Coordinates of Pell equations in various sequences

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Pell equations

Let d be a positive integer which is not a square. The Pell equation corresponding to d is the equation

$$X^2 - dY^2 = \pm 1 \tag{1}$$

to be solved in positive integers (X, Y).

It is known that (1) always has positive integer solutions. Letting (X_1, Y_1) be the smaller positive integer solution of it, all other solutions are of the form (X_n, Y_n) with

$$X_n + \sqrt{d}Y_n = (X_1 + \sqrt{d}Y_1)^n \quad \text{for all} \quad n \geq 1.$$

Our problem: first attempt

Let $\mathcal{U}$ be your favorite set of positive integers. What can one say about d such that the equation

$$X_n \in \mathcal{U} \quad \text{for some} \quad n? \tag{2}$$

Unfortunately, if one formulates it in this way, the above problem is trivial. Namely, let $u \in \mathcal{U}$. Write

$$u^2 + 1 = dv^2,$$

for some squarefree integer d. Then

$$u^2 - dv^2 = -1,$$

so $u = X_n$ for some $n \geq 1$ corresponding to d. If $u > 1$, we can play the same game with

$$u^2 - 1 = dv^2.$$

Our problem: second attempt

Since our first attempt seemed to have a trivial answer, we try the following potentially more interesting problem:

What can we say about d such that

$$X_n \in \mathcal{U}$$

holds for at least **two** *different values of n?*

That is, we now look for values of the squarefree integer d such that the equation

$$U^2 - dV^2 = \pm 1,$$

has two different positive integer solutions $(U, V) \neq (U', V')$ with $\{U, U'\} \subset \mathcal{U}$.

When $\mathcal{U}$ are the base 10-repdigits

Take

$$\mathcal{U} = \left\{ a\left(\frac{10^m - 1}{9}\right); 1 \leq a \leq 9,\ m \geq 1 \right\}.$$

The elements of $\mathcal{U}$ are base 10 repdigits since

$$a\left(\frac{10^m - 1}{9}\right) = \underbrace{\overline{aa\cdots a}}_{m \text{ times}}.$$

Theorem

*(Dossavi-Yovo, L., Togbé, **2016**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = 1.$$

The equation $X_n \in \mathcal{U}$ has at most one solution n except:

(i) $d = 2$ *for which* $n \in \{1, 3\}$;

(ii) $d = 3$ *for which* $n \in \{1, 2\}$.

Appolinaire Dossavi-Yovo

When $\mathcal{U}$ are the Fibonacci numbers

Let $\mathcal{U}$ be the sequence of all Fibonacci numbers given by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$.

Theorem

*(L., Togbé, **2016**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 1$$

The equation $X_n \in \mathcal{U}$ has at most one solution n except for $d = 2$ in which case $n \in \{1, 2\}$.

The above result can be reformulated by saying that the only nontrivial solutions of the Diophantine equation

$$(F_n^2 \pm 1)(F_m^2 \pm 1) = \square$$

are $(n, m) = (1, 4),\ (2, 4)$.

Variations: Repdigits in an arbitrary base

Let $g \geq 2$ be an integer and

$$\mathcal{U}_g = \left\{ a\left(\frac{g^m - 1}{g - 1}\right); 1 \leq a \leq g - 1,\ m \geq 1 \right\}.$$

Members of $\mathcal{U}_g$ are called base-g-repdigits.

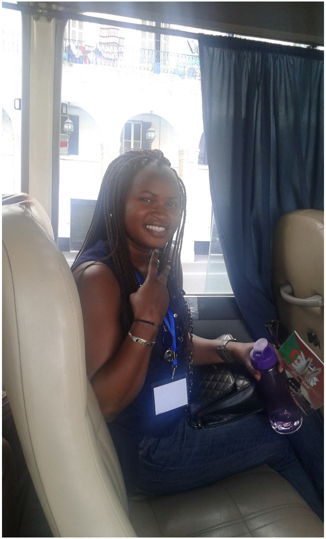
Theorem

(Faye, L. **2016***). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = 1.$$

If $X_n \in \mathcal{U}$ has two solutions n, then

$$d < \exp\left((10g)^{10^5}\right).$$



Bernadette Faye

Variations: with Fibonacci numbers

Let again

$$\mathcal{U} = \{F_n : n \geq 4\}.$$

Theorem

*(Kafle, L., Togbé, **2016**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 4. \tag{3}$$

The equation $X_n \in \mathcal{U}$ has at most one solution n.

