Elliptic hypergeometric series and applications

Christian Krattenthaler

Universität Wien
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
Alin Bostan (11 April 2018):

We would like to invite you to give a plenary talk to a conference that we will organize next year in Brasov (Romania), from 12 to 17 May 2019. The conference is attached to the ERC grant "Elliptic Combinatorics"

http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.
http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.

C.K. (3 May 2018):

Je suis tres honnore par l’invitation.
Il sera un plaisir pour moi d’en faire partie.
Or, en materiere d’"elliptic", je n’ai rien d’en parler ; "combinatorics", normalement, ca va.
Its general topic is algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory.

C.K. (3 May 2018):

Je suis tres honnore par l’invitation.
Il sera un plaisir pour moi d’en faire partie.
Or, en materiere d’"elliptic", je n’ai rien d’en parler ; "combinatorics", normalement, ca va.

Alin Bostan (4 May 2018):

merci beaucoup ! Pourrais-tu aussi bientôt nous envoyer un titre et un résumé ?
Alin Bostan (4 May 2018):

merci beaucoup ! Pourrais-tu aussi bientôt nous envoyer un titre et un résumé ?
Alin Bostan (4 May 2018):
merci beaucoup ! Pourrais-tu aussi bientôt nous envoyer un titre et un résumé ?

C.K. (31 March 2019):
Je réfléchis toujours ...
Alin Bostan (4 May 2018):

merci beaucoup ! Pourrais-tu aussi bientôt nous envoyer un titre et un résumé ?

C.K. (31 March 2019):

Je réfléchis toujours ...

Alin Bostan (31 March 2019):

Si je peux me permettre une suggestion :
il y a un an, tu nous écrivais, avec beaucoup de modestie et d’humour :
> en matière d’"elliptic", je n’ai rien d’en parler ;
> "combinatorics", normalement, ça va.
Or, nous sommes au courant de ton papier elliptico-combinatoire :

"The Determinant of an Elliptic Sylvesteresque Matrix"

n’aurais-tu pas envie de nous parler de ça ?
Elliptic hypergeometric series and applications
Elliptic hypergeometric series and applications

- Hypergeometric Series
Elliptic hypergeometric series and applications

– Hypergeometric Series
– Basic Hypergeometric Series
Elliptic hypergeometric series and applications

- Hypergeometric Series
- Basic Hypergeometric Series
- Elliptic Hypergeometric Series
Elliptic hypergeometric series and applications

– Hypergeometric Series
– Basic Hypergeometric Series
– Elliptic Hypergeometric Series
– Three Applications
The "building blocks" are the shifted factorials (Pochhammer symbols) \((\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\).

The (generalised) hypergeometric series is defined by
\[
\sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{(b_1)_l \cdots (b_s)_l} z^l.
\]

If one chooses \(a_i = -n\) for some \(i\), where \(n\) is a non-negative integer, then the infinite sum becomes a terminating sum since \((-n)_l = (-n)(-n+1)\cdots(1)\cdot0\cdot1\cdots(-n+l-1) = 0\) for \(l\) large enough.
The “building blocks” are the *shifted factorials* (*Pochhammer symbols*)

\[(\alpha)_m = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + m - 1).\]

The *(generalised) hypergeometric* series is defined by

\[rF_s[[a_1, \ldots, a_r; b_1, \ldots, b_s; z]] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{l! (b_1)_l \cdots (b_s)_l} z^l.\]
The “building blocks” are the *shifted factorials* (Pochhammer symbols)

\[(\alpha)_m = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + m - 1).\]

The (generalised) hypergeometric series is defined by

\[rF_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{l! (b_1)_l \cdots (b_s)_l} z^l.\]

If one chooses \(a_i = -n\) for some \(i\), where \(n\) is a non-negative integer, then the infinite sum becomes a terminating sum since

\[(-n)_l = (-n)(-n + 1) \cdots (-1) \cdot 0 \cdot 1 \cdots (-n + l - 1) = 0\]

for \(l\) large enough.
Hypergeometric Series

The (generalised) hypergeometric series is defined by

\[ rF_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \\ \end{array} ; z \right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{l! (b_1)_l \cdots (b_s)_l} z^l. \]
The (generalised) hypergeometric series is defined by

\[ rF_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right] = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_r)_l}{l! (b_1)_l \cdots (b_s)_l} z^l. \]

Hypergeometric series are everywhere:

- \[ e^x = 0F_0 \left[ \begin{array}{c} - \\ - \end{array} ; x \right], \]
- \[ \log x = x \, _2F_1 \left[ \begin{array}{c} 1, 1 \\ \frac{1}{2} \end{array} ; -x \right], \]
- \[ \cos x = 0F_1 \left[ \begin{array}{c} \frac{1}{2} \\ -\frac{x^2}{4} \end{array} ; -\right], \]
- \[ J_\alpha(x) = \left( \frac{x}{2} \right)^\alpha 0F_1 \left[ \begin{array}{c} - \\ \alpha + 1; -\frac{x^2}{4} \end{array} \right]. \]
Binomial sums can (usually) be expressed in terms of hypergeometric series:
Binomial sums can (usually) be expressed in terms of hypergeometric series:

\[ \sum_{k=0}^{n} \binom{m}{k} \binom{n}{k} = \, _2F_1\left[ -m, -n; 1 \right]. \]

This is a special case of the classical Gauss hypergeometric series \( \, _2F_1 \), which converges for \(|x| < 1\), and also for \( x = 1 \) and \( \Re(c-a-b) > 0 \).
Binomial sums can (usually) be expressed in terms of hypergeometric series:

\[
\sum_{k=0}^{n} \binom{m}{k} \binom{n}{k} = \binom{2F1}{-m, -n; 1}.
\]
Binomial sums can (usually) be expressed in terms of hypergeometric series:

\[
\sum_{k=0}^{n} \binom{m}{k} \binom{n}{k} = 2F_1\left[\begin{array}{c}
-m, -n \\
1
\end{array} ; 1 \right].
\]

This is a special case of the classical Gauß hypergeometric series

\[
2F_1\left[\begin{array}{c}
a, b \\
c
\end{array} ; x \right],
\]

which converges for \(|x| < 1\), and also for \(x = 1\) and \(\Re(c - a - b) > 0\).
Gauß already proved that
\[ _2F_1 \left[ a, b; c; 1 \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} , \]
provided \( \Re(c-a-b) > 0. \)

If one specialises \( b = -n, \) one obtains the Chu–Vandermonde summation formula:
\[ _2F_1 \left[ a, -n; c; 1 \right] = \left( c-a \right)^{n} (c)^{n} , \]
where \( n \) is a non-negative integer.
Summation formulas

Gauß already proved that

\[
2F_1 \left[ \begin{array}{c} a, b \\ c \\ \end{array} ; 1 \right] = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)},
\]

provided \( \Re(c - a - b) > 0 \).
Gauß already proved that

\[ 2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} ; 1 \right] = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \]

provided \( \Re(c - a - b) > 0 \).

If one specialises \( b = -n \), one obtains the *Chu–Vandermonde summation formula*:

\[ 2F_1 \left[ \begin{array}{c} a, -n \\ c \end{array} ; 1 \right] = \frac{(c - a)_n}{(c)_n}, \]

where \( n \) is a non-negative integer.
Summation formulas

The Pfaff–Saalschütz summation formula:

\[
\binom{a\,\,b\,\,-n\,\,1+a+b-c-n}{c-a\,\,c-b\,\,c-a-b+n}
\]

provided \(n\) is a non-negative integer.

Dougall's summation formula:

\[
\binom{a\,\,a/2+1\,\,b\,\,c\,\,d\,\,a/2\,\,1+a-b\,\,1+a-c\,\,1+a-d\,\,1+2a-b-c-d+n}{-n-a+b+c+d\,\,a+1+n}
\]

provided \(n\) is a non-negative integer.
Summation formulas

The Pfaff–Saalschütz summation formula:

\[ \sum_{n=0}^{\infty} \binom{a, b, -n}{c, 1 + a + b - c - n'} = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}, \]

provided \( n \) is a non-negative integer.
Hypergeometric Series

Summation formulas

The Pfaff–Saalschütz summation formula:

\[
\binom{3}{2} F_2 \left[ \begin{array}{c} a, b, -n \\ c, 1 + a + b - c - n \end{array} ; 1 \right] = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n},
\]

provided \( n \) is a non-negative integer.

Dougall’s summation formula

\[
\binom{7}{6} F_6 \left[ \begin{array}{c} a, a/2 + 1, b, c, d, \\ a/2, 1 + a - b, 1 + a - c, 1 + a - d, \\ 1 + 2a - b - c - d + n, -n \\ -a + b + c + d - n, a + 1 + n \end{array} ; 1 \right] = \frac{(1 + a)_n (1 + a - b - c)_n (1 + a - b - d)_n (1 + a - c - d)_n}{(1 + a - b)_n (1 + a - c)_n (1 + a - d)_n (1 + a - b - c - d)_n},
\]

provided \( n \) is a non-negative integer.
Hypergeometric Series

Transformation formulas

Euler's transformation formula:
\[ \genfrac{[]}{-1pt}{1}{a,b,c}{2F1} \left[ a, b, c; z \right] = (1-z)^{c-a-b} \genfrac{[]}{-1pt}{1}{c-a,b,c}{2F1} \left[ c-a, c-b, c; z \right], \quad |z| < 1.\]

Kummer's transformation formula:
\[ \genfrac{[]}{-1pt}{1}{a,b,c}{3F2} \left[ a, b, c, d, e; 1 \right] = \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-a) \Gamma(d+e-b-c)} \genfrac{[]}{-1pt}{1}{a,d-b,d-c}{3F2} \left[ d-b, d-c, d+e-b-c; 1 \right], \quad \text{provided both series converge.} \]

Whipple's transformation formula:
\[ \genfrac{[]}{-1pt}{1}{a,b,c}{4F3} \left[ a, b, c, -n,e,f,1+a+b+c-e-f-n; 1 \right] = \left( \frac{e-a}{n} \right)^n \left( \frac{f-a}{n} \right)^n \left( \frac{e}{n} \right)^n \genfrac{[]}{-1pt}{1}{-n,a,1+a+b+c-e-f-n}{4F3} \left[ -n, a, 1+a+b-c-e-f-n; 1 \right]. \]
Hypergeometric Series

Transformation formulas

Euler’s transformation formula:

\[ \begin{align*}
2F1\left[ \begin{array}{c}
a, b \\
c
\end{array} ; z \right] &= (1 - z)^{c-a-b} 2F1\left[ \begin{array}{c}
c - a, c - b \\
c
\end{array} ; z \right],
\end{align*} \]

provided \(|z| < 1\).

Kummer’s transformation formula:

\[ \begin{align*}
3F2\left[ \begin{array}{c}
a, b, c \\
d, e
\end{array} ; 1 \right] &= \frac{\Gamma(e) \Gamma(d + e - a - b - c)}{\Gamma(e - a) \Gamma(d + e - b - c)} 3F2\left[ \begin{array}{c}
a, d - b, d - c \\
d, d + e - b - c
\end{array} ; 1 \right],
\end{align*} \]

provided both series converge.

