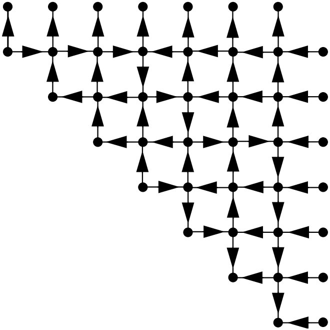
$n = 1$:	1	$= 1$
$n = 2$:	2	$= 2$
$n = 3$:	5	$= 5$
$n = 4$:	16	$= 2^4$
$n = 5$:	67	$= 67$
$n = 6$:	368	$= 2^4 \cdot 23$
$n = 7$:	2630	$= 2 \cdot 5 \cdot 263$
$n = 8$:	24376	$= 2^3 \cdot 11 \cdot 277$
$n = 9$:	293770	$= 2 \cdot 5 \cdot 29 \cdot 1013$
$n = 10$:	4610624	$= 2^6 \cdot 61 \cdot 1181$
$n = 11$:	94080653	$= 4679 \cdot 20107$
$n = 12$:	2492747656	$= 2^3 \cdot 7 \cdot 2063 \cdot 21577$
$n = 13$:	85827875506	$= 2 \cdot 29 \cdot 73 \cdot 20271109$
$n = 14$:	3842929319936	$= 2^13 \cdot 7 \cdot 67015369$
$n = 15$:	223624506056156	$= 2^2 \cdot 67 \cdot 7547 \cdot 110563111$
$n = 16$:	16901839470598576	$= 2^4 \cdot 13 \cdot 12343 \cdot 6583394929$
$n = 17$:	1659776507866213636	$= 2^2 \cdot 263 \cdot 1577734323066743$
$n = 18$:	211853506422044996288	$= 2^6 \cdot 13 \cdot 254631618295727159$
$n = 19$:	35137231473111223912310	$= 2 \cdot 5 \cdot 1601 \cdot 2194705276271781631$
$n = 20$:	$7569998079873075147860464$	$= 2^4 \cdot 473124879992067196741279$

More numbers of DSASMs

$n = 1$: 1
$n = 2$: 2
$n = 3$: 5
$n = 4$: 16
$n = 5$: 67
$n = 6$: 368
$n = 7$: 2630
$n = 8$: 24376
$n = 9$: 293770
$n = 10$: 4610624
$n = 11$: 94080653
$n = 12$: 2492747656
$n = 13$: 85827875506
$n = 14$: 3842929319936
$n = 15$: 223624506056156
$n = 16$: 16901839470598576
$n = 17$: 1659776507866213636
$n = 18$: 211853506422044996288
$n = 19$: 35137231473111223912310
$n = 20$: 7569998079873075147860464
$n = 21$: 2118828647238536587298828059
$n = 22$: 770673990478689868790208726560
$n = 23$: 364208960426686596537755921369204
$n = 24$: 223584747192421482835961510980545856
$n = 25$: 178316052131585634729418453829335054936
$n = 26$: 184783530298259828417276720806393969283072
$n = 27$: 248781049045288108090429913919027669986240120
$n = 28$: 435100651401570160848422958340223675778192652096
$n = 29$: 988575714030048455539523534323540042730905113789712
$n = 30$: 2918268808268512558661062634728957323873319565816585216
$n = 31$: 11191977556595306787688924514392387401160522562509404752480
$n = 32$: 55758361084253080973582654412549007021500164037613319497525120
$n = 33$: 360873619910724091463085427354561273174898813007073520286270385312
$n = 34$: 3034428094821742605659499518550072906286077860183906562736432838885888
$n = 35$: 33147799463054177459141651964502103162909097171877989104468852980628830640
$n = 36$: 470387500964073698951254185900620768347698608335597100251088754974122927207552
$n = 37$: 8671511724276778346310776146790389784220888235304663578971016972018061943458672488
$n = 38$: 207681592952826862584324117895542918615491954128661522770180883458509427151696694500864
$n = 39$: 6461752399123675883811274927599968774322623175693197510050988605401080962582639480482415880
$n = 40$: 26117199759046554015131876293093328870231850526369000539434590011958212123141820697862488 7208000

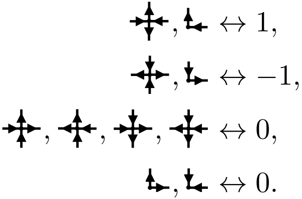
→ With a new implementation of the Bousquet-Mélou-Habsieger algorithm: $\mathrm{DSASM}(n)$ for n up to 40.