Allowing also the "small Fibonacci numbers", we get that the equation $X_n \in \mathcal{U}$, where (X_n, Y_n) satisfies (8) has only one solution n except when $d \in \{2, 5\}$ for which all n have $n \leq 4$.

Bir Kafle

Proofs so far: The Fibonacci case

Assume that (X_n, Y_n) is the nth solution to

$$X^2 - dY^2 = 1,$$

and say $X_n = F_m$ for some m. Assume first that n is even. Then

$$X_n = 2X_{n/2}^2 - 1.$$

So, we need to find Fibonacci numbers F_m such that $F_m = 2X^2 - 1$. Via the relation

$$L_m^2 - 5F_m^2 = \pm 4,$$

where $\{L_m\}_{m\geq 1}$ is the Lucas companion of $\{F_m\}_{m\geq 1}$, we get

$$Y^2 = 5(2X^2 - 1)^2 \pm 4.$$

This reduces to finding all integer points on some elliptic curves. This gives a few d and for each d, we find all the solutions n.

Next let $n_1 < n_2$ and $m_1 < m_2$ such that

$$X_{n_1} = F_{m_1} \quad \text{and} \quad X_{n_2} = F_{m_2}.$$

We apply gcd getting that

$$\gcd(X_{n_1}, X_{n_2}) = \gcd(F_{m_1}, F_{m_2}).$$

In the left we have $X_{\gcd(n_1,n_2)}$. In the right we have $F_{\gcd(m_1,m_2)}$. So, we can replace (n_1, n_2) by $(\gcd(n_1, n_2), n_2)$ and then $X_{\gcd(n_1,n_2)}$ b y X_1. Also, (m_1, m_2) by $(m_1, m_1 t)$. So, with

$$\alpha = X_1 + \sqrt{d} Y_1, \quad \beta = X_1 - \sqrt{d} Y_1,$$

we get

$$\begin{aligned} X_1 &= \frac{\alpha + \beta}{2} = F_{m_1}; \\ X_n &= \frac{\alpha^n + \beta^n}{2} = F_{m_1 t}. \end{aligned}$$

From now on, the argument has an analytic part and an arithmetic part. The analytic part shows that putting

$$(\gamma, \delta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right),$$

we have

$$\begin{aligned} \left|\alpha - \frac{2}{\sqrt{5}}\gamma^{m_1}\right| &< \frac{4}{\gamma^{m_1}}; \\ \left|\alpha^n \frac{\sqrt{5}}{2}\gamma^{-m_1 t} - 1\right| &< \frac{2\sqrt{5}}{\gamma^{2m_1 t}}. \end{aligned}$$

Also, $n > t$.

The arithmetic part claims that

$$\gamma^{m_1} < 6n^2.$$

To do this, one shows that

$$\frac{X_n}{X_1} \equiv \pm n \pmod{X_1}.$$

Secondly,

$$\left(\frac{F_{m_1 t}}{F_{m_1}}\right)^2 \equiv \pm t^2 \pmod{F_{m_1}}.$$

So, from our equations, we get that

$$F_{m_1} \mid n^2 \pm t^2,$$

which implies the arithmetic inequality.

Feeding these into some lower bound for a linear form in logarithms, we get

$$n \ll \log(m_1 t) \ll \log n,$$

which gives a bound on n. Specifically, $n \leq 7 \times 10^{14}$ and $m_1 \leq 145$. The rest is just calculations.
It is clear that this method also works for the case of (X_n, Y_n) being solutions to

$$X^2 - dY^2 = \pm 4$$

when dealing with $X_n \in \{F_m : m \geq 1\}$ having two solutions n.

Proofs so far: The repdigit case

This is similar. Indeed, say

$$X^2 - dY^2 = 1$$

and

$$X_n = a\left(\frac{10^m - 1}{9}\right).$$

with n even. Assuming $a \neq 9$, writing $m = r_0 + 3m_0$, $X := X_{n/2}$ and $Y := 10^{m_0}$, we get

$$2X^2 - 1 = a\left(\frac{10^{r_0}Y^3 - 1}{9}\right) \quad 1 \leq a \leq 8,$$

which reduces to integer points on elliptic curves.