Whipple’s transformation formula:

\[ \begin{align*}
4F3\left[ \begin{array}{c}
a, b, c, -n \\
e, f, 1 + a + b + c - e - f - n
\end{array} ; 1 \right] &= \frac{(e - a)_n (f - a)_n}{(e)_n (f)_n} \\
\times 4F3\left[ \begin{array}{c}
-n, a, 1 + a + c - e - f - n, 1 + a + b - e - f - n \\
1 + a + b + c - e - f - n, 1 + a - e - n, 1 + a - f - n
\end{array} ; 1 \right]
\end{align*} \]

where \(n\) is a non-negative integer.
Transformation formulas

A transformation between two $\text{}_{9}F_{8}$-series:

\[
\text{$_{9}F_{8}$} \left[ \begin{array}{c}
a, \frac{a}{2} + 1, b, c, d, e, f, \\
a, \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, \\
\lambda + a + 1 + m - e - f, -m, \\
e + f - m - \lambda, a + 1 + m ; 1
\end{array} \right]
= \frac{(1 + a)_m (1 + a - e - f)_m (1 + \lambda - e)_m (1 + \lambda - f)_m}{(1 + \lambda)_m (1 + \lambda - e - f)_m (1 + a - e)_m (1 + a - f)_m}
\times \text{$_{9}F_{8}$} \left[ \begin{array}{c}
\lambda, \frac{\lambda}{2} + 1, \lambda + b - a, \lambda + c - a, \lambda + d - a, e, f, \\
\lambda, \frac{\lambda}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + \lambda - e, 1 + \lambda - f, \\
\lambda + a + 1 + m - e - f, -m, \\
e + f - m - a, \lambda + 1 + m ; 1
\end{array} \right],
\]

where $\lambda = 2a + 1 - b - c - d$ and $m$ is a non-negative integer.
Aren't these series extremely clumsy, random? Who can remember these formulas? Aren't these too many?
 Interrupt

Aren’t these series extremely clumsy, random?
Who can remember these formulas?
Aren’t these too many?
Interruption

Aren’t these series extremely clumsy, random?

\[ \binom{a, b, -n}{c, 1 + a + b - c - n}; 1 = \frac{(c - a)n}{(c - b)n} \frac{(c)n}{(c - a - b)n}. \]
Aren’t these series extremely clumsy, random?

No!

\[\begin{align*}
3F2\left[\begin{array}{c}
a, b, -n \\
c, 1 + a + b - c - n
\end{array} ; 1 \right] &= (c - a)\; n\; (c - b)\; n\; (c - a - b)\; n.
\end{align*}\]

This hypergeometric series is **terminating** and **balanced**, which by definition means that 
\((\text{sum of upper parameters}) + 1 = (\text{sum of lower parameters})\).
Aren’t these series extremely clumsy, random?

No!

Let us look at the Pfaff–Saalschütz summation:

\[
\begin{align*}
3F2 \left[ \begin{array}{c}
a, b, -n \\
c, 1 + a + b - c - n
\end{array} ; 1 \right] &= \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}.
\end{align*}
\]

This hypergeometric series is \textit{terminating} and \textit{balanced}, which by definition means that

\[
(\text{sum of upper parameters}) + 1 = (\text{sum of lower parameters}).
\]
Let us look at Dougall’s summation:

$$\begin{align*}
\, _7F_6 & \left[ \begin{array}{c}
 a, a/2 + 1, b, c, d, \\
a/2, 1 + a - b, 1 + a - c, 1 + a - d, \\
1 + 2a - b - c - d + n, -n \\
-a + b + c + d - n, a + 1 + n; 1
\end{array} \right] \\
= & \frac{(1 + a)_n (1 + a - b - c)_n (1 + a - b - d)_n (1 + a - c - d)_n}{(1 + a - b)_n (1 + a - c)_n (1 + a - d)_n (1 + a - b - c - d)_n}.
\end{align*}$$

This series is terminating and very-well-poised, which by definition means that there is a leading (upper) parameter $a$, and the other upper/lower parameters form pairs all of which sum to $a + 1$, one of these pairs being $(a^2 + 1, a^2)$. The “effect” of the latter pair in the series is the “very-well-poised” factor $(a^2 + 1)^{l (a^2)} = a + 2$.
Let us look at Dougall’s summation:

\[
\begin{aligned}
\pFq{7}{6}{a, a/2 + 1, b, c, d; a/2, 1 + a - b, 1 + a - c, 1 + a - d, 1 + 2a - b - c - d + n, -n; 1}{1 + a - b - c - d + n, -n, a + 1 + n; 1} = \\
\frac{(1 + a)_n (1 + a - b - c)_n (1 + a - b - d)_n (1 + a - c - d)_n}{(1 + a - b)_n (1 + a - c)_n (1 + a - d)_n (1 + a - b - c - d)_n}.
\end{aligned}
\]

This series is *terminating* and *very-well-poised*, which by definition means that there is a *leading (upper)* parameter \( a \), and the other upper/lower parameters form pairs all of which sum to \( a + 1 \), one of these pairs being \( (\frac{a}{2} + 1, \frac{a}{2}) \). The “effect” of the latter pair in the series is the “very-well-poised” factor

\[
\frac{(\frac{a}{2} + 1)_l}{(\frac{a}{2})_l} = \frac{a + 2l}{a}.
\]
The \( \text{9F}_8 \)-transformation formula

\[
\text{9F}_8 \left[ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, \\
\lambda + a + 1 + m - e - f, -m; 1 \right] \\
= \frac{(1 + a)_m (1 + a - e - f)_m (1 + \lambda - e)_m (1 + \lambda - f)_m}{(1 + \lambda)_m (1 + \lambda - e - f)_m (1 + a - e)_m (1 + a - f)_m} \\
\times \text{9F}_8 \left[ \frac{\lambda}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + \lambda - e, 1 + \lambda - f, \\
\lambda + a + 1 + m - e - f, -m; 1 \right]
\]

is a transformation formula between two terminating, very-well-poised \( \text{9F}_8 \)-series.
There is even more “order/structure”:
There is even more “order/structure”:
The $9\ _8F_8$-transformation formula implies all the other summation formulas we have seen.
There is even more “order/structure”:
The $_9F_8$-transformation formula implies all the other summation formulas we have seen.
Namely, we have:

$_9F_8$-transformation formula $\Rightarrow$ Dougall

$\Rightarrow$ Pfaff–Saalschütz $\Rightarrow$ Gauß $\Rightarrow$ Chu–Vandermonde.
In the $9F_8$-transformation formula

$$9F_8\left[\begin{array}{c}
\frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, \\
\lambda + a + 1 + m - e - f, -m \\
e + f - m - \lambda, a + 1 + m
\end{array}; 1\right]$$

$$= \frac{(1 + a)_m (1 + a - e - f)_m (1 + \lambda - e)_m (1 + \lambda - f)_m}{(1 + \lambda)_m (1 + \lambda - e - f)_m (1 + a - e)_m (1 + a - f)_m}$$

$$\times 9F_8\left[\begin{array}{c}
\frac{\lambda}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + \lambda - e, 1 + \lambda - f, \\
\lambda + a + 1 + m - e - f, -m \\
e + f - m - a, \lambda + 1 + m
\end{array}; 1\right],$$

where $\lambda = 2a + 1 - b - c - d$, make the substitution $b \rightarrow a + 1 - c$. This implies $\lambda = a - d$. Thus, the right-hand series “collapses” to just one term, the left-hand series reduces to a $7F_6$-series, and one obtains Dougall’s summation.
In Dougall’s formula

\[ \binom{7}{F_6} \left[ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, \right. \\
\left. \frac{1}{2}, 1 + a - b, 1 + a - c, 1 + a - d, \right. \\
\left. 1 + 2a - b - c - d + n, -n \\
\left. -a + b + c + d - n, a + 1 + n \right] ; 1 \right] \\
= \frac{(1 + a)_n (1 + a - b - c)_n (1 + a - b - d)_n (1 + a - c - d)_n}{(1 + a - b)_n (1 + a - c)_n (1 + a - d)_n (1 + a - b - c - d)_n}. \]

make the substitutions \( a \rightarrow a - n - 1, \ d = -m \) with \( m \) a non-negative integer, and then let \( n \rightarrow \infty \);
In Dougall’s formula

\[ \begin{aligned}
\ {}_7F_6 \left[ \begin{array}{c}
a, a/2 + 1, b, c, d, \\
a/2, 1 + a - b, 1 + a - c, 1 + a - d,
\end{array} \right] \\
&= \frac{(1 + a)_n (1 + a - b - c)_n (1 + a - b - d)_n (1 + a - c - d)_n}{(1 + a - b)_n (1 + a - c)_n (1 + a - d)_n (1 + a - b - c - d)_n}.
\end{aligned} \]

make the substitutions \( a \to a - n - 1 \), \( d = -m \) with \( m \) a non-negative integer, and then let \( n \to \infty \); the result is

\[ \begin{aligned}
\ {}_3F_2 \left[ \begin{array}{c}
b, c, -m \\
a, 1 + b + c - a - m
\end{array} \right] &= \frac{(a - b)_m (a - c)_m}{(a)_m (a - b - c)_m}.
\end{aligned} \]

This is the Pfaff–Saalschütz summation!
In the Pfaff–Saalschütz summation
\[ 3F_2 \left[ \begin{array}{c} a, b, -n \\ c, 1 + a + b - c - n' \end{array} ; 1 \right] = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n} \]
let \( n \to \infty \).
In the Pfaff–Saalschütz summation
\[
\,\!_3F_2\left[\begin{array}{c}
 a, b, -n \\
 c, 1 + a + b - c - n
\end{array}; 1\right] = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}
\]
let \( n \to \infty \). This yields
\[
\,\!_2F_1\left[\begin{array}{c}
 a, b \\
 c
\end{array}; 1\right] = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.
\]
This is Gauß’ summation formula!
In the Pfaff–Saalschütz summation
\[
\begin{aligned}
\binom{3}{2}
\begin{bmatrix}
a, b, -n \\
c, 1 + a + b - c - n
\end{bmatrix}
\end{aligned}
= \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}
\]
let \(n \to \infty\). This yields
\[
\begin{aligned}
\binom{2}{1}
\begin{bmatrix}
a, b \\
c
\end{bmatrix}
\end{aligned}
= \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.
\]
This is Gauß' summation formula!

We have already seen that by specialising \(b = -n\), where \(n\) is a non-negative integer, we get the Chu–Vandermonde summation formula.
Hypergeometric Series

Interruption

Who can remember these formulas?
Who can remember these formulas?

Nobody.
Who can remember these formulas?

Nobody.

Aren’t these too many?
Hypergeometric Series

Interruption

Who can remember these formulas?

Nobody.

Aren’t these too many?

No!
Hypergeometric Series

**Interruption**

Who can remember these formulas?

Nobody.

Aren’t these too many?

No!

