Truncated Theta Series, Partitions Inequalities and Rogers-Ramanujan Functions

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My collaboration with George E. Andrews on the truncated version of Euler's pentagonal number theorem has opened up a new study on truncated theta series. Since then over twenty papers on this topic have followed and several partition inequalities are derived in this way. Today, I will talk about:

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I. Truncated theta series and linear partitions inequalities
II. Linear homogeneous partition inequalities and PTE problem

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I. Truncated theta series and linear partitions inequalities
II. Linear homogeneous partition inequalities and PTE problem
III. Twenty new identities involving the Rogers-Ramanujan functions

Introduction

Introduction

I. Truncated theta series and linear partitions inequalities

II. A new method for proving the linear homogeneous partition inequalities

III. Twenty new identities involving the Rogers-Ramanujan functions

Introduction

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n.

$$\begin{aligned} &4 \\ &3+1 \\ &2+2 \\ &2+1+1 \\ &1+1+1+1 \end{aligned}$$

Let $p(n)$ be the number of partitions of n with $p(0)=1$.

Introduction

The theory of partitions began with Euler in the mid-eighteenth century. In order to understand certain aspects of partitions, Euler introduced the idea of a generating function of a sequence $\{a_n\}$, namely

$$\sum_{n=0}^{\infty} a_n x^n.$$

In particular, Euler showed that the generating function of the partition function $p(n)$, can be expressed as an elegant infinite product:

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Introduction

Here and throughout this talk, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n;$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Introduction

We sometimes use the following compressed notations:

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2, q)_n \cdots (a_r; q)_n,$$
$$(a_1, a_2, \ldots, a_r; q)_\infty = (a_1; q)_\infty (a_2, q)_\infty \cdots (a_r; q)_\infty.$$

Because the infinite product

$$(a; q)_\infty$$

diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a; q)_\infty$ appears in a formula, we shall assume

$$|q| < 1$$

.

Introduction

Using strict binary tree structures, we produced in 2011 the fastest known algorithm for the generation of the partitions of n. More details about this algorithm can be found in

Merca, M.:
Fast Algorithm for Generating Ascending Compositions
Journal of Mathematical Modelling and Algorithms, 11(1), 89–104 (2012) MR2910461.

Merca, M.:
Binary Diagrams for Storing Ascending Composition,
The Computer Journal, 56(11): 1320–1327 (2013).

Introduction

To prove that my algorithm is the fastest, we needed the following linear partition inequality: for $n > 0$,

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqslant 0.$$

In September 29, 2011, I asked G. E. Andrews about this inequality, and he told me that it is not in the literature. To prove this inequality, we needed to show that, except for the constant term, all the coefficients in

$$\frac{1 - q - q^2 + q^5}{(q;q)_\infty}$$

are non-positive. That's how my collaboration started with Andrews on the truncated theta series.

I. Truncated theta series and linear partitions inequalities

I.1. **Truncated pentagonal number series**

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I.2. **Truncated Gauss's theta series**

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I. Truncated theta series and linear partitions inequalities

I.1. **Truncated pentagonal number series**

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I.3. **Truncated Jacobi triple product series**

I.4. **Truncated Watson quintuple product series**

Truncated pentagonal number series

I.1. **Truncated pentagonal number series** $n(3n-1)/2, \quad n \in \mathbb{Z}$

I.1.1. Truncated Euler's pentagonal number theorem

I.1.2. Shank's formula and partition inequalities

I.1.3. Partitions with nonnegative rank

I.1.4. Partitions with positive rank

I.2. Truncated Gauss's theta series

I.3. Truncated Jacobi triple product series

I.4. Truncated Watson quintuple product series

Truncated Euler's pentagonal number theorem

The pentagonal number theorem relates the product and the series representations of the Euler function $(q;q)_\infty$.

THEOREM 2.1. [Euler's Pentagonal Number Theorem]

For $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

This theorem leads to an efficient method of computing the partition function $p(n)$, i.e.,

$$\sum_{k=-\infty}^{\infty} (-1)^k p\big(n - k(3k-1)/2\big) = \delta_{0,n},$$

where $p(n) = 0$ for any negative integer n and $\delta_{i,j}$ is the Kronecker delta.