In case

$$X_{n_1} = a_1\left(\frac{10^{m_1}-1}{9}\right), \quad X_{n_2} = a_2\left(\frac{10^{m_2}-1}{9}\right),$$

with n_1 and n_2 odd, we use the fact that

$$\gcd\left(a_1\left(\frac{10^{m_1}-1}{9}\right), a_2\left(\frac{10^{m_2}-1}{9}\right)\right) = a_3\left(\frac{10^{m_3}-1}{9}\right),$$

where $m_3 = \gcd(m_1, m_2)$ and

$$a_3 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 21, 63\}.$$

Up to substitutions we get

$$\begin{aligned} X_1 &= a\left(\frac{10^{m_1}-1}{9}\right), \quad a \in \{1, 2, \ldots, 9, 21, 63\}; \\ X_n &= b\left(\frac{10^{mt}-1}{9}\right), \quad b \in \{1, \ldots, 9\}. \end{aligned}$$

looking very similar to the system of two equations from the Fibonacci case.

Example

(i)

$$\gcd(333333, 777777777) = 2331 = 21 \cdot 111.$$

(ii)

$$\gcd(999999, 777777777777777777777777777) = 63 \cdot 111.$$

Now an analytic argument shows that

$$0 < \left| \frac{2b}{9} 10^{mt} \alpha^{-n} - 1 \right| < \frac{1}{\alpha^{n-1}},$$

and an arithmetic argument shows that $m < 4 \times 10^{16} \log n$. Together they give

$$t < 6 \times 10^{15}, \quad m \le 2 \times 10^{18}, \quad n \le 3 \times 10^{15}.$$

The rest is just calculations. The method works for any base. There, combined with theoretical lower bounds for linear forms in logs gives the main result of Faye, L.

How about for the Tribonacci sequence

Let $\mathcal{U}$ be the sequence of Tribonacci numbers given by $T_1 = T_2 = 1, \ T_3 = 2$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 1$.

Theorem

(L., Montejano, Szalay, Togbé, **2016***). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 1. \tag{4}$$

The equation $X_n = T_m$ has at most one solution (n, m) except:

(i) $(n, m) = (1, 3)$ *and* $(2, 5)$ *in the* $+$ *case* $(d = 3)$;

(ii) $(n, m) = (1, 1), \ (1, 2), \ (3, 5)$ *in the* $-$ *case* $(d = 2)$.

The ALFA team.

This problem is different from the previous ones because:

We have no idea how to solve

$$T_m = 2X^2 - 1.$$

We have no idea how to relate

$$\gcd(T_m, T_n)$$

back to members of the Tribonacci sequence.

New idea

Instead we just used two linear forms in logarithms and a clever linear combination of them. Here is at work. Say

$$(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right), \quad \delta = X_1 + \sqrt{d}Y_1.$$

Then $F_n = X_m$ is equivalent to

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\delta^m + \delta^{-m}}{2}.$$

This implies

$$n \log \alpha + \log(2/\sqrt{5}) - m \log \delta = O\left(\min\left\{\frac{1}{\alpha^n}, \frac{1}{\delta^m}\right\}\right).$$

Linear forms in logs give $m \ll \log n$ and $n \ll \log \delta \log n$. Unfortunately we don't know δ.

But say we have another such relation $F_{n'} = X_{m'}$. Then also

$$n' \log \alpha + \log(2/\sqrt{5}) - m' \log \delta = O\left(\min\left\{\frac{1}{\alpha^{n'}}, \frac{1}{\delta^{m'}}\right\}\right).$$

Then we do linear algebra and assuming $n < n'$, we get

$$(n'm - m'n) \log \alpha - (m - m') \log(2/\sqrt{5}) = O\left(\frac{n'}{\alpha^n}\right).$$

This gives $n \ll \log n'$. Since $n \gg \log \delta$, we get that $\log \delta \ll \log n'$. Thus, $n' \ll (\log \delta) \log n' \ll (\log n')^2$, so everything is bounded.

This idea ended up being very fruitful. The next few slides show results obtained using it.

With sums of two Fibonacci numbers

Let $2\mathcal{F} = \mathcal{F} + \mathcal{F}$ be the set of numbers which can be written as a sum of two Fibonacci numbers.

Theorem

*(C. A. Gómez Ruiz, L., **2018**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 1. \tag{5}$$

The equation $X_n \in 2\mathcal{F}$ has at most one solution n except for $d \in \{2, 3, 5, 11, 30\}$.

Is it true that for every $k \geq 3$ there are only finitely many d such that $X_n \in k\mathcal{F}$ has more than one solution n? Here

$$k\mathcal{F} = \mathcal{F} + \mathcal{F} + \cdots + \mathcal{F}.$$

We have no idea. If we replace $k\mathcal{F}$ by having at most k ones in their binary expansion the answer is NO.