An extensive table of summation and transformation formulas (encompassing the tables in the standard book by Lucy Slater) can be found in the documentation for my *Mathematica* package HYP at

https://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html

Moreover, HYP allows you to use these formulas in order to manipulate hypergeometric series.
Remark

Every once in a while, authors discover a “new binomial identity”.

\[ q \sum_{j=0}^{2j} j! (2j−1)!! (−q)^j (q−m)^j = \binom{2m}{m^2 q} \binom{mq}{m}. \]
Remark

Every once in a while, authors discover a “new binomial identity”. It is very, very unlikely that this identity is really new!
Remark

Every once in a while, authors discover a “new binomial identity”. It is very, very unlikely that this identity is really new! Here is a random example from the arXiv:

$$\sum_{j=0}^{q} \frac{2^j}{j! (2j - 1)!!} (-q)_j (q - m)_j = \frac{\binom{2m}{2q}}{\binom{m}{q}}.$$  

The authors say that they were not able to find the identity in the literature.
Remark

Every once in a while, authors discover a “new binomial identity”. It is very, very unlikely that this identity is really new! Here is a random example from the arXiv:

\[
\sum_{j=0}^{q} \frac{2^j}{j! (2j-1)!!} \frac{(-q)_j (q-m)_j}{(m)_j} = \binom{2m}{2q} \binom{m}{q}.
\]

The authors say that they were not able to find the identity in the literature. Obviously so! It is absolutely necessary to write the binomial sum in hypergeometric notation:

\[
_2F_1 \left[ \begin{array}{c} -q, q-m \\ 1/2 \end{array} \right]; 1.
\]
Remark

It is absolutely necessary to write the binomial sum in hypergeometric notation:

$$\, _2 F_1 \left[ \begin{array}{l} q - m, -q \\ 1/2 \end{array} ; 1 \right].$$
Remark

It is absolutely necessary to write the binomial sum in hypergeometric notation:

\[ \binom{q - m, -q}{1/2; 1}. \]

We recognise this series as a special instances of a \( \binom{a, -n}{c; 1} \)-series that can be evaluated by means of the Chu–Vandermonde summation formula

\[ \binom{a, -n}{c; 1} = \frac{(c - a)_n}{(c)_n}, \]

where \( n \) is a non-negative integer.
Motivated by work on Wigner and Racah coefficients, L. C. Biedenharn and J. D. Louck initiated a theory of multidiensional hypergeometric series which they called $U(n)$ hypergeometric series.
The "building blocks" are the $q$-shifted factorials $(\alpha; q)_m = (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \cdots (1 - \alpha q^m - 1)$.

The basic hypergeometric series is defined by

$$r \phi s \left[ a_1, \ldots, a_r \mid b_1, \ldots, b_s ; q, z \right] = \sum_{l=0}^{\infty} \left( a_1 ; q \right)_l \cdots \left( a_r ; q \right)_l \left( q ; q \right)_l \left( b_1 ; q \right)_l \cdots \left( b_s ; q \right)_l \left( -1 \right)^{s-r+1} q^{l^2} z^l.$$
The "building blocks" are the \( q \)-shifted factorials

\[
(\alpha; q)_m = (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \cdots (1 - \alpha q^{m-1}).
\]

The basic hypergeometric series is defined by

\[
{\phi}_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right]_{q, z} = \sum_{l=0}^{\infty} \frac{(a_1; q)_l \cdots (a_r; q)_l}{(q; q)_l (b_1; q)_l \cdots (b_s; q)_l} \left( (-1)^l q^{l(l+1)} \right)^{s-r+1} z^l.
\]
The “building blocks” are the \( q \)-shifted factorials

\[
(\alpha; q)_m = (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \cdots (1 - \alpha q^{m-1}).
\]

The basic hypergeometric series is defined by

\[
\begin{align*}
\phi_r \left[\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right]_{q, z} \\
&= \sum_{l=0}^{\infty} \frac{(a_1; q)_l \cdots (a_r; q)_l}{(q; q)_l(b_1; q)_l \cdots (b_s; q)_l} \frac{(-1)^l q^{l(\binom{l}{2})}}{q^{s-r+1}} z^l.
\end{align*}
\]

If one chooses \( a_i = q^{-n} \) for some \( i \), where \( n \) is a non-negative integer, then the infinite sum becomes a terminating sum since

\[
(q^{-n})_l = (1 - q^{-n}) \cdots (1 - q^{-1}) \cdot (1 - q^0) \cdot (1 - q) \cdots (1 - q^{-n+l-1}) = 0
\]

for \( l \) large enough.
The *basic hypergeometric series* is defined by

\[
_r\phi_s \left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right] = \sum_{l=0}^{\infty} \frac{(a_1; q)_l \cdots (a_r; q)_l}{(q; q)_l (b_1; q)_l \cdots (b_s; q)_l} \left( (-1)^l q^{\binom{l}{2}} \right)^{s-r+1} z^l.
\]
Basic Hypergeometric Series

The *basic hypergeometric series* is defined by

\[ r\phi_s\left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right] = \sum_{l=0}^{\infty} \frac{(a_1; q)_l \cdots (a_r; q)_l}{(q; q)_l (b_1; q)_l \cdots (b_s; q)_l} \left( \frac{-1}{q} \right)^l q^{\binom{l}{2}} (q; q)_s^{s-r+1} z^l. \]

If one makes the substitution \( a_i \rightarrow q^{a_i}, b_i \rightarrow q^{b_i}, z \rightarrow z/(1 - q)^{r-s-1} \), then the above basic hypergeometric series reduces to the (ordinary) hypergeometric series

\[ rF_s\left[ \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right]. \]
Basic Hypergeometric Series

Summation formulas

The $q$-analogue of Gauß' summation formula is Heine's identity

$$2\phi_1\left[ a, b; c; q, c; ab \right] = \left( \frac{c}{a}; q \right)_\infty \left( \frac{c}{b}; q \right)_\infty \left( \frac{c}{q}; q \right)_\infty \left( \frac{c/ab}{q}; q \right)_\infty.$$ 

If one specialises $b = q^{-n}$, one obtains the $q$-Chu–Vandermonde summation formula:

$$2\phi_1\left[ a, q^{-n}c; q, cq^n/a \right] = \left( \frac{c}{a}; q \right)_n \left( \frac{c}{q}; q \right)_n \left( \frac{c/ab}{q}; q \right)_\infty,$$

where $n$ is a non-negative integer.
Summation formulas

The $q$-analogue of Gauß’ summation formula is Heine’s identity

$$2\phi_1\left[a, b \atop c, \frac{c}{ab}ight]_q = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}. $$

If one specialises $b = q - n$, one obtains the $q$-Chu–Vandermonde summation formula:

$$2\phi_1\left[a, q^{-n}c \atop c, \frac{c}{ab}ight]_q = \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n (c/ab; q)_n}. $$
The $q$-analogue of Gauß’ summation formula is Heine’s identity

\[
2\phi_1\left[\begin{array}{c}a, b \\ c \end{array}; q, \frac{c}{ab}\right] = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}.
\]

If one specialises $b = q^{-n}$, one obtains the $q$-Chu–Vandermonde summation formula:

\[
2\phi_1\left[\begin{array}{c}a, q^{-n} \\ c \end{array}; q, \frac{cq^n}{a}\right] = \frac{(c/a; q)_n}{(c; q)_n},
\]

where $n$ is a non-negative integer.
The $q$-Pfaff–Saalschütz summation formula:

$$3\phi_2\left[ a, b, q^{-n}c, abq^{1-n}/c ; q, q \right] = (c/a; q)_n (c/b; q)_n (c; q)_n (c/ab; q)_n,$$

provided $n$ is a non-negative integer. This hypergeometric series is terminating and balanced, which by definition means that $(\text{product of upper parameters}) \times q = (\text{product of lower parameters})$. 

Christian Krattenthaler

Elliptic hypergeometric series
The \( q\)-Pfaff–Saalschütz summation formula:

\[
\begin{align*}
\sum_{n=0}^{\infty} & \left[ \begin{array}{c} a, b, q^{-n} \\ c, abq^{1-n}/c \\ \end{array} \right] (c/a; q)_n (c/b; q)_n \\
& = \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n (c/ab; q)_n},
\end{align*}
\]

provided \( n \) is a non-negative integer.

This hypergeometric series is terminating and balanced, which by definition means that \((\text{product of upper parameters}) \times q = (\text{product of lower parameters})\).
Basic Hypergeometric Series

Summation formulas

The \( q \)-Pfaff–Saalschütz summation formula:

\[
\genfrac{[}{]}{0pt}{}{3}{2} \left[ \begin{array}{c} a, b, q^{−n} \vspace{1ex} \\
\end{array} \right] \begin{array}{c} \vspace{1ex} \begin{array}{c} \vspace{1ex} \\
\end{array} \begin{array}{c} c, abq^{1−n}/c \end{array} \end{array} \right] = \frac{(c/a; q)_n(c/b; q)_n}{(c; q)_n(c/ab; q)_n},
\]

provided \( n \) is a non-negative integer.

This hypergeometric series is \textit{terminating} and \textit{balanced}, which by definition means that

\[(\text{product of upper parameters}) \times q = (\text{product of lower parameters}).\]
Jackson’s summation formula (the $q$-analogue of Dougall’s formula):

$$8\phi_7\left[\begin{array}{c}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n}
\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcd/aq^n, aq^{1+n}; q, q\end{array}\right]$$

$$= \frac{(aq; q)_n(aq/bc; q)_n(aq/bd; q)_n(aq/cd; q)_n}{(aq/b; q)_n(aq/c; q)_n(aq/d; q)_n(aq/bcd; q)_n},$$

provided $n$ is a non-negative integer.
Summation formulas

Jackson’s summation formula (the $q$-analogue of Dougall’s formula):

$$8\phi_7 \left[ \begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2 q^{1+n} / bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcd/aq^n, aq^{1+n}; q, q \end{array} \right] = \frac{(aq; q)_n(aq/bc; q)_n(aq/bd; q)_n(aq/cd; q)_n}{(aq/b; q)_n(aq/c; q)_n(aq/d; q)_n(aq/bcd; q)_n},$$

provided $n$ is a non-negative integer.

This series is terminating and very-well-poised, which by definition means that there is a leading (upper) parameter $a$, and the other upper/lower parameters form pairs whose product is always $aq$, two of these pairs being $(q\sqrt{a}, \sqrt{a})$ and $(-q\sqrt{a}, -\sqrt{a})$. 
Summation formulas

This series is \textit{terminating} and \textit{very-well-poised}, which by definition means that there is a \textit{leading (upper) parameter} $a$, and the other upper/lower parameters form pairs whose product is always $aq$, two of these pairs being $(q\sqrt{a}, \sqrt{a})$ and $(-q\sqrt{a}, -\sqrt{a})$. The “effect” of the latter pairs in the series is the “very-well-poised” factor

$$\frac{(q\sqrt{a}; q)_l (-q\sqrt{a}; q)_l}{(\sqrt{a}; q)_l (-\sqrt{a}; q)_l} = \frac{1 - aq^{2l}}{1 - a}.$$
The $q$-analogue of the $9F_8$-transformation formula:

$$\begin{align*}
10\phi_9 & \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m} \right. \\
 & \left. \sqrt{a}, -\sqrt{a},aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, q \right] \\
= & \frac{(aq; q)_m (aq/ef; q)_m (\lambda q/e; q)_m (\lambda q/f; q)_m}{(\lambda q; q)_m (\lambda q/ef; q)_m (aq/e; q)_m (aq/f; q)_m} \\
& \times 10\phi_9 \left[ \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m} \right. \\
& \left. \sqrt{\lambda}, -\sqrt{\lambda},aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, q \right],
\end{align*}$$

where $\lambda = a^2q/bcd$ and $m$ is a non-negative integer.
The $q$-analogue of the $9F_8$-transformation formula:

\[
10\phi_9\left[a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m}, \\
\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, q\right] = \\
\frac{(aq; q)_m (aq/ef; q)_m (\lambda q/e; q)_m (\lambda q/f; q)_m}{(\lambda q; q)_m (\lambda q/ef; q)_m (aq/e; q)_m (aq/f; q)_m} \\
\times 10\phi_9\left[\lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m}, \\
\sqrt{\lambda}, -\sqrt{\lambda}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, q\right],
\]

where $\lambda = a^2q/bcd$ and $m$ is a non-negative integer.