Truncated Euler's pentagonal number theorem

THEOREM 2.2. [Andrews-M, 2012]

For $k \geqslant 1$,

$$\frac{(-1)^{k-1}}{(q;q)_\infty} \sum_{n=-(k-1)}^{k} (-1)^n q^{n(3n-1)/2} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Andrews, G. E., Merca, M.:
The truncated pentagonal number theorem,
J. Combin. Theory Ser. A, **119** (2012) 1639–1643.

Truncated Euler's pentagonal number theorem

THEOREM 2.2. [Andrews-M, 2012]

For $k \geqslant 1$,

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An immediate consequence owing to the positivity of the sum on the right hand side of this identity is given by the following infinite family of linear partition inequalities.

Truncated Euler's pentagonal number theorem

Corollary 2.3. [Andrews-M, 2012]

For $n > 0$, $k \geqslant 1$,

$$(-1)^{k-1} \sum_{j=-(k-1)}^{k} (-1)^j p\big(n - j(3j-1)/2\big) \geqslant 0,$$

with strict inequality if $n \geqslant k(3k+1)/2$. For example,

$$p(n) - p(n-1) \geqslant 0,$$
$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqslant 0, \text{ and}$$
$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) \geqslant 0.$$

Regarding this corollary, we recall the following partition theoretic interpretation.

Truncated Euler's pentagonal number theorem

THEOREM 2.4. [Andrews-M, 2012]

For $n, k > 0$,

$$(-1)^{k-1} \sum_{j=-(k-1)}^{k} (-1)^j p\big(n - j(3j-1)/2\big) = M_k(n),$$

where $M_k(n)$ is the number of partitions of n in which:

- k is the least integer that is not a part and
- there are more parts $> k$ than there are $< k$.

Shank's formula and partition inequalities

Shank's formula and partition inequalities

THEOREM 2.2. [Andrews-M, 2012]

For $k \geqslant 1$,

$$\frac{(-1)^{k-1}}{(q;q)_\infty} \sum_{n=-(k-1)}^{k} (-1)^n q^{n(3n-1)/2} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Theorem 2.2 was not the first truncated form of the pentagonal number theorem. In 1951, D. Shanks provided the following identity.

Shank's formula and partition inequalities

THEOREM 2.5. [D. Shanks, 1951]

For $k \geqslant 1$,

$$\sum_{n=-k}^{k} (-1)^n q^{n(3n-1)/2} = \sum_{n=0}^{k} (-1)^n \frac{q^{\binom{n+1}{2}+kn}(q;q)_k}{(q;q)_n}.$$

Shanks, D.:
A short proof of an identity of Euler,
Proc. Amer. Math. Soc., **2** (1951) 747–749.

Shank's formula and partition inequalities

Motivated by Shanks's formula, Andrews and Merca considered Theorem 2.2 and proved the following result.

THEOREM 2.6. [Andrews-M, 2012]

For $k \geqslant 1$,

$$\sum_{n=-(k-1)}^{k} (-1)^n q^{n(3n-1)/2} = \sum_{n=0}^{k-1} (-1)^n \frac{q^{\binom{n}{2}+(k+1)n}(q;q)_k}{(q;q)_n}.$$

Shank's formula and partition inequalities

THEOREM 2.5. [D. Shanks, 1951]

$$\sum_{n=-k}^{k}(-1)^n q^{n(3n-1)/2} = \sum_{n=0}^{k}(-1)^n \frac{q^{\binom{n+1}{2}+kn}(q;q)_k}{(q;q)_n}.$$

THEOREM 2.6. [Andrews-M, 2012]

For $k \geqslant 1$,

$$\sum_{n=-(k-1)}^{k}(-1)^n q^{n(3n-1)/2} = \sum_{n=0}^{k-1}(-1)^n \frac{q^{\binom{n}{2}+(k+1)n}(q;q)_k}{(q;q)_n}.$$

Shank's formula and partition inequalities

Related to Shanks's identity, we recently obtained the following result.

THEOREM 2.7. [M, 2019]

For $k \geqslant 1$,

$$\frac{(-1)^k}{(q;q)_\infty} \sum_{n=-k}^{k} (-1)^n q^{n(3n-1)/2}$$
$$= (-1)^k + \frac{q^{k(3k+7)/2+2}}{(q,q^3;q^3)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(3n+3k+5)}}{(q^3;q^3)_n (q^2;q^3)_{n+k+1}}$$
$$+ \frac{q^{k(3k+5)/2+1}}{(q^2,q^3;q^3)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(3n+3k+4)}}{(q^3;q^3)_n (q;q^3)_{n+k+1}}.$$

Shank's formula and partition inequalities

To prove this theorem, we consider the following identity.