Six-vertex model



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The degree-4 vertices have two incoming and two outgoing edges.
- The top vertical edges point up.
- The rightmost horizontal edges point to the left.

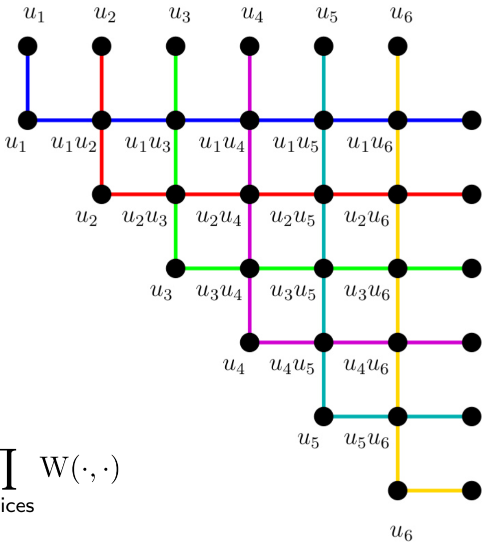


Vertex weights

Bulk weights	Left boundary weights	Matrix entry
$\mathrm{W}($$, u) = 1$	$\mathrm{W}($$, u) = \frac{\beta\, qu + \gamma\, \bar{q}\bar{u}}{\sigma(q^2)}$	1
$\mathrm{W}($$, u) = 1$	$\mathrm{W}($$, u) = \frac{\gamma\, qu + \beta\, \bar{q}\bar{u}}{\sigma(q^2)}$	-1
$\mathrm{W}($$, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$\mathrm{W}($$, u) = \alpha\, \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	0
$\mathrm{W}($$, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$\mathrm{W}($$, u) = \delta\, \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	0
$\mathrm{W}($$, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		0
$\mathrm{W}($$, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		0

Notation: $\bar{x} = x^{-1}$ and $\sigma(x) = x - \bar{x}$.

Vertex labeling



Partition function:

$$Z_n(u_1,\ldots,u_n) = \sum_{\text{conf's}} \prod_{\text{vertices}} \mathrm{W}(\cdot,\cdot)$$

Partition function

The even-order partition function is $Z_{2n}(u_1, \ldots, u_{2n}) =$

$$\sigma(q^4)^{-n(2n-1)}\sigma(q^2)^{-2n} \prod_{1\le i<j\le 2n} \frac{\sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times$$

$$\operatorname*{Pf}_{1\le i<j\le 2n} \left(\frac{\sigma(u_i\bar{u}_j)\big(\alpha\delta\sigma(q^4)\sigma(q^2u_i^2)\sigma(q^2u_j^2)+\sigma(q^2\bar{u}_i\bar{u}_j)(\beta q u_i+\gamma\bar{q}\bar{u}_i)(\beta q u_j+\gamma\bar{q}\bar{u}_j)\big)}{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)} \right)$$

Partition function

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$$\operatorname*{Pf}_{1\le i<j\le 2n} \left(\frac{\sigma(u_i\bar{u}_j)\big(\alpha\delta\sigma(q^4)\sigma(q^2u_i^2)\sigma(q^2u_j^2)+\sigma(q^2\bar{u}_i\bar{u}_j)(\beta q u_i+\gamma\bar{q}\bar{u}_i)(\beta q u_j+\gamma\bar{q}\bar{u}_j)\big)}{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)} \right)$$

The odd-order partition function is $Z_{2n-1}(u_1, \ldots, u_{2n-1}) =$

$$\sigma(q^4)^{-(n-1)(2n-1)} \sigma(q^2)^{-2n+1} \prod_{1\le i<j\le 2n-1} \frac{\sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times$$

$$\operatorname*{Pf}_{0\le i<j\le 2n-1} \left(\begin{cases} \beta q u_j + \gamma \bar{q}\bar{u}_j, & \text{if } i = 0, \\ \frac{\sigma(u_i\bar{u}_j)\big(\alpha\delta\sigma(q^4)\sigma(q^2u_i^2)\sigma(q^2u_j^2)+\sigma(q^2\bar{u}_i\bar{u}_j)(\beta q u_i+\gamma\bar{q}\bar{u}_i)(\beta q u_j+\gamma\bar{q}\bar{u}_j)\big)}{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)} \end{cases} \right)$$