It is a transformation formula between terminating, very-well-poised hypergeometric series.
Also here we have:

\[ \phi_9^{10} \text{-transformation formula} \Rightarrow \text{Jackson} \]

\[ \Rightarrow q\text{-Pfaff–Saalschütz} \Rightarrow q\text{-Gauß} \Rightarrow q\text{-Chu–Vandermonde}. \]
Basic Hypergeometric Series

Summation and transformation formulas

An extensive table of summation and transformation formulas (encompassing the tables in the standard book by George Gasper and Mizan Rahman) can be found in the documentation for my Mathematica package HYPQ at

https://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html

Moreover, HYPQ allows you to use these formulas in order to manipulate hypergeometric series.
Multidimensional basic hypergeometric series

Stephen Milne, Bob Gustafson, and followers extended the work of Biedenharn and Louck to develop a theory of multi-dimensional basic hypergeometric series associated with root systems.
Elliptic Hypergeometric Series

Given a complex number $p$ with $|p| < 1$, we define
\[
\theta(x; p) := \infty \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^j + 1/x).
\]

Note: $\theta(x; 0) = 1 - x$.

The building blocks are "elliptic shifted factorials":
\[
(a; q, p)_m := \theta(a; p) \theta(aq; p) \cdots \theta(aq^{m-1}; p),
\]

Note: $(a; q, 0)_m = (1 - a)(1 - aq) \cdots (1 - aq^{m-1}) = (a; q)_m$.

We also employ the short notation
\[
(a_1, a_2, \ldots, a_k; q, p)_m = (a_1; q, p)_m (a_2; q, p)_m \cdots (a_k; q, p)_m.
\]
Elliptic Hypergeometric Series

Given a complex number $p$ with $|p| < 1$, we define

$$\theta(x; p) := \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x).$$

Note: $\theta(x; 0) = 1 - x$. 
Given a complex number $p$ with $|p| < 1$, we define

$$\theta(x; p) := \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x).$$

Note: $\theta(x; 0) = 1 - x$.

The building blocks are “elliptic shifted factorials”:

$$(a; q, p)_m := \theta(a; p) \theta(aq; p) \cdots \theta(aq^{m-1}; p),$$

Note: $(a; q, 0)_m = (1 - a)(1 - aq) \cdots (1 - aq^{m-1}) = (a; q)_m$.  

Elliptic Hypergeometric Series
Given a complex number $p$ with $|p| < 1$, we define

$$
\theta(x; p) := \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x).
$$

Note: $\theta(x; 0) = 1 - x$.

The building blocks are “elliptic shifted factorials”:

$$(a; q, p)_m := \theta(a; p) \theta(aq; p) \cdots \theta(aq^{m-1}; p),$$

Note: $(a; q, 0)_m = (1 - a)(1 - aq) \cdots (1 - aq^{m-1}) = (a; q)_m$.

We also employ the short notation

$$(a_1, a_2, \ldots, a_k; q, p)_m = (a_1; q, p)_m (a_2; q, p)_m \cdots (a_k; q, p)_m.$$
A transformation formula

In work on solutions to the Yang–Baxter equation, Frenkel and Turaev found the elliptic hypergeometric transformation formula

$$\sum_{k=0}^{\infty} q^k \theta^k \left( aq^2; p \right) \left( a, b, c, d, e, f; \lambda aq^{1+m/ef}, q^{-m}; q, p \right) k \theta^k \left( a, p \right) \left( q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/\lambda; q, p \right) k = \left( aq, aq/ef, \lambda q/e, \lambda q/f; q, p \right) m$$

where $\lambda = a^2 q^{2bcd}$ and $m$ is a non-negative integer.

It generalises the earlier $10\phi 9$-transformation formula.

By setting $b = aq/c$ the right-hand sum "collapses" to the term for $k = 0$. 

Christian Krattenthaler
In work on solutions to the Yang–Baxter equation, Frenkel and Turaev found the *elliptic hypergeometric transformation formula*

\[
\sum_{k=0}^{\infty} q^k \frac{\theta(aq^{2k}; p)(a, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, p)_k} = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_m}{(\lambda q, \lambda q/ef, aq/e, aq/f; q, p)_m}
\]

\[
\times \sum_{k=0}^{\infty} q^k \frac{\theta(\lambda q^{2k}; p)(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(\lambda; p)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, p)_k}
\]

where \(\lambda = a^2 q/\lambda q/ef\) and \(m\) is a non-negative integer.

It generalises the earlier \(10\phi_9\)-transformation formula.
A transformation formula

In work on solutions to the Yang–Baxter equation, Frenkel and Turaev found the elliptic hypergeometric transformation formula

\[ \sum_{k=0}^{\infty} q^k \frac{\theta(aq^{2k}; p)(a, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, p)_k} \]

\[ = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_m}{(\lambda q, \lambda q/ef, aq/e, aq/f; q, p)_m} \]

\[ \times \sum_{k=0}^{\infty} q^k \frac{\theta(\lambda q^{2k}; p)(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(\lambda; p)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, p)_k} \]

where \( \lambda = a^2 q / bcd \) and \( m \) is a non-negative integer.

It generalises the earlier \( 10\phi_9 \)-transformation formula.

By setting \( b = aq/c \) the right-hand sum “collapses” to the term for \( k = 0 \).
A summation formula

This yields *Frenkel and Turaev’s summation formula*

\[
\sum_{k=0}^{\infty} q^k \frac{\theta(a q^{2k}; p)(a, d, e, f, a^2 q^{1+m}/def, q^{-m}; q, p)_k}{\theta(a; p)(q, aq/d, aq/e, aq/f, defq^{-m}/a, aq^{1+m}; q, p)_k} = \frac{(aq, aq/ef, aq/de, aq/df; q, p)_m}{(aq/d, aq/def, aq/e, aq/f; q, p)_m},
\]

provided \(m\) is a non-negative integer.

It generalises Jackson’s \(8\phi_7\)-summation formula.
A summation formula

This yields Frenkel and Turaev's summation formula

\[
\sum_{k=0}^{\infty} q^k \frac{\theta(aq^{2k}; p)(a, d, e, f, a^2 q^{1+m}/def, q^{-m}; q, p)_k}{\theta(a; p)(q, aq/d, aq/e, aq/f, defq^{-m}/a, aq^{1+m}; q, p)_k} = \frac{(aq, aq/ef, aq/de, aq/df; q, p)_m}{(aq/d, aq/def, aq/e, aq/f; q, p)_m},
\]

provided \(m\) is a non-negative integer.

It generalises Jackson's \(8\phi_7\)-summation formula.

However: Since one cannot take limits, respectively infinite elliptic hypergeometric series are not well-defined since they diverge, there are no elliptic analogues of the \((q-)\)Pfaff–Saalschütz summation or the \((q-)\)Gauß summation!
Nonetheless, Ole Warnaar, Eric Rains, Hjalmar Rosengren, Michael Schlosser, Vyacheslav Spiridonov and followers developed the results of Frenkel and Turaev into a rich theory, and in particular so a theory of *multi-dimensional elliptic hypergeometric series associated with root systems*. This multi-dimensional theory contains much of the multi-dimensional (basic) hypergeometric theory as special cases.
A Multidimensional Elliptic Hypergeometric Series

\[
\prod_{\lambda=1}^{\lambda} \prod_{r=1}^{r} \left( \prod_{\theta=1}^{\theta} \lambda \right) a_{\lambda}^{(\lambda - p_{\lambda})} \left( \prod_{\theta=1}^{\theta} \left( \prod_{\theta=1}^{\theta} \theta \right) \right) \prod_{i=1}^{i} \frac{a_{\lambda}^{(p_{\lambda})}}{a_{\lambda}^{(q_{\lambda})}} \left( \prod_{j=1}^{j} \theta_{\lambda}^{(p_{\lambda})} \right) \prod_{e=1}^{e} \frac{a_{\lambda}^{(f_{\lambda})}}{a_{\lambda}^{(g_{\lambda})}} \left( \prod_{f=1}^{f} \theta_{\lambda}^{(p_{\lambda})} \right) \prod_{r=1}^{r} \frac{a_{\lambda}^{(m_{\lambda})}}{a_{\lambda}^{(n_{\lambda})}} \left( \prod_{m=1}^{m} \theta_{\lambda}^{(p_{\lambda})} \right) \prod_{i=1}^{i} \frac{a_{\lambda}^{(f_{\lambda})}}{a_{\lambda}^{(g_{\lambda})}} \left( \prod_{k=1}^{k} \theta_{\lambda}^{(p_{\lambda})} \right) \prod_{j=1}^{j} \frac{a_{\lambda}^{(m_{\lambda})}}{a_{\lambda}^{(n_{\lambda})}} \left( \prod_{l=1}^{l} \theta_{\lambda}^{(p_{\lambda})} \right). \]

Christian Krattenthaler
Elliptic hypergeometric series
A Multidimensional Elliptic Hypergeometric Series

\[
\sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(aq^{k_i+k_j}; p)^2
\]

\[
\times \prod_{i=1}^r \frac{\theta(aq^{2k_i}; p)(a, b, c, d, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{r-1-m}/\lambda, aq^{1+m}; q, p)_{k_i}}
\]

\[
= \prod_{i=1}^r \frac{(b, c, d, ef/a; q, p)_{i-1}}{(\lambda b/a, \lambda c/a, \lambda d/a, ef/\lambda; q, p)_{i-1}}
\]

\[
\times \prod_{i=1}^r \frac{(aq; q, p)_m (aq/ef; q, p)_{m+1-r} (\lambda q/e, \lambda q/f; q, p)_{m-i+1}}{(\lambda q; q, p)_m (\lambda q/ef; q, p)_{m+1-r} (aq/e, aq/f; q, p)_{m-i+1}}
\]

\[
\times \sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(\lambda q^{k_i+k_j}; p)^2
\]

\[
\times \prod_{i=1}^r \frac{\theta(\lambda q^{2k_i}; p) (\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{\theta(\lambda; p) (q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{r-1-m}/a, \lambda q^{1+m}; q, p)_{k_i}}
\]

where \( \lambda = a^2 q^{2-r}/bcd \).
The one-dimensional special case \((r = 1)\) of this transformation formula is

\[
\sum_{k=0}^{m} q^k \frac{\theta(aq^{2k}; p) (a, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(a; p) (q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, p)_k} \\
= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_m}{(\lambda q, \lambda q/ef, aq/e, aq/f; q, p)_m} \\
\times \sum_{k=0}^{m} q^k \frac{\theta(\lambda q^{2k}; p) (\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_k}{\theta(\lambda; p) (q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, p)_k}
\]

where \(\lambda = a^2 q^{2-r}/bcd\).
The one-dimensional special case ($r = 1$) of this transformation formula is

$$\sum_{k=0}^{m} q^{k} \frac{\theta(aq^{2k}; p) (a, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_{k}}{\theta(a; p) (q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m}; q, p)_{k}}$$

$$= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_{m}}{(\lambda q, \lambda q/ef, aq/e, aq/f; q, p)_{m}}$$

$$\times \sum_{k=0}^{m} q^{k} \frac{\theta(\lambda q^{2k}; p) (\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m}; q, p)_{k}}{\theta(\lambda; p) (q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m}; q, p)_{k}}$$

where $\lambda = a^2 q^{2-r} / bcd$.