LEMMA 2.8. [M, 2019]

For $|q| < 1$, $t \neq 0$,

$$\sum_{n=0}^{\infty}(-1)^n t^n q^{\alpha n^2+\beta n} = (tq^{\alpha+\beta}; q^{2\alpha})_\infty \sum_{n=0}^{\infty} \frac{q^{n(2\alpha n+\alpha+\beta)} t^n}{(q^{2\alpha}, tq^{\alpha+\beta}; q^{2\alpha})_n}.$$

Proof. To prove the lemma, we take into account

Shank's formula and partition inequalities

The Gauss hypergeometric series

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n $$

and

The second identity by Heine's transformation of ${}_2\phi_1$

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(c/b;q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty} {}_2\phi_1\left(\begin{matrix} abz/c, b \\ bz \end{matrix}; q, c/b\right). $$

Shank's formula and partition inequalities

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n t^n q^{\alpha n^2+\beta n} &= \lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta}/z;q^{2\alpha})_n t^n z^n}{(z;q^{2\alpha})_n}\\
&= \lim_{z\to 0} {}_2\phi_1\left(\begin{matrix} q^{2\alpha}, q^{\alpha+\beta}/z \\ z \end{matrix}; q^{2\alpha}, tz\right)\\
&= \lim_{z\to 0}\frac{(z^2/q^{\alpha+\beta}, tq^{\alpha+\beta};q^{2\alpha})_\infty}{(z,tz;q^{2\alpha})_\infty}\, {}_2\phi_1\left(\begin{matrix} tq^{3\alpha+\beta}/z, tq^{\alpha+\beta}/z \\ q^{\alpha+\beta} \end{matrix}; q^{2\alpha}, \frac{z^2}{q^{\alpha+\beta}}\right)\\
&= (tq^{\alpha+\beta};q^{2\alpha})_\infty \lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(tq^{3\alpha+\beta}/z, q^{\alpha+\beta}/z;q^{2\alpha})_n}{(q^{2\alpha}, tq^{\alpha+\beta};q^{2\alpha})_n}\left(\frac{z^2}{q^{\alpha+\beta}}\right)^n\\
&= (tq^{\alpha+\beta};q^{2\alpha})_\infty \lim_{z\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\alpha n(n+1)}(q^{\alpha+\beta}/z;q^{2\alpha})_n t^n z^n}{(q^{2\alpha}, tq^{\alpha+\beta};q^{2\alpha})_n}\\
&= (tq^{\alpha+\beta};q^{2\alpha})_\infty \sum_{n=0}^{\infty}\frac{q^{2\alpha n^2+n(\alpha+\beta)}t^n}{(q^{2\alpha}, tq^{\alpha+\beta};q^{2\alpha})_n}
\end{aligned}
$$

Shank's formula and partition inequalities

Corollary 2.9. [M, 2019]

For $n > 0$, $k \geqslant 1$,

$$(-1)^k \sum_{j=-k}^{k} (-1)^j p(n - j(3j-1)/2) \geqslant 0,$$

with strict inequality if $n > k(3k+5)/2$. For example,

$$p(n) - p(n-1) - p(n-2) \leqslant 0,$$
$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) \geqslant 0, \quad \text{and}$$
$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) \\ - p(n-12) - p(n-15) \leqslant 0.$$

Partitions with nonnegative rank

Partitions with nonnegative rank

Ramanujan proved that for every positive integer n, we have:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In order to explain the last two congruences combinatorially, Dyson introduced the rank of a partition.

Dyson, F.:
Some guesses in the theory of partitions,
Eureka (Cambridge), **8** (1944) 10–15.

Partitions with nonnegative rank

The rank of a partition

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k$$

is defined to be its largest part minus the number of its parts

$$\lambda_1 - k.$$

Partition	Largest part	Number of parts	Rank
5	5	1	4
4+1	4	2	2
3+2	3	2	1
3+1+1	3	3	0
2+2+1	2	3	-1
2+1+1+1	2	4	-2
1+1+1+1+1	1	5	-4

Partitions with nonnegative rank

We define $N(n)$ to be the number of partitions of n with non-negative rank.