Carlos Alexis Gómez Ruiz

With products of two Fibonacci numbers

Let $\mathcal{F}^2 = \mathcal{F} \cdot \mathcal{F}$ be the sequence of numbers which are products of two Fibonacci numbers.

Theorem

*(L., Montejano, Szalay, Togbé, **2018**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 1. \tag{6}$$

The equation $X_n \in \mathcal{F}^2$ has at most one solution n except for $d \in \{2, 3, 5\}$.

With generalized k-Fibonacci numbers

For an integer $k \geq 2$ consider the following generalization of the Fibonacci sequence $\mathcal{F}^{(k)} = \{F_n^{(k)}\}_{n \geq -(k-2)}$ given by

$$F_n = F_{n-1} + \cdots + F_{n-k} \qquad n \geq 2,$$

where $F_{2-k} = F_{3-k} = \cdots = F_0 = 0,\ F_1 = 1$. When $k = 2,\ 3$ one obtains the Fibonacci and Tribonacci sequences, respectively.

Theorem

*(Ddamulira, L., **2018**). Let $k \geq 4$ be a fixed integer. Let $d \geq 2$ be a square-free integer. Assume that*

$$X_{n_1} = F_{m_1}^{(k)}, \qquad \text{and} \quad X_{n_2} = F_{m_2}^{(k)} \tag{7}$$

for positive integers $m_2 > m_1 \geq 2$ and $n_2 > n_1 \geq 1$, where X_n is the x–coordinate of the nth solution of the Pell equation

$$X^2 - dY^2 = \pm 1.$$

Put $\epsilon = X_1^2 - dY_1^2$. Then, either:

(i) $n_1 = 1,\ n_2 = 2$, $m_1 = (k+3)/2$, $m_2 = k+2$ *and* $\epsilon = 1$*; or*

(ii) $n_1 = 1,\ n_2 = 3,\ k = 3 \times 2^{a+1} + 3a - 5,\ m_1 = 3 \times 2^a + a - 1,\ m_2 = 9 \times 2^a + 3a - 5$ *for some positive integer a and* $\epsilon = 1$.

Explanations for the exceptions

For $k \geq 2$ one has

$$\begin{array}{rcll} F_n^{(k)} & = & 2^{n-2} & \text{for} \quad n \in [2, k+1]; \\ F_n^{(k)} & = & 2^{n-2} - (n-k)2^{n-k-3} & \text{for} \quad n \in [k+2, 2k+1]. \end{array}$$

For suitable n and k it might happen that

$$F_n^{(k)} = 2^{n-2} - (n-k)2^{n-k-3} = 2x^2 - 1, \ 4x^3 - 3x$$

for some positive integer x which is necessarily a power of 2.

Mahadi Ddamulira

With factorials

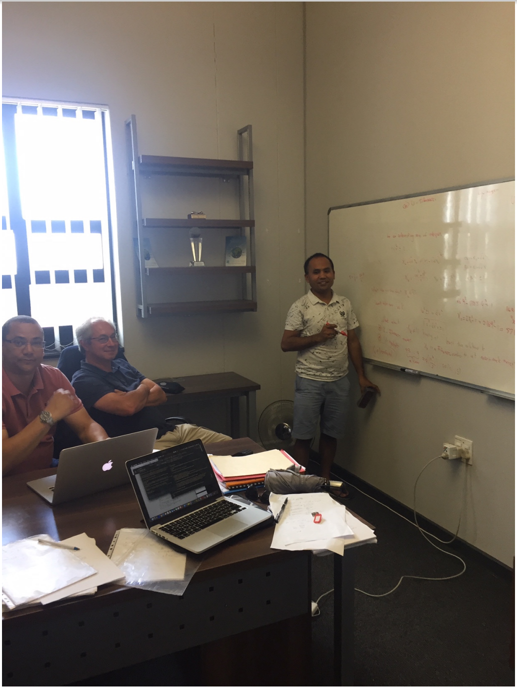
Let $\mathcal{F}act = \{m! : m \geq 1\}$.

Theorem

*(Laishram, L., Sias, **2019**). Let (X_n, Y_n) be the nth solution of the Diophantine equation*

$$X^2 - dY^2 = \pm 1. \tag{8}$$

The equation $X_n \in \mathcal{F}act$ implies $n = 1$.