This is Frenkel and Turaev’s transformation formula!
A Multidimensional Elliptic Hypergeometric Series

\[
\sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} q^{\sum_{i=1}^{r}(2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(aq^{k_i+k_j}; p)^2
\]

\[
\times \prod_{i=1}^{r} \theta(aq^{2k_i}; p)(a, b, c, d, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}
\]

\[
\times \frac{\prod_{i=1}^{r} \theta(aq^{2k_i}; p)(a, b, c, d, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{r-1-m}/\lambda, aq^{1+m}; q, p)_{k_i}}
\]

\[
= \prod_{i=1}^{r} \frac{(b, c, d, ef/a; q, p)_{i-1}}{(\lambda b/a, \lambda c/a, \lambda d/a, ef/\lambda; q, p)_{i-1}}
\]

\[
\times \prod_{i=1}^{r} \frac{(aq; q, p)_m (aq/ef; q, p)_{m+1-r} (\lambda q/e, \lambda q/f; q, p)_{m-i+1}}{(\lambda q; q, p)_m (\lambda q/ef; q, p)_{m+1-r} (aq/e, aq/f; q, p)_{m-i+1}}
\]

\[
\times \sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} q^{\sum_{i=1}^{r}(2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(\lambda q^{k_i+k_j}; p)^2
\]

\[
\times \prod_{i=1}^{r} \frac{\theta(\lambda q^{2k_i}; p) (\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{\theta(\lambda; p)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{r-1-m}/a, \lambda q^{1+m}; q, p)_{k_i}}
\]

where \( \lambda = a^2 q^{2-r}/bcd \).
A Multidimensional Elliptic Hypergeometric Series

\[ \langle \text{big expression} \rangle = \langle \text{another big expression} \rangle \]

This identity was discovered conjecturally by Ole Warnaar in 2000, and later proved independently by Rains and by Coskun and Gustafson. It was new even when specialised to the $q$-case, or to the (ordinary) hypergeometric case.
\[ \langle \text{big expression} \rangle = \langle \text{another big expression} \rangle \]

This identity was discovered conjecturally by Ole Warnaar in 2000, and later proved independently by Rains and by Coskun and Gustafson.

It was new even when specialised to the $q$-case, or to the (ordinary) hypergeometric case.
3 applications

- Enumeration of standard tableaux of skew shape
- Discrete analogues of Macdonald–Mehta integrals
- Best polynomial approximation

Christian Krattenthaler

Elliptic hypergeometric series
3 applications

1. Enumeration of standard tableaux of skew shape
3 applications

1. Enumeration of standard tableaux of skew shape
2. Discrete analogues of Macdonald–Mehta integrals
3 applications

1. Enumeration of standard tableaux of skew shape
2. Discrete analogues of Macdonald–Mehta integrals
3. Best polynomial approximation
The first application: Counting standard Young tableaux

(joint work with Michael Schlosser)
Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) be two \( n \)-tuples of non-negative integers which are in non-increasing order and satisfy \( \lambda_i \geq \mu_i \) for all \( i \).

A **standard Young tableau** of skew shape \( \lambda/\mu \) is an arrangement of the numbers \( 1, 2, \ldots, \sum_{i=1}^{n}(\lambda_i - \mu_i) \) of the form

\[
\begin{array}{cccc}
\pi_1,\mu_1+1 & \cdots & \cdots & \pi_1,\lambda_1 \\
\pi_2,\mu_2+1 & \cdots & \pi_2,\mu_1+1 & \cdots & \pi_2,\lambda_2 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\pi_n,\mu_n+1 & \cdots & \cdots & \cdots & \pi_n,\lambda_n \\
\end{array}
\]

such that numbers along rows and columns are increasing.
A standard Young tableau of skew shape $\lambda/\mu$ is an arrangement of the numbers $1, 2, \ldots, \sum_{i=1}^{n} (\lambda_i - \mu_i)$ of the form

<table>
<thead>
<tr>
<th>$\pi_{1,\mu_1+1}$</th>
<th>$\cdots$</th>
<th>$\pi_{1,\lambda_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_{2,\mu_2+1}$</td>
<td>$\cdots$</td>
<td>$\pi_{2,\mu_1+1}$</td>
</tr>
<tr>
<td>$\pi_{n,\mu_n+1}$</td>
<td>$\cdots$</td>
<td>$\pi_{n,\lambda_n}$</td>
</tr>
</tbody>
</table>

such that numbers along rows and columns are increasing.
A standard Young tableau of skew shape $\lambda/\mu$ is an arrangement of the numbers $1, 2, \ldots, \sum_{i=1}^{n}(\lambda_i - \mu_i)$ of the form

\[
\begin{array}{cccc}
\pi_{1,\mu_1+1} & \cdots & \cdots & \pi_{1,\lambda_1} \\
\pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots & \pi_{2,\lambda_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\pi_{n,\mu_n+1} & \cdots & \cdots & \cdots & \pi_{n,\lambda_n}
\end{array}
\]

such that numbers along rows and columns are increasing.

A standard Young tableau of shape $(6, 5, 4, 3, 2, 1)/(3, 3, 0, 0, 0, 0)$:

\[
\begin{array}{ccc}
2 & 5 & 13 \\
3 & 9 \\
1 & 4 & 8 & 12 \\
6 & 11 & 15 \\
7 & 14 \\
10
\end{array}
\]
JOHN STEMBRIDGE:

My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?
John Stembridge:

My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?
We shall do something more general than DeWitt here: we shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.
Our goal: Let $N$, $n$, $m$, $r$ be non-negative integers. Compute the number of all standard Young tableaux of shape $(N, N - 1, \ldots, N - n + 1)/(m^r)$, where $(m^r)$ stands for $(m, m, \ldots, m, 0, \ldots, 0)$ with $r$ components $m$.

$N$

$r$

$n$

$m$
Aitken’s Formula

The number of all standard Young tableaux of shape $\lambda/\mu$ equals

$$\left( \sum_{i=1}^{n} (\lambda_i - \mu_i) \right)! \cdot \det_{1 \leq i, j \leq n} \left( \frac{1}{(\lambda_i - i - \mu_j + j)!} \right).$$
The first application: Counting standard Young tableaux

We substitute in Aitken’s formula:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)! \det_{1 \leq i,j \leq n} \begin{pmatrix}
1 & j \leq r \\
\frac{1}{(N + 1 - 2i - m + j)!} & j > r
\end{pmatrix}.
\]
The first application: Counting standard Young tableaux

We substitute in Aitken’s formula:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)! \det_{1 \leq i, j \leq n} \left( \begin{array}{c}
\frac{1}{(N+1-2i-m+j)!} & j \leq r \\
\frac{1}{(N+1-2i+j)!} & j > r
\end{array} \right).
\]

We now do a Laplace expansion with respect to the first \( r \) columns:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \ldots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^{r} k_i} \det_{1 \leq i, j \leq r} \left( \frac{1}{(N+1-2k_i-m+j)!} \right)
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\}} \left( \frac{1}{(N+1-2i+j)!} \right) \cdot \det_{r+1 \leq j \leq n} \left( \frac{1}{(N+1-2i+j)!} \right).
\]
The first application: Counting standard Young tableaux

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \cdots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^{r} k_i} \det_{1 \leq i, j \leq r} \left( \frac{1}{(N + 1 - 2k_i - m + j)!} \right)
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\}} \left( \frac{1}{(N + 1 - 2i + j)!} \right).
\]
The first application: Counting standard Young tableaux

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \cdots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^{r} k_i} \det_{1 \leq i, j \leq r} \left( \frac{1}{(N + 1 - 2k_i - m + j)!} \right)
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\},\ r+1 \leq j \leq n} \left( \frac{1}{(N + 1 - 2i + j)!} \right).
\]

Both determinants can be evaluated by means of

\[
\det_{1 \leq i, j \leq s} \left( \frac{1}{(X_i + j)!} \right) = \prod_{i=1}^{s} \frac{1}{(X_i + s)!} \prod_{1 \leq i < j \leq s} (X_i - X_j),
\]
After a lot of simplification, one arrives at

\[
(-1)^{\binom{r}{2}} 2^\binom{r}{2} + \binom{n-r}{2} \left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\]

\[
\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)! (N-m+r-1)!}
\]

\[
\times \sum_{0 \leq k_1 < \cdots < k_r \leq n-1} \prod_{1 \leq i < j \leq r} (k_j - k_i)^2
\]

\[
\times \prod_{i=1}^{r} \frac{\binom{-N-m+r-1}{2} k_i}{\binom{-N+n-1}{2} k_i} \frac{\binom{-N-m+r-2}{2} k_i}{\binom{-N+n-2}{2} k_i} (-n+1) k_i!
\]
The first application: Counting standard Young tableaux

After a lot of simplification, one arrives at

\[(−1)^{r}\binom{r}{2}2\binom{r}{2} + \binom{n-r}{2}\left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr\right)!
\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)!(N-m+r-1)!}
\times \sum_{0 \leq k_1 < \cdots < k_r \leq n-1 \ 1 \leq i < j \leq r} \prod \frac{(k_j - k_i)^2}{(−N−m+r−1)_{k_i} (−N−m+r−2)_{k_i} (−n+1)_{k_i}} \cdot \prod_{i=1}^{r} \frac{(-N+n-1)_{k_i} (-N+n-2)_{k_i} k_i}{(−N+n-1)_{k_i} (−N+n-2)_{k_i} k_i} \cdot \]

Michael Schlosser immediately saw that our multi-dimensional transformation formula can be applied!
In the elliptic multi-dimensional transformation formula, we let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$. Next we perform the substitutions $b \to q^b$, $c \to q^c$, etc., we divide both sides of the identity obtained so far by $(1 - q)^{r_2}$, and we let $q \to 1$. 
The first application: Counting standard Young tableaux

Corollary

For all non-negative integers \( m, r \) and \( s \), we have

\[
\sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \frac{(d + k_i)_s (b)_k (-m)_{k_i}}{k_i! (f)_{k_i}} \\
= \frac{(-1)^{r(r-1)/2}}{(r + s - 1)!^{s-1}} \prod_{i=1}^{r} (b)_{i-1} \frac{(-f + b + s + 2r - i - m)_{m-r+1}}{(-f - m + i)_{m-i+1}} \\
\times \prod_{i=1}^{r+s-1} \frac{(i - 1)! m!}{(m - i)!} \prod_{i=r}^{r+s-1} \frac{(d - b + 1 - r)_i}{(r + s - i - 1)! (d)_{i-r} (f - b - s + 1 - r)_i} \\
\times \sum_{0 \leq \ell_1 < \ell_2 < \ldots < \ell_s \leq r+s-1} \prod_{1 \leq i < j \leq s} (\ell_i - \ell_j)^2 \\
\times \prod_{i=1}^{s} \frac{(d)_{\ell_i} (f - b - s + 1 - r)_{\ell_i} (-r - s + 1)_{\ell_i}}{\ell_i! (d - b + 1 - r)_{\ell_i} (-m)_{\ell_i}}.
\]
Corollary

For all non-negative integers \(m\), \(r\) and \(s\), we have

\[
\sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \frac{(d + k_i)^s (b) k_i (-m) k_i}{k_i! (f) k_i}
\]

\[\times \prod_{i=1}^{r+s-1} \frac{(i-1)!}{(m-i)!} \prod_{i=r}^{r+s-1} \frac{(d - b + 1 - r)_i}{(r + s - i - 1)! (d)_{i-r} (f - b - s + 1 - r)_i}
\]

\[\times \sum_{0 \leq \ell_1 < \ell_2 < \ldots < \ell_s \leq r+s-1} \prod_{1 \leq i < j \leq s} (\ell_i - \ell_j)^2 \prod_{i=1}^{s} \frac{(d)_{\ell_i} (f - b - s + 1 - r)_{\ell_i} (-r - s + 1)_{\ell_i}}{\ell_i! (d - b + 1 - r)_{\ell_i} (-m)_{\ell_i}}.
\]
The first application: Counting standard Young tableaux

Theorem

If $N - n$ is even, the number of standard Young tableaux of shape $(N, N - 1, \ldots, N - n + 1)/(m^r)$ equals

\[
(-1)^{\left(\frac{(N-n)}{2}\right)} + \frac{1}{2} r(N-n) 2^{\left(\frac{n}{2}\right)} + (N-n-m) r \left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr\right)!
\]

\[
\times \frac{1}{(r + \frac{N-n-2}{2})!(N-n)/2 \left(\frac{N+n-2}{2}\right)!(N-n)/2} \prod_{i=1}^{(N+n)/2} (i - 1)!
\]

\[
\times \prod_{i=1}^{r} \left(\frac{N-n}{2} + i - 1\right)! (n + m - r + 2i - 1)! \left(\frac{n+m-r}{2} + i\right)_{(N-n)/2}
\]

\[
\times \prod_{i=1}^{N-n/2} \left(\prod_{1 \leq i < j \leq \frac{N-n}{2}} (\ell_i - \ell_j)^2\right)
\]

\[
\cdot \prod_{i=1}^{\frac{N-n}{2}} \left(\frac{N-n-2}{2} - \ell_i\right)_{\ell_i} \left(\frac{n+m-r+1}{2} - i\right)_{r+i-\ell_i-1} (N-m-r+2)/2 - i)_{r+i-\ell_i-1}
\]

and there is a similar statement if $N - n$ is odd.
The first application: Counting standard Young tableaux

In the case of a full staircase (i.e., $n = N$), the formula reduces to DeWitt’s original result.