Pfaffian

For an array $A = (a_{i,j})_{1 \le i < j \le 2n}$ its Pfaffian is defined as

$$\mathrm{Pf}(A) = \sum_{\pi = \{\{i_1,j_1\},\ldots,\{i_n,j_n\}\}} \mathrm{sgn}(\pi) \cdot \prod_{k=1}^{n} a_{i_k,j_k},$$

where $\pi = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ is a perfect matching of $\{1, 2, \ldots, 2n\}$, assuming $i_k < j_k$ and

$$\mathrm{sgn}\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\} := \mathrm{sgn}(i_1 j_1 \ldots i_n j_n).$$

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$$\mathrm{sgn}\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\} := \mathrm{sgn}(i_1 j_1 \ldots i_n j_n).$$

- Setting $a_{i,j} = -a_{j,i}$ if $i > j$, we need not require $i_k < j_k$.
- For a skew-symmetric matrix we have $\mathrm{Pf}(A)^2 = \det(A)$.

Evaluate the partition function

Note that all weights become 1 under the substitution

$$\begin{aligned} \alpha &= \delta = 1, \\ \beta &= \gamma = \sigma(q), \\ q &= \exp\left(\tfrac{2\pi i}{12}\right) = \tfrac{1}{2}\left(\sqrt{3} + i\right), \\ u_1 &= \cdots = u_n = 1. \end{aligned}$$

Hence, after this substitution, Z_n gives the number $\mathrm{DSASM}(n)$.

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Hence, after this substitution, Z_n gives the number $\mathrm{DSASM}(n)$.

Problem: One first has to evaluate the Pfaffian and simplify before the substitution $u_1 = \cdots = u_n$ is admissible.

Example of partition function

The even-order partition function for $n = 1$ is

$$Z_2(u_1, u_2) = \frac{1}{\left(q^4-1\right)^3\left(q^4+1\right)u_1^2u_2^2}$$
$$\times\Big(\gamma^2q^8 - \alpha\delta + \alpha\delta q^{16}u_1^4u_2^4 - \alpha\delta q^{12}u_1^4 - \alpha\delta q^{12}u_2^4 + \beta^2q^{12}u_1^2u_2^2$$
$$+ \beta\gamma q^{10}u_1^2 + \beta\gamma q^{10}u_2^2 + \alpha\delta q^8 - \alpha\delta q^8u_1^4u_2^4 - \beta^2q^8u_1^4u_2^4$$
$$- \beta\gamma q^6u_1^2u_2^4 - \beta\gamma q^6u_1^4u_2^2 + \alpha\delta q^4u_1^4 + \alpha\delta q^4u_2^4 - \gamma^2q^4u_1^2u_2^2\Big)$$

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$$\times \Big(\gamma^2q^8 - \alpha\delta + \alpha\delta q^{16}u_1^4u_2^4 - \alpha\delta q^{12}u_1^4 - \alpha\delta q^{12}u_2^4 + \beta^2q^{12}u_1^2u_2^2$$
$$+ \beta\gamma q^{10}u_1^2 + \beta\gamma q^{10}u_2^2 + \alpha\delta q^8 - \alpha\delta q^8u_1^4u_2^4 - \beta^2q^8u_1^4u_2^4$$
$$- \beta\gamma q^6u_1^2u_2^4 - \beta\gamma q^6u_1^4u_2^2 + \alpha\delta q^4u_1^4 + \alpha\delta q^4u_2^4 - \gamma^2q^4u_1^2u_2^2\Big)$$

Setting $u_1 = u_2 = 1$ yields

$$\frac{1}{(q^4-1)^3(q^4+1)} \cdot \Big(-\alpha\delta + \alpha\delta q^{16} - 2\alpha\delta q^{12} + \beta^2q^{12} + 2\beta\gamma q^{10}$$
$$- \beta^2q^8 + \gamma^2q^8 - 2\beta\gamma q^6 + 2\alpha\delta q^4 - \gamma^2q^4\Big)$$