Laishram, L., Sias

Lucas sequences

Let r, s be coprime integers, $r^2 + 4s \neq 0$. Let α, β with $|\alpha| \geq |\beta|$ be the roots of

$$x^2 - rx - s = 0.$$

Assume $(r, s) \neq (1, -1),\ (-1, -1)$. The Lucas sequence of the first kind and second kind $\{U_n\}_{n\geq 0}$, and $\{V_n\}_{n\geq 0}$ of parameters (r, s), respectively, have its general terms given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geq 0.$$

Alternatively, one can define them by setting $U_0 = 0,\ U_1 = 1,\ V_0 = 2,\ V_1 = r$ and imposing that the recurrence

$$W_{n+2} = rW_{n+1} + sW_n \qquad \text{holds for all} \qquad n \geq 0$$

is satisfied for both $\{W_n\}_{n\geq 0} \in \{\{U_n\}_{n\geq 0}, \{V_n\}_{n\geq 0}\}$.

Products of factorials

Theorem

(L., Stanica, **2006***). The largest solution of the equation*

$$F_{n_1}F_{n_2}\cdots F_{n_t} = m_1!\cdots m_k!$$

in integers $1 \le n_1 < \cdots < n_t$ *and* $1 \le m_1 \le m_2 \le \cdots \le m_k$ *is*

$$F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$$

Letting

$$\mathcal{P}F = \{\prod_{i=1}^{k} m_i! : k \geq 0,\ m_i \geq 1\}$$

be the set of positive integers which are products of factorials, one can prove easily that if $\{U_n\}_{n\geq 0}$ is a Lucas sequence, then

$$|U_n| \in \mathcal{P}F \tag{9}$$

has only finitely many solutions n. In fact, let me prove it. Write

$$|U_n| = m_1! \cdots m_k!, \qquad 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k.$$

The left–hand side is $\leq 2|\alpha|^n$. For $n \geq 31$, the left–hand side has, by the Primitive Divisor Theorem, proved by Bilu, Hanrot, Voutier in **2001**, a prime factor $p \geq n-1$, which must divide $m_k!$. Thus, $m_k \geq n-1$, so $m_k! \geq 2((n-1)/e)^{n-1}$. Hence, we got

$$2|\alpha|^n \geq 2((n-1)/e)^{n-1},$$

so $n \leq N(|\alpha|)$.

Members of Lucas sequences which are products of factorials

Theorem

(Laishram, L. Sias, **2019***). In equation* (9)*, we have:*

(i) $n \leq 3 \times 10^5$.

(ii) *If additionally, the roots* α, β *are real, then* $n \leq 210$.

(iii) *Further, if* $s = \pm 1$*, then* $n \leq 150$.

The same bounds hold if we replace U_n *by* V_n *in* (9).

Idea

The general idea is to study on one hand the size of the two sides of (9) and on the other hand the arithmetic information obtained from considering the multiplicative contribution to the sides of (9) of the *primitive prime factors* of U_n.

The size is easy. On the one–hand,

$$\log |U_n| \le \log 2 + n \log |\alpha|.$$

On the other hand

$$\log \left(\prod_{i=1}^{k} m_i! \right) \ge \log 2 + \sum_{i=1}^{k} (m_i - 1)(\log m_i - 1).$$

So,

$$n \log |\alpha| \ge \sum_{i=1}^{k} (m_i - 1)(\log m_i - 1). \tag{10}$$

Now for the primitive prime factors. These are the primes $p \mid U_n$ and $p \nmid U_m$ for any $1 \le m < n$. Also, as a technical condition, we assume that $p \nmid |\Delta|$. It is known that in the left,

$$\sum_{\substack{p^\alpha \| U_n \\ p \text{ primitive}}} \log p^\alpha \ge \log\left(\frac{|\Phi_n(\alpha,\beta)|}{n}\right).$$

The left–hand side can be lower bounded as

$$\log\left(\frac{|\Phi_n(\alpha,\beta)|}{n}\right) \ge (\phi(n)-1)\log|\alpha| - C_1 2^{\omega(n)}(\log n)^2 \log|\alpha| \tag{11}$$

with some explicit constant C_1. In the right–hand side, these primes are at most the primes $p \equiv \pm 1 \pmod n$, which divide $m_1!m_2!\cdots m_k!$. Using the known formula for the contribution of a prime in a factorial, we get that in the right this is at most

$$\le \sum_{m_i \ge n-1} (m_i - 1) \sum_{\substack{p\equiv \pm 1 \pmod n \\ p \le m_i}} \frac{\log p}{p-1}.$$