**Corollary**

*The number of standard Young tableaux of shape $(n, n - 1, \ldots, 1)/(m^r)$ equals*

\[
2^{(\begin{array}{c} n \\ 2 \end{array}) - rm} \left( \binom{n+1}{2} - mr \right)! \prod_{i=1}^{n} \frac{(i - 1)!}{(2i - 1)!} \times \prod_{i=1}^{r} \frac{(n + m - r + 2i - 1)! (i - 1)!}{(m + i - 1)! (n - m - r + 2i - 1)!}
\]
The “next” case:

Corollary

The number of standard Young tableaux of shape $(n+1, n, \ldots, 2)/(m^r)$ equals

\[
2^{\binom{n}{2}-(m-1)r} \left( \binom{n+2}{2} - mr - 1 \right)! \prod_{i=1}^{n} \frac{(i-1)!}{(2i)!} \\
\times \prod_{i=1}^{r} \frac{(n + m - r + 2i - 1)! (i - 1)!}{(m + i - 1)! (n - m - r + 2i)!} \\
\times \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} \frac{(n - \ell + 1)_{\ell} \binom{n+m-r}{2}_{r-\ell} \binom{n-m-r+1}{2}_{r-\ell}}{\binom{n+m-r+1}{2}_{r-\ell}}.
\]
In general:
The number of standard Young tableaux of shape 
\((N, N - 1, \ldots, N - n)/(m^r)\) equals an \(\lceil(N - n)/2\rceil\)-fold hypergeometric sum.
The first application: Counting standard Young tableaux

John Stembridge:
The first application: Counting standard Young tableaux

John Stembridge:

I think her approach is much simpler;
John Stembridge:

I think her approach is much simpler; but I don’t think it would extend to the ‘‘next case’’ you mention.
The second application: Discrete M–M-integrals

(joint work with Richard Brent and Ole Warnaar)

Ole Warnaar (15 May 2015):
Together with Richard Brent, I have recently been looking at sums of the form
\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_\alpha_i - k_\alpha_j) \right| \prod_{i=1}^{\gamma r} |k_i| \delta (2n + k_i),
\]
which we call “discrete Mehta-type integrals”.
At least, for \( \alpha, \gamma \in \{1, 2\} \) and small \( \delta \), we believe that these sums can be evaluated in closed form.

Christian Krattenthaler
Elliptic hypergeometric series
Ole Warnaar (15 May 2015):

Together with Richard Brent, I have recently been looking at sums of the form

$$\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right|^{\gamma} \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n+k_i},$$

which we call "discrete Mehta-type integrals".
Ole Warnaar (15 May 2015):

Together with Richard Brent, I have recently been looking at sums of the form

$$\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right| \prod_{i=1}^{r} |k_i|^{\delta} \binom{2n}{n+k_i},$$

which we call "discrete Mehta-type integrals".

At least, for $\alpha, \gamma \in \{1,2\}$ and small $\delta$, we believe that these sums can be evaluated in closed form.
The second application: Discrete M–M-integrals

The Mehta integral

\[
(2\pi)^{-r/2} \int_{\mathbb{R}} \left| \prod_{1 \leq i < j \leq r} (t_i - t_j) \right| \gamma^r \prod_{i=1}^r e^{-t_i^2/2} \, dt_1 \cdots dt_r = \prod_{i=1}^r \Gamma(1 + i \gamma/2) \Gamma(1 + \gamma/2).
\]
The second application: Discrete M–M-integrals

The Mehta integral

\[
(2\pi)^{-r/2} \int_{\mathbb{R}^r} \left| \prod_{1 \leq i < j \leq r} (t_i - t_j) \right|^\gamma \prod_{i=1}^{r} e^{-t_i^2/2} \, dt_1 \cdots dt_r
= \prod_{i=1}^{r} \frac{\Gamma(1 + i\gamma/2)}{\Gamma(1 + \gamma/2)}.
\]
The second application: Discrete M–M-integrals

The Mehta integral

\[
(2\pi)^{-r/2} \int_{\mathbb{R}^r} \left| \prod_{1 \leq i < j \leq r} (t_i - t_j) \right|^\gamma \prod_{i=1}^r e^{-t_i^2/2} \, dt_1 \cdots dt_r
\]

\[
= \prod_{i=1}^r \frac{\Gamma(1 + i\gamma/2)}{\Gamma(1 + \gamma/2)}. 
\]

“Discrete Mehta integrals”

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i - k_j) \right|^\gamma \prod_{i=1}^r \binom{2n}{n + k_i} = ??
\]
The second application: Discrete M–M-integrals

The Mehta integral

\[(2\pi)^{-r/2} \int_{\mathbb{R}^r} \left| \prod_{1 \leq i < j \leq r} (t_i - t_j) \right|^{\gamma} \prod_{i=1}^{r} e^{-t_i^2/2} \, dt_1 \cdots dt_r = \prod_{i=1}^{r} \frac{\Gamma(1 + i\gamma/2)}{\Gamma(1 + \gamma/2)}.
\]

“Discrete Mehta-type integrals”

\[\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right|^{\gamma} \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n + k_i} = ??
\]
The second application: Discrete M–M-integrals

The Mehta integral

\[(2\pi)^{-r/2} \int_{\mathbb{R}^r} \prod_{1 \leq i < j \leq r} (t_i - t_j)^{\gamma} \prod_{i=1}^{r} e^{-t_i^2/2} \, dt_1 \cdots dt_r \]

\[= \prod_{i=1}^{r} \frac{\Gamma(1 + i \gamma/2)}{\Gamma(1 + \gamma/2)}. \]

“Discrete analogues of Macdonald–Mehta integrals”

\[\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha)^{\gamma} \prod_{i=1}^{r} |k_i|^{\delta} \left( \begin{array}{c} 2n \\ n + k_i \end{array} \right) = ?? \]
Ole Warnaar:

Needless to tell you that the case $\alpha = 1$, $\gamma = 2$, $\delta = 0$ follows from specialising a rectangular Schur functions in two sets of variables.
Ole Warnaar:

Needless to tell you that the case $\alpha = 1$, $\gamma = 2$, $\delta = 0$ follows from specialising a rectangular Schur functions in two sets of variables.

Indeed,

\[
\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i} \binom{2m}{m+k_i}
\]

\[
= \prod_{i=1}^{r} \frac{(m+n)^2}{(i-1)} \frac{2n}{i-1} \frac{2m}{i-1} (2m + 2n - i - r + 2)! (i - 1)!^5
\]

can be proved in various ways, one of which is by the use of Schur functions (and a $q$-analogue as well), as I pointed out in a paper 15 years ago.
Ole Warnaar:

Needless to tell you that the case $\alpha = 1$, $\gamma = 2$, $\delta = 0$ follows from specialising a rectangular Schur functions in two sets of variables.

Indeed,

$$\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i} \binom{2m}{m+k_i}$$

$$= \prod_{i=1}^{r} \binom{m+n}{i-1}^2 \binom{2n}{i-1} \binom{2m}{i-1} (2m + 2n - i - r + 2)! (i - 1)!^5$$

can be proved in various ways, one of which is by the use of Schur functions (and a $q$-analogue as well), as I pointed out in a paper 15 years ago.
Indeed,

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i} \binom{2m}{m+k_i}
= \prod_{i=1}^{r} \binom{m+n}{i-1}^2 \binom{2n}{i-1} \binom{2m}{i-1} (2m + 2n - i - r + 2)! (i-1)!^5.
\]
The second application: Discrete M–M-integrals

Indeed,

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \binom{2n}{n + k_i} \binom{2m}{m + k_i}
\]

\[
= \prod_{i=1}^{r} \binom{m + n}{i - 1}^2 \binom{2n}{i - 1} \binom{2m}{i - 1} (2m + 2n - i - r + 2)! (i - 1)!^5.
\]

But, say,

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n + k_i} \binom{2m}{m + k_i} = ??
\]
The second application: Discrete M–M-integrals

Indeed,

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \left( \begin{array}{c} 2n \\ n + k_i \end{array} \right) \left( \begin{array}{c} 2m \\ m + k_i \end{array} \right) = \prod_{i=1}^{r} \left( \frac{m + n}{i - 1} \right)^2 \left( \begin{array}{c} 2n \\ i - 1 \end{array} \right) \left( \begin{array}{c} 2m \\ i - 1 \end{array} \right) \left(2m + 2n - i - r + 2\right)! (i - 1)!^5.
\]

But, say,

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \left( \begin{array}{c} 2n \\ n + k_i \end{array} \right) \left( \begin{array}{c} 2m \\ m + k_i \end{array} \right) = ??
\]

And what about a q-analogue?
The second application: Discrete M–M-integrals

Our discrete analogue of Macdonald–Mehta integrals:

\[
\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right|^{\gamma} \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n + k_i}.
\]

We found (and proved) closed form evaluations in the following cases:

\[
\begin{array}{ccc}
\alpha & \gamma & \delta \\
1 & 1 & 0 \\
1 & 2 & 0 \\
1 & 2 & 3 \\
2 & 2 & 0 \\
2 & 1 & 2 \\
2 & 2 & 3 \\
2 & 3 & 3 \\
\end{array}
\]

We also (eventually) found \( q \)-analogues in most cases.
The second application: Discrete M–M-integrals

Our discrete analogue of Macdonald–Mehta integrals:

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n + k_i}.
\]

We found (and proved) closed form evaluations in the following cases:

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\gamma)</th>
<th>(\delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0, 1, 2, 3</td>
</tr>
</tbody>
</table>
The second application: Discrete M–M-integrals

Our discrete analogue of Macdonald–Mehta integrals:

\[
\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right| \prod_{i=1}^r |k_i|^{\delta} \binom{2n}{n+k_i}.
\]

We found (and proved) closed form evaluations in the following cases:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0, 1, 2, 3</td>
</tr>
</tbody>
</table>

We also (eventually) found $q$-analogues in most cases.
We struggled longest with the discrete analogues of Macdonald–Mehta integrals with $\gamma = 2$:

$$\sum_{k_1, \ldots, k_r \in \mathbb{Z}} \left| \prod_{1 \leq i < j \leq r} (k_i^\alpha - k_j^\alpha) \right|^2 \prod_{i=1}^r |k_i|^\delta \left( \frac{2n}{n + k_i} \right).$$
The second application: Discrete M–M-integrals

For example, we observed that

\[
\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i}
\]

\[
= \sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 (k_i + k_j)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i}
\]

evaluates “nicely”.