Example of partition function

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$$\times \Big(\gamma^2 q^8 - \alpha\delta + \alpha\delta q^{16} u_1^4 u_2^4 - \alpha\delta q^{12} u_1^4 - \alpha\delta q^{12} u_2^4 + \beta^2 q^{12} u_1^2 u_2^2$$
$$+ \beta\gamma q^{10} u_1^2 + \beta\gamma q^{10} u_2^2 + \alpha\delta q^8 - \alpha\delta q^8 u_1^4 u_2^4 - \beta^2 q^8 u_1^4 u_2^4$$
$$- \beta\gamma q^6 u_1^2 u_2^4 - \beta\gamma q^6 u_1^4 u_2^2 + \alpha\delta q^4 u_1^4 + \alpha\delta q^4 u_2^4 - \gamma^2 q^4 u_1^2 u_2^2\Big)$$

Setting $u_1 = u_2 = 1$ yields

$$\frac{1}{\left(q^4-1\right)^3\left(q^4+1\right)} \cdot \Big(-\alpha\delta + \alpha\delta q^{16} - 2\alpha\delta q^{12} + \beta^2 q^{12} + 2\beta\gamma q^{10}$$
$$- \beta^2 q^8 + \gamma^2 q^8 - 2\beta\gamma q^6 + 2\alpha\delta q^4 - \gamma^2 q^4\Big)$$

Setting $\alpha = \delta = 1$ and $\beta = \gamma = q - q^{-1}$ yields $\dfrac{q^4+q^2+1}{q^4+1}$,
which is 2 for q any primitive 12-th root of unity.

Elimination of variables

After lots of manipulations. . . . .

Theorem. Let $g(u,v) = \dfrac{(u-v)\left(\frac{i}{\sqrt{3}}(uv-1) - u - v\right)}{1 + uv + u^2v^2}$

Then the number of $(2n \times 2n)$ DSASMs is equal to

$$3^{n^2} \operatorname*{Pf}_{1 \le i,j \le 2n}\Big([u^{i-1}v^{j-1}]g(u+1,v+1)\Big),$$

while the number of $(2n+1) \times (2n+1)$ DSASMs is equal to

$$3^{(n+1)n}(-1)^n \operatorname*{Pf}_{1 \le i,j \le 2n}\Big([u^i v^j]g(u+1,v+1)\Big).$$

Compute matrix entries

$[u^k v^n] g(u+1, v+1)$ for $g(u,v) = \dfrac{(u-v)\left(\frac{i}{\sqrt{3}}(uv-1) - u - v\right)}{1 + uv + u^2v^2}$

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$$g(u+1, v+1) = \left(\frac{i}{\sqrt{3}}(uv + u + v) - u - v - 2\right)\bar{g}(u,v)$$

with $\qquad \bar{g}(u,v) := \dfrac{u-v}{(u+1)^2(v+1)^2 + (u+1)(v+1) + 1}$

Compute matrix entries

$$[u^k v^n]g(u+1, v+1) \text{ for } g(u,v) = \frac{(u-v)\left(\frac{i}{\sqrt{3}}(uv-1) - u - v\right)}{1 + uv + u^2v^2}$$

$$g(u+1, v+1) = \left(\tfrac{i}{\sqrt{3}}(uv+u+v) - u - v - 2\right)\bar{g}(u,v)$$

$$\text{with} \qquad \bar{g}(u,v) := \frac{u-v}{(u+1)^2(v+1)^2 + (u+1)(v+1) + 1}$$

Apply Cauchy's integral formula:

$$\bar{c}(k,n) := [u^k v^n]\bar{g}(u,v) = \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{\bar{g}(u,v)}{u^{k+1}v^{n+1}}\,\mathrm{d}u\,\mathrm{d}v$$

$$c(k,n) := [u^k v^n]g(u+1,v+1) = \tfrac{i}{\sqrt{3}}\bar{c}(k+1,n+1) + \\ + \left(\tfrac{i}{\sqrt{3}} - 1\right)\left(\bar{c}(k+1,n) + \bar{c}(k,n+1)\right) - 2\,\bar{c}(k,n)$$

Recurrences for matrix entries

Employ HolonomicFunctions to derive recurrences for $\bar{c}(k,n)$:

- int1 $= \frac{\bar{g}(u,v)}{u^{k+1}v^{n+1}}$
- int2 $=$ CreativeTelescoping[int1, Der[u], {Der[v], S[k], S[n]}]
- CreativeTelescoping[int2[[1]], Der[v]][[1]]

$$\{(k+1)(n-k+1)S_k+(n+1)(k-n+1)S_n-(k-n-1)(k-n+1),\\
3(n+3)(n+4)(k-n-3)(k-n-2)(k-n-1)(k-n)S_n^4\\
-3(n+3)(k-3n-7)(k-n-4)(k-n-2)(k-n-1)(k-n)S_n^3\\
+(k-n-4)(k-n-3)(k-n-1)(k-n)\\
\quad\times\left(k^2-4kn-7k+10n^2+38n+36\right)S_n^2\\
-(n+1)(k-5n-7)(k-n-4)(k-n-3)(k-n-2)(k-n)S_n\\
+n(n+1)(-k+n+1)(-k+n+2)(-k+n+3)(-k+n+4)\}$$

Chinese remaindering

- Choose primes p such that $x^2 + 3 = 0$ has a solution in $\mathbb{Z}_p$
- compute the matrix entries with the recurrences
- replace $i\sqrt{3}$ by such a solution
- evaluate the determinant mod p and use chinese remaindering

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This way we computed $\mathrm{DSASM}(n)$ for n up to 600:

$$\mathrm{DSASM}(600) = 149641 \ldots\ldots 695744 \approx 1.49641 \cdot 10^{20523}$$

Refined enumeration of DSASMs

For an ASM A let $\rho(A)$ denote the column of the 1 in the top row.

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-1&1\\0&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\quad\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\quad\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&-1&1\\0&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1&0\\0&1&-1&1\\1&-1&1&0\\0&1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Example: for $n = 4$ one gets $F_4(y) = 5y + 5y^2 + 4y^3 + 2y^4$.

Refined enumeration of DSASMs

For an ASM A let $\rho(A)$ denote the column of the 1 in the top row.

Proposition. Let $q = \frac{\sqrt{3}+i}{2}$. Then

$$F_n(y) = \sum_{A \text{ DSASM of order } 2n} y^{\rho(A)} = \sum_{k=1}^{n} \mathrm{DSASM}(n,k) \cdot y^k = \frac{1}{1-2y}$$

$$\times \Bigg(y(1-y)\,\mathrm{DSASM}(2n-1) + (-1)^n i 3^{n(n-1)} \sqrt{3} y^{2n+1} (1-y)^{-2n+1}$$

$$\times \sum_{l=1}^{2n-1} (-1)^{l-1} [u^{i-1}] g\left(u+1, \frac{1+q^4 y}{q^4+y}\right)$$

$$\times \mathrm{Pf}_{1 \le i < j \le 2n-1, i,j \neq l} \left([u^{i-1} v^{j-1}] g(u+1, v+1) \right) \Bigg).$$

Example: for $n = 4$ one gets $F_4(y) = 5y + 5y^2 + 4y^3 + 2y^4$.

Refined enumeration of ASMs

This approach is motivated by the fact that for ASMs, one also has nice closed forms for the refined counting, and that there is a nice relation between ASMs with the top 1 at neighboring positions:

$$(n-k)(n+k-1)\cdot \mathrm{ASM}(n,k) = k(2n-k-1)\cdot \mathrm{ASM}(n,k+1)$$

Challenge: Find a combinatorial proof!

Top 1 at first, second, or last position

We have $\mathrm{DSASM}(n,1) = \mathrm{DSASM}(n,2) = \mathrm{DSASM}(n-1)$
and $\mathrm{DSASM}(n,n) = \mathrm{DSASM}(n-2)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & & & & \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

Top 1 at third position

Lemma. $\mathrm{DSASM}(n,3) = 2 \cdot \mathrm{DSASM}(n-1) - 3 \cdot \mathrm{DSASM}(n-2)$.

Proof. We construct two sets, M_1 and M_2, of $(n \times n)$ matrices.