Using the Montgomery–Vaughan bound

$$\pi(x; b, a) \leq \frac{2x}{\phi(b)\log(x/b)} \qquad \text{valid for all} \qquad x \geq b,$$

for the number of primes $p \equiv a \pmod{b}$ with $p \leq x$ and Abel summation formula, we get the following upper bound on the right–hand side

$$\left(\frac{4(1+\log\log n)}{\phi(n)}\right) \sum_{m_i \geq n-1} (m_i - 1)(\log m_i - 1),$$

which combined with (10) gives an upper bound of

$$\left(\frac{4(1+\log\log n)}{\phi(n)}\right) n \log|\alpha|$$

on the contribution of the primitive primes from the right–hand side.

Comparing the last bound with (11), we get

$$(\phi(n)-1)\log|\alpha|-C_1 2^{\omega(n)}\log|\alpha|(\log n)^2 < \left(\frac{4(1+\log\log n)}{\phi(n)}\right) n\log|\alpha|.$$

Voutier shows that one can take $C_1 = 73$. This gives $n < 18 \times 10^6$. Then one goes down easily to about 2×10^6. For 3×10^5 more ingredients are needed.

What about X-coordinates of Pell equations and factorials?

Well say, $X_k = n!$. Then

$$X_k = \frac{1}{2}(\alpha^k + \beta^k), \quad \alpha = X_1 + d\sqrt{Y_1} = X_1 + \sqrt{X_1^2 - \varepsilon}, \quad \varepsilon \in \{\pm 1\},$$

and β is the conjugate of α. So, we get

$$\alpha^k + \beta^k = V_k = 2!n!,$$

therefore by the previous results, $k < 150$. One may assume that k is prime. If $k = 2$, then $X_2 = 2X_1^2 \pm 1$ is odd and > 1, so it is not a product a product of factorials.
Say $k = p$ and $p \in [3, 150]$ is a prime. Then $X_p = P_p(X_1)$ is a polynomial of degree p in X_1.

Take $p = 3$, $\varepsilon = 1$. Then we need to solve

$$X_3 = X_1(4X_1^2 - 3) = n!$$

In the left, the only factors that divide $4X_1^2 - 3$ are (aside possibly from 3 to exponent exactly 1), only primes q such that $(3/q) = 1$. These occupy two of the four possible progressions modulo 12 which may contain infinitely many primes, so half of all the primes, and they contribute the factor $4X_1^2 - 3$ of X_3 so multiplicatively about

$$X_3^{2/3}.$$

In the right, these primes, by the equidistribution of the primes in progressions modulo 12, will contribute about

$$n!^{1/2} = X_3^{1/2}.$$

We get a contradiction for large X_3. To quantify what large means we need explicit estimates for primes in progressions with ratio 12. More generally, we need for all other all primes $p \in [5, 150]$ explicit estimates for the number of primes $q \leq x$ which are in a certain progression modulo p.

We used the following result.

Theorem

*(Bennet, Martin, O'Bryant, Rechnitzer, **2018**). Let* $m \leq 1200$, $\gcd(a, m) = 1$. *For all* $x \geq 50m^2$ *we have*

$$\frac{x}{\phi(m)\log x} < \pi(x; m, a) < \frac{x}{\phi(m)\log x}\left(1 + \frac{5}{2\log x}\right).$$

What about Y-coordinates of Pell equations in sequences?

How about $Y_n \in U$ for your favourite set U? Here is the problem is slightly different.

There are infinitely many binary recurrences $\mathcal{U}$ such that $Y_n \in U$ has two solutions n. For example, this is so if $1 \in \mathcal{U}$ and $\mathcal{U}$ contains infinitely many even numbers. It is also so for $U = \{2^m - 1 : m \geq 1\}$ since taking $d = 2^{2a} - 1$ for some a, then both $Y_1 = 1$ and $Y_3 = 2^{2a+2} - 1$ are in $\mathcal{U}$. However, this is best possible:

Theorem

*(B. Faye, F. Luca, **2019**). If $\mathcal{U} = \{U_m\}_{m\geq 1}$ is a binary recurrent sequence of integers, then $Y_n = U_m$ has at most two solutions (n, m) provided $d > d_0(\mathcal{U})$, where $d_0(\mathcal{U})$ is effectively computable.*

In case $\mathcal{U} = \{2^m - 1 : m \geq 1\}$, one can take $d_0(\mathcal{U}) = 1$.

THANK YOU!