Christian Krattenthaler
Elliptic hypergeometric series
The second application: Discrete M–M-integrals

For example, we observed that

$$\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n + k_i} \binom{2m}{m + k_i}$$

$$= \sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 (k_i + k_j)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n + k_i} \binom{2m}{m + k_i}$$

evaluates “nicely”.

How would a $q$-analogue look like? Wouldn’t it contain

$$\sum_{k_1,\ldots,k_r \in \mathbb{Z}} \prod_{1 \leq i < j \leq r} (1 - q^{k_i-k_j})^2 (1 - q^{k_i+k_j})^2 \times \text{stuff}$$
The second application: Discrete M–M-integrals

How did our gigantic transformation formula (with $p = 0$) look like?
The second application: Discrete M–M-integrals

How did our gigantic transformation formula (with \( p = 0 \)) look like?

\[
\sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} q^{\sum_{i=1}^{r} (2i-1)k_i} \prod_{1 \leq i < j \leq r} (1 - q^{k_i - k_j})^2 (1 - aq^{k_i + k_j})^2
\]

\[
\times \prod_{i=1}^{r} \frac{(1 - aq^{2k_i})(a, b, c, d, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q)_{k_i}}{(1 - a)(q, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{r-1-m}/\lambda, aq^{1+m}; q)_{k_i}}
\]

\[
= \prod_{i=1}^{r} \frac{(b, c, d, ef/a; q)_{i-1}}{(\lambda b/a, \lambda c/a, \lambda d/a, ef/\lambda; q)_{i-1}}
\]

\[
\times \prod_{i=1}^{r} \frac{(aq; q)_m (aq/ef; q)_{m+1-r} (\lambda q/e, \lambda q/f; q)_{m-i+1}}{(\lambda q; q)_m (\lambda q/ef; q)_{m+1-r} (aq/e, aq/f; q)_{m-i+1}}
\]

\[
\times \sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} q^{\sum_{i=1}^{r} (2i-1)k_i} \prod_{1 \leq i < j \leq r} (1 - q^{k_i - k_j})^2 (1 - \lambda q^{k_i + k_j})^2
\]

\[
\times \prod_{i=1}^{r} \frac{(1 - \lambda q^{2k_i})(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{2-r+m}/ef, q^{-m}; q)_{k_i}}{(1 - \lambda)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{r-1-m}/a, \lambda q^{1+m}; q)_{k_i}}
\]
So,

\[ \text{Theorem} \]

For all non-negative integers \( m \) and \( n \) and a positive integer \( r \), we have

\[
\sum_{k_1,\ldots,k_r} = -n \prod_{1 \leq i < j \leq r} (k_{2i} - k_{2j})^2 \prod_{i=1}^r k_{2i} \left( \frac{n}{2} + k_i \right) \left( \frac{m}{2} + k_i \right) \Gamma(m + n - i + 2) \Gamma(m + n - i - r + 2) \times \Gamma(2m + 2n - 2i - 2r + 3) \Gamma(2m - 2i + 2) \Gamma(2n - 2i + 2).
\]
The second application: Discrete M–M-integrals

So, if one chooses \( a = q^2 \), \( d = q^{1-n} \), \( e = q^{1-m} \), and \( f = q^2 \), and then lets \( q \to 1 \) in this transformation formula, and one gets:

**Theorem**

For all non-negative integers \( m \) and \( n \) and a positive integer \( r \), we have

\[
\sum_{k_1, \ldots, k_r = -n}^{n} \prod_{1 \leq i < j \leq r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i} = r! 2^{-r} \prod_{i=1}^{r} \left( \frac{\Gamma_{q^{1/2}}(2i) \Gamma(2n+1) \Gamma(2m+1)}{\Gamma(m+n-i+2) \Gamma(m+n-i-r+2)} \right. \\
\left. \times \frac{\Gamma(2m+2n-2i-2r+3)}{\Gamma(2n-2i+2) \Gamma(2m-2i+2)} \right). 
\]
And, if we don’t let $q \to 1$, we get a $q$-analogue for free:

**Theorem**

For all non-negative integers $m$ and $n$ and a positive integer $r$, we have

$$
\sum_{k_1,\ldots,k_r=-n}^{n} \prod_{1 \leq i < j \leq r} [k_j - k_i]_q^2 [k_i + k_j]_q^2 \\
\times \prod_{i=1}^{r} q^{k_i^2-\left(2i-\frac{1}{2}\right)k_i} \left[\frac{k_i}{2}\right]_q [k_i]_q^2 \left[\begin{array}{c}
2n \\
\frac{m + k_i}{q}
\end{array}\right]_q \left[\begin{array}{c}
2m \\
\frac{n + k_i}{q}
\end{array}\right]_q
$$

$$
= r! \left(\frac{2}{\left[2\right]_q}\right)^r [2]_{q^{1/2}}^{-r} q^{-2\left(\frac{r+1}{3}\right)-\frac{1}{2}\left(\frac{r+1}{2}\right)}
\times \prod_{i=1}^{r} \left(\frac{\Gamma_q^{1/2}(2i) \Gamma_q(2n + 1) \Gamma_q(2m + 1)}{\Gamma_q(m + n - i + 2) \Gamma_q(m + n - i - r + 2)}
\times \frac{\Gamma_q^{1/2}(2m + 2n - 2i - 2r + 3)}{\Gamma_q^{1/2}(2n - 2i + 2) \Gamma_q^{1/2}(2m - 2i + 2)}\right).
$$
Similar specialisations worked for the other cases with $\gamma = 2$:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0, 1, 2, 3</td>
</tr>
</tbody>
</table>

except for $\alpha = \delta = 1$. For that case, we have a (lengthy) ad hoc argument, but we don’t know a $q$-analogue.
Yuan Xu (26 August 2017): In work in approximation theory, I encountered a certain determinant (see the attachment). On the basis of computer experiments, I believe that this determinant can be evaluated in closed form. Have you seen it before?
Yuan Xu (26 August 2017):

In work in approximation theory, I encountered a certain determinant (see the attachment). On the basis of computer experiments, I believe that this determinant can be evaluated in closed form. Have you seen it before?
The third application: Best polynomial approximation

The determinant

Let
\[ f(s_1, s_2, r, i, j) := \binom{r}{j - i} \frac{(s_1 + i)_{j-i}}{(s_1 + s_2 + i + j - 1)_{j-i} (s_1 + s_2 + r + 2i)_{j-i}}. \]

Form the matrix
\[ M(r) := \begin{cases} 
  f(s_1, s_2, r, i, j) & \text{for } 0 \leq i < r \\
  (-1)^{j-i-r} f(s_2, s_1, r, i - r, j) & \text{for } r \leq i < 2r 
\end{cases} \]
for \( 0 \leq i, j \leq 2r - 1 \).

Then \( \det M(r) \) seems to be "nice".
The determinant

For example, the matrix $M(2)$ is

$$
\begin{pmatrix}
1 & \frac{2s_1}{(S)(S+2)} & \frac{s_1(s_1+1)}{(S+1)(S+2)^2(S+3)} & 0 \\
0 & 1 & \frac{2(s_1+1)}{(S+2)(S+4)} & \frac{(s_1+1)(s_1+2)}{(S+3)(S+4)^2(S+5)} \\
1 & -\frac{2s_2}{(S)(S+2)} & \frac{2(s_2+1)}{(S+1)(S+2)^2(S+3)} & 0 \\
0 & 1 & -\frac{(s_2+1)(s_2+2)}{(S+2)(S+4)} & \frac{(s_2+1)(s_2+2)}{(S+3)(S+4)^2(S+5)}
\end{pmatrix},
$$

with $S = s_1 + s_2$. 
The third application: Best polynomial approximation

A generalised determinant

Let
\[ f(s_1, s_2, r, i, j) := \binom{r}{j - i} \frac{(s_1 + i)_{j-i}}{(s_1 + s_2 + i + j - 1)_{j-i} (s_1 + s_2 + r + 2i)_{j-i}}. \]

Form the matrix
\[
M(r_1, r_2) := \begin{pmatrix}
    f(s_1, s_2, r_1, i, j) & \text{for } 0 \leq i < r_2 \\
    (-1)^{j-i-r_2} f(s_2, s_1, r_2, i - r_2, j) & \text{for } r_2 \leq i < r_1 + r_2
\end{pmatrix}
\]

Then \( \det M(r_1, r_2) \) seems to be “nice”.

Christian Krattenthaler
Elliptic hypergeometric series
Where does this come from?
Where does this come from?
Consider the triangle

$$\triangle := \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$ 

Define the Jacobi-type weight function

$$\varpi_{\alpha, \beta, \gamma}(x, y) := x^\alpha y^\beta (1 - x - y)^\gamma, \quad \alpha, \beta, \gamma > -1.$$ 

Define

$$E_n(f)_{\alpha, \beta, \gamma} = E_n(f)_{L^2(\varpi_{\alpha, \beta, \gamma})} := \inf_p \| f - p \|_{L^2(\varpi_{\alpha, \beta, \gamma})},$$

where the minimum is over all polynomials in two variables of degree at most \(n\).
The main theorem

Theorem

Let $\alpha, \beta, \gamma > -1$, and let $r$ be a positive integer. For $f \in W_2^r(\varpi_{\alpha, \beta, \gamma})$, we have

$$E_n(f)_{\alpha, \beta, \gamma} \leq \frac{c}{n^r} \left[ E_{n-r}(\partial_1^r f)_{\alpha+r, \beta, \gamma+r} + E_{n-r}(\partial_2^r f)_{\alpha, \beta+r, \gamma+r} + E_{n-r}(\partial_3^r f)_{\alpha+r, \beta+r, \gamma} \right]$$

for $n \geq 3r$, where $c$ is a constant independent of $n$ and $f$. Here, $W_2^r(\varpi_{\alpha, \beta, \gamma})$ is a certain Sobolev space.
The third application: Best polynomial approximation

Which are the main ingredients?

(1) The polynomials

\[ J_{\alpha,\beta,\gamma}^k(x, y) = (x+y)^k J_{\alpha,\beta}^k(y-x) J_{2k+\alpha+\beta+1,\gamma}^{n-k} (1-2x-2y), \quad 0 \leq k \leq n, \]

are orthogonal for the 2-dimensional Jacobi-type weight \( \varpi_{\alpha,\beta,\gamma} \) on the triangle \( \triangle \), where

\[ J_{\alpha,\beta}^n(t) = \frac{(n+\alpha+\beta+1)_n}{n!} P_{\alpha,\beta}^n(t), \]

with \( P_{\alpha,\beta}^n \) the usual Jacobi polynomials.

Christian Krattenthaler
Elliptic hypergeometric series
The third application: Best polynomial approximation

Which are the main ingredients?