M_1:

0	0	1	0	$\cdots$	0
0					
1		-1			
0			DSASM		
$\vdots$					
0					

M_2:

0	0	1	0	$\cdots$	0
0	$+1$	-1			
1	-1				
0			DSASM		
$\vdots$					
0					

Observations:

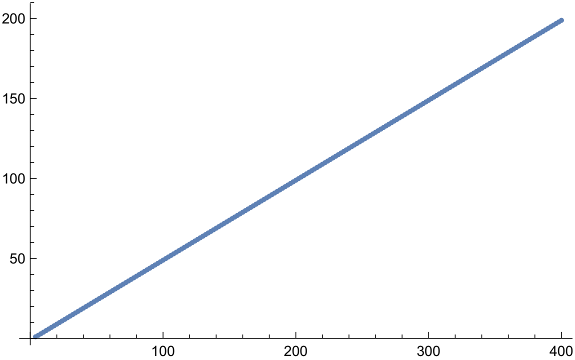
- $|M_1| = |M_2| = \mathrm{DSASM}(n-1)$
- $|M_1 \cap M_2| = \mathrm{DSASM}(n-2)$
- $\mathrm{DSASM}(n-2)$ elements in M_1 are not ASM
- $\mathrm{DSASM}(n-2)$ elements in M_2 are not ASM

What about the fourth position?

Look at the coefficient of y^3 in the transformed generating function

$$\bar{F}_n(y) = \frac{1}{3\ \mathrm{DSASM}(n-2)} \left(\frac{2y-1}{y} F_n(y) - \mathrm{DSASM}(n-1) \cdot (y-1) \right)$$

For $n = 4, 6, 8, \ldots$ plot $\quad q_3(n)$



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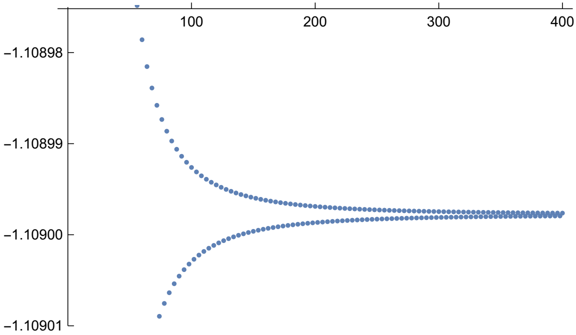
$$\bar{F}_n(y) = \frac{1}{3\ \mathrm{DSASM}(n-2)} \left(\frac{2y-1}{y} F_n(y) - \mathrm{DSASM}(n-1) \cdot (y-1) \right)$$

For $n = 4, 6, 8, \ldots$ plot $\quad q_3(n) - \frac{n}{2}$



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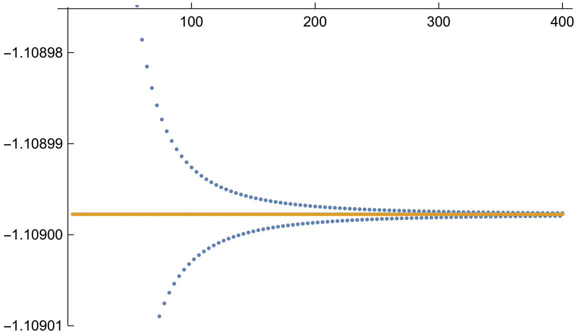
For $n = 4, 6, 8, \ldots$ plot $\quad q_3(n) - \frac{n}{2} + 1.10899775$



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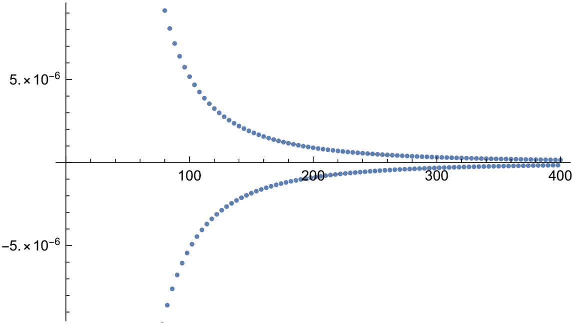
$$\bar{F}_n(y) = \frac{1}{3\ \mathrm{DSASM}(n-2)} \left(\frac{2y-1}{y} F_n(y) - \mathrm{DSASM}(n-1) \cdot (y-1) \right)$$

For $n = 4, 6, 8, \ldots$ plot $\quad q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi}$