(1) The polynomials

\[ J_{k,n}^{\alpha,\beta,\gamma}(x, y) := (x + y)^k J_k^{\alpha,\beta} \left( \frac{y - x}{x + y} \right) J_{n-k}^{2k+\alpha+\beta+1,\gamma}(1 - 2x - 2y), \]

\[ 0 \leq k \leq n, \]

are orthogonal for the 2-dimensional Jacobi-type weight \( \varpi_{\alpha,\beta,\gamma} \) on the triangle \( \triangle \), where

\[ J_n^{\alpha,\beta}(t) = \frac{1}{(n + \alpha + \beta + 1)_n} P_n^{(\alpha,\beta)}(t), \]

with \( P_n^{(\alpha,\beta)} \) the usual Jacobi polynomials.
The third application: Best polynomial approximation

Which are the main ingredients?
The third application: Best polynomial approximation

Which are the main ingredients?

(2) The following determinant evaluation:

**Theorem**

With $f(s_1, s_2, r, i, j)$ as defined before and

$$M(r) := \begin{pmatrix} f(s_1, s_2, r, i, j) & \text{for } 0 \leq i < r \\ (-1)^{j-i-r} f(s_2, s_1, r, i-r, j) & \text{for } r \leq i < 2r \end{pmatrix}_{0 \leq i, j \leq 2r-1}.$$

the determinant of $M(r)$ equals

$$(-1)^r \prod_{j=1}^{r} \frac{1}{(s_1 + s_2 + 2r + j - 2)_r}.$$
Let
\[ f(s_1, s_2, r, i, j) := \binom{r}{j-i} \frac{(s_1 + i)_{j-i}}{(s_1 + s_2 + i + j - 1)_{j-i} (s_1 + s_2 + r + 2i)_{j-i}} \]
and
\[ M(r_1, r_2) := \begin{pmatrix} f(s_1, s_2, r_1, i, j) & \text{for } 0 \leq i < r_2 \\ (-1)^{j-i-r_2} f(s_2, s_1, r_2, i - r_2, j) & \text{for } r_2 \leq i < r_1 + r_2 \end{pmatrix}_{0 \leq i, j \leq r_1 + r_2 - 1} \]
The third application: Best polynomial approximation

Let
\[ f(s_1, s_2, r, i, j) := \binom{r}{j-i} \frac{(s_1 + i)_{j-i}}{(s_1 + s_2 + i + j - 1)_{j-i} (s_1 + s_2 + r + 2i)_{j-i}} \]

and
\[ M(r_1, r_2) := \begin{pmatrix}
 f(s_1, s_2, r_1, i, j) & \text{for } 0 \leq i < r_2 \\
 (-1)^{j-i-r_2} f(s_2, s_1, r_2, i - r_2, j) & \text{for } r_2 \leq i < r_1 + r_2
\end{pmatrix}_{0 \leq i, j \leq r_1 + r_2 - 1} \]

How to calculate the determinant of \( M(r_1, r_2) \)?
The third application: Best polynomial approximation

Let
\[ f(s_1, s_2, r, i, j) := \binom{r}{j - i} \frac{(s_1 + i)_{j - i}}{(s_1 + s_2 + i + j - 1)_{j - i} (s_1 + s_2 + r + 2i)_{j - i}} \]
and
\[ M(r_1, r_2) := \begin{cases} f(s_1, s_2, r_1, i, j) & \text{for } 0 \leq i < r_2 \\ (-1)^{j - i - r_2} f(s_2, s_1, r_2, i - r_2, j) & \text{for } r_2 \leq i < r_1 + r_2 \end{cases} \]

How to calculate the determinant of \( M(r_1, r_2) \)?

Laplace expansion again!
Laplace expansion

\[ \text{det} M = \sum_{0 \leq k_0 < \ldots < k_{r_2 - 1} \leq r_1 + r_2 - 1} (-1)^{k_2^2} \cdot \prod_{i=0}^{r_2-1} \text{det} M_{k_0, \ldots, k_{r_2-1} r_1, \ldots, r_1 + r_2 - 1} \]
The third application: Best polynomial approximation

Laplace expansion

Write $M$ for $M(r_1, r_2)$ for short.

Then

$$\det M = \sum_{0 \leq k_0 < \cdots < k_{r_2-1} \leq r_1 + r_2 - 1} (-1)^{\binom{r_2}{2} + \sum_{i=0}^{r_2-1} k_i} \det M_{0, \ldots, r_2-1}^{k_0, \ldots, k_{r_2-1}}$$

$$\cdot \det M_{r_2, \ldots, r_1 + r_2 - 1}^{l_0, \ldots, l_{r_1-1}},$$

where $M^{a_1, \ldots, a_r}_{b_1, \ldots, b_r}$ denotes the submatrix of $M$ consisting of rows $a_1, \ldots, a_r$ and columns $b_1, \ldots, b_r$, and $\{l_0, \ldots, l_{r_1-1}\}$ is the complement of $\{k_0, \ldots, k_{r_2-1}\}$ in $\{0, 1, \ldots, r_1 + r_2 - 1\}$. 
Laplace expansion

Write $M$ for $M(r_1, r_2)$ for short.

Then

$$
\det M = \sum_{0 \leq k_0 < \cdots < k_{r_2-1} \leq r_1 + r_2 - 1} (-1)^{r_2 \choose 2} + \sum_{i=0}^{r_2-1} k_i \det M_{k_0, \ldots, k_{r_2-1}}^{k_0, \ldots, r_2-1} \cdot \det M_{l_0, \ldots, l_{r_1-1}}^{l_0, \ldots, r_1-1, r_2-1},
$$

where $M_{b_1, \ldots, b_r}^{a_1, \ldots, a_r}$ denotes the submatrix of $M$ consisting of rows $a_1, \ldots, a_r$ and columns $b_1, \ldots, b_r$, and $\{l_0, \ldots, l_{r_1-1}\}$ is the complement of $\{k_0, \ldots, k_{r_2-1}\}$ in $\{0, 1, \ldots, r_1 + r_2 - 1\}$.

Also here, it turns out that it is not difficult to evaluate the minors which appear in this sum.
After a lot of simplification, one arrives at

\[
(-1)^{r_1 r_2} \prod_{i=0}^{r_1+r_2-1} \frac{(s_2)_i (s_1 + s_2 + i - 2)! (i + s_1 + s_2 - 1)_i}{(s_1 + s_2 + 2i - 2)! (r_1 + r_2 - i - 1)! (s_1 + s_2 + r_1 + r_2 + i - 2)!} \\
\times \prod_{i=0}^{r_2-1} \frac{(s_1 + s_2 + r_1 + 2i - 2)! (s_1 + s_2 + r_1 + 2i - 1)! (r_1 + i)!}{(s_1)_i (s_1 + s_2 + r_1 + i - 2)! (r_1 + r_2 - 1)! (s_1 + s_2)_{r_1+r_2-1}} \\
\times \prod_{i=0}^{r_1-1} \frac{(s_1 + s_2 + r_2 + 2i - 2)! (s_1 + s_2 + r_2 + 2i - 1)! (r_2 + i)!}{(s_2)_i (s_1 + s_2 + r_2 + i - 2)!} \\
\times \sum_{0 \leq k_0 < \ldots < k_{r_2-1} \leq r_1+r_2-1} (-1)^{\sum_{i=0}^{r_2-1} k_i} \prod_{0 \leq i < j \leq r_2-1} (k_j-k_i)^2 (k_i+k_j+s_1+s_2-1)^2 \\
\cdot \prod_{i=0}^{r_2-1} \frac{(s_1 + s_2 - 1 + 2k_i)}{(s_1 + s_2 - 1)} \cdot \frac{(s_1 + s_2 - 1)_{k_i} (s_1)_{k_i} (-r_1 - r_2 + 1)_{k_i}}{k_i! (s_2)_{k_i} (s_1 + s_2 + r_1 + r_2 - 1)_{k_i}}.
\]
After a lot of simplification, one arrives at

\[
(-1)^{r_1r_2} \prod_{i=0}^{r_1+r_2-1} \frac{(s_2)_i (s_1 + s_2 + i - 2)! (i + s_1 + s_2 - 1)_i}{(s_1 + s_2 + 2i - 2)! (r_1 + r_2 - i - 1)! (s_1 + s_2 + r_1 + r_2 + i - 2)!} \\
\times \prod_{i=0}^{r_2-1} \frac{(s_1 + s_2 + r_1 + 2i - 2)! (s_1 + s_2 + r_1 + 2i - 1)! (r_1 + i)!}{(s_1)_i (s_1 + s_2 + r_1 + i - 2)! (r_1 + r_2 - 1)! (s_1 + s_2)_{r_1+r_2-1}} \\
\times \prod_{i=0}^{r_1-1} \frac{(s_1 + s_2 + r_2 + 2i - 2)! (s_1 + s_2 + r_2 + 2i - 1)! (r_2 + i)!}{(s_2)_i (s_1 + s_2 + r_2 + i - 2)!} \\
\times \sum_{0 \leq k_0 < \cdots < k_{r_2-1} \leq r_1+r_2-1} (-1)^{\sum_{i=0}^{r_2-1} k_i} \prod_{0 \leq i < j \leq r_2-1} (k_j - k_i)^2 (k_i + k_j + s_1 + s_2 - 1)^2 \\
\cdot \prod_{i=0}^{r_2-1} \frac{(s_1 + s_2 - 1 + 2k_i)}{(s_1 + s_2 - 1)} \cdot \frac{(s_1 + s_2 - 1)_k (s_1)_k (-r_1 - r_2 + 1)_k}{k_i! (s_2)_k (s_1 + s_2 + r_1 + r_2 - 1)_k}.
\]

Now apply the \( p = 0, \ q \to 1 \) case of the transformation formula.
The third application: Best polynomial approximation

An elliptic generalisation:

**Theorem**

We consider the \((r_1 + r_2) \times (r_1 + r_2)\) matrix \(F = (f_{ij}')\), where

\[
f_{ij}' = \begin{cases} 
    f_{ij}(s_1, s_2, t_1, t_2, r_1, r_2), & \text{for } 0 \leq i \leq r_2 - 1, \\
    f_{i-r_2,j}(s_2, s_1, s_1s_2/t_1, s_1s_2/t_2, r_2, r_1), & \text{for } r_2 \leq i \leq r_1 + r_2 - 1,
\end{cases}
\]

with

\[
f_{ij} = f_{ij}(s_1, s_2, t_1, t_2, r_1, r_2) := q^{(j-i)/2} + r_2(j-i) \frac{(q; q, p)_{r_1}}{(q; q, p)_{r_1 - j + i}} \frac{(q; q, p)_{r_1}}{(q; q, p)_{r_1 - j + i}} \times \frac{(s_1 q^i, t_1 q^i, t_2 q^i, s_1 s_2^2 q^{r_1-r_2+i}/t_1 t_2; q, p)_{j-i}}{(q, s_1 s_2 q^{i+j-1}, s_1 s_2 q^{r_1+2i}; q, p)_{j-i}}.
\]
The third application: Best polynomial approximation

Then

\[
\det F = (-1)^{r_1 r_2} \left( \frac{t_1 t_2}{s_2} \right)^{r_1 r_2} q^{\frac{1}{2} r_1 r_2 (r_1 + 4 r_2 - 3)} \times \prod_{j=1}^{r_1} \frac{(s_2 q^{-r_2 + j} / t_1, s_2 q^{-r_2 + j} / t_2, s_1 s_2 q^{-r_2 + j} / t_1 t_2; q, p)_{r_2}}{(s_1 s_2 q^{r_1 + r_2 + j - 2}; q, p)_{r_2}}.
\]
Elliptic hypergeometric series
Transient Transcendence in Transylvania?

Christian Krattenthaler

Elliptic hypergeometric series