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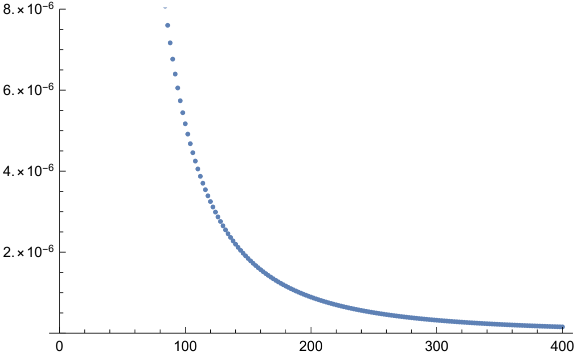
For $n = 4, 6, 8, \ldots$ plot $\left(q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi}\right) \cdot (-1)^{n/2}$



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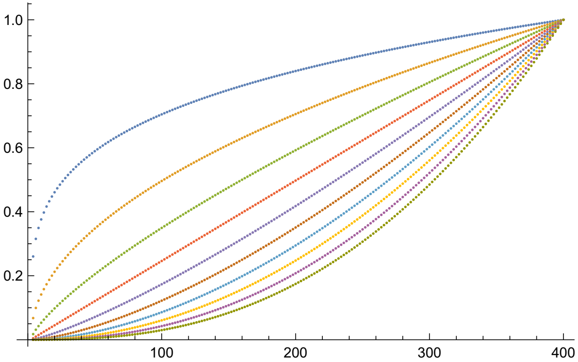
For $n = 4, 6, 8, \ldots$ plot $\left(\left(q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi} \right) \cdot (-1)^{n/2} \right)^{-j/10}$



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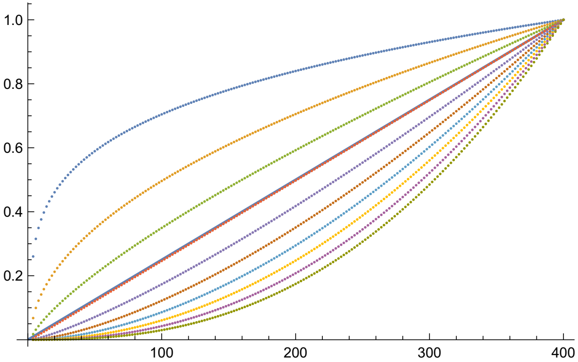
For $n = 4, 6, 8, \dots$ plot $\left(\left(q_3(n) - \frac{n}{2} + \frac{5}{6} + \frac{\sqrt{3}}{2\pi} \right) \cdot (-1)^{n/2} \right)^{-j/10}$



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Look at the coefficient of y^3 in the transformed generating function

$$\bar{F}_n(y) = \frac{1}{3\ \mathrm{DSASM}(n-2)} \left(\frac{2y-1}{y} F_n(y) - \mathrm{DSASM}(n-1) \cdot (y-1) \right)$$

These experiments suggest an asymptotic expansion of the form

$$q_3(n) \approx \frac{n}{2} - \frac{5}{6} - \frac{\sqrt{3}}{2\pi} + c \cdot (-1)^{n/2} \cdot n^{-5/2} + \dots$$

- Confirmed by Jay Pantone, using differential approximants
- He estimates $c \approx 0.086873868...$
- For DASASMs, these coefficients have a closed form as a rational function

Robbins

"Apparently these numbers do not factor into small primes, so a simple product formula seems unlikely. Of course this does not rule out other very simple formulas, but these would be more difficult to discover (let alone prove)."

Announcement

July 22 – 26, 2019 **OPSFA** **Hagenberg, Austria**

15th International Symposium on Orthogonal Polynomials, Special Functions and Applications

News	General	Committees	Registration	Program	Travel

General Information

The 15th International Symposium on Orthogonal Polynomials, Special Functions and Applications (OPSFA) will take place in Hagenberg, Austria, from July 22 to July 26, 2019 (arrival day is July 21). It is organized by the Research Institute for Symbolic Computation (RISC) of the Johannes Kepler University Linz (JKU) and the Johann Radon Institute for Computational and Applied Mathematics (RICAM) of the Austrian Academy of Sciences (ÖAW).

https://www3.risc.jku.at/conferences/opsfa2019