It is well known that the generating function of $N(n)$, can be expressed as

$$\sum_{n=0}^{\infty} N(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

On the other hand, it is known the identity

$$\frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} = 1 + \sum_{n=1}^{\infty} q^n \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}.$$

The following result contains two truncated forms of this identity.

Partitions with nonnegative rank

THEOREM 2.10. [M, 2019]

For $|q|<1$ and $k\geqslant 1$, there holds

$$(i)\qquad \frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2} = 1+\sum_{j=1}^{\infty} q^j \begin{bmatrix}2j-1\\ j-1\end{bmatrix} + (-1)^{k-1}\frac{q^{k(3k+1)/2}}{(q,q^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(3j+3k+2)}}{(q^3;q^3)_j(q^2;q^3)_{k+j}},$$

$$(ii)\qquad \frac{1}{(q;q)_\infty}\sum_{j=1}^{k}(-1)^{j-1} q^{j(3j-1)/2} = \sum_{j=1}^{\infty} q^j \begin{bmatrix}2j-1\\ j-1\end{bmatrix} + (-1)^{k-1}\frac{q^{k(3k+5)/2+1}}{(q^2,q^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(3j+3k+4)}}{(q^3;q^3)_j(q;q^3)_{k+j+1}}.$$

Partitions with nonnegative rank

As a consequence of the second identity in Theorem 2.10, we obtain the following infinite family of linear partition inequalities.

Corollary 2.11. [M, 2019]

For $n > 0$, $k \geqslant 1$,

$$(-1)^{k-1}\left(\sum_{j=1}^{k}(-1)^{j}p(n-j(3j-1)/2)-N(n)\right)\geqslant 0.$$

with strict inequality if $n \geqslant k(3k+5)/2+1$. For example,

$$p(n-1) \geqslant N(n),$$
$$p(n-1)-p(n-5) \leqslant N(n), \text{ and}$$
$$p(n-1)-p(n-5)+p(n-12) \geqslant N(n).$$

Partitions with positive rank

Partitions with positive rank

We define $R(n)$ to be the number of partitions of n with positive rank.

It is well known that

$$\sum_{n=0}^{\infty} R(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(3n+1)/2}$$

and

$$\frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(3n+1)/2} = \sum_{n=1}^{\infty} q^{n+1} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}.$$

The following theorem contains two truncated versions of this identity.

Partitions with positive rank

THEOREM 2.13. [M, 2019]

For $|q| < 1$ and $k > 1$, there holds

$$(i) \qquad \frac{1}{(q;q)_\infty}\sum_{j=1}^{k-1}(-1)^{j+1}q^{j(3j+1)/2} = \sum_{j=1}^{\infty} q^{j+1}\begin{bmatrix} 2j \\ j-1 \end{bmatrix} + (-1)^k \frac{q^{k(3k+1)/2}}{(q,q^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(3j+3k+2)}}{(q^3;q^3)_j(q^2;q^3)_{k+j}};$$

$$(ii) \qquad \frac{1}{(q;q)_\infty}\sum_{j=0}^{k}(-1)^{j}q^{j(3j-1)/2} = 1 + \sum_{j=1}^{\infty} q^{j+1}\begin{bmatrix} 2j \\ j-1 \end{bmatrix} + (-1)^k \frac{q^{k(3k+5)/2+1}}{(q^2,q^3;q^3)_\infty}\sum_{j=0}^{\infty}\frac{q^{j(3j+3k+4)}}{(q^3;q^3)_j(q;q^3)_{k+j+1}}.$$

As a consequence of Theorem 2.13, we remark the following equivalent form of Corollary 2.12.

Corollary 2.14. [M, 2019]

For $n \geqslant 0$, $k > 1$,

$$(-1)^k \left(\sum_{j=1}^{k-1} (-1)^{j+1} p(n - j(3j+1)/2) - R(n) \right) \geqslant M_k(n),$$

with strict inequality if $n \geqslant k(3k+5)/2 + 1$. For example,

$$p(n-2) \geqslant R(n) + M_2(n),$$
$$p(n-2) - p(n-7) \leqslant R(n) + M_3(n),$$
$$p(n-2) - p(n-7) + p(n-15) \geqslant R(n) + M_4(n), \text{ and}$$
$$p(n-2) - p(n-7) + p(n-15) - p(n-26) \leqslant R(n) + M_5(n).$$

Truncated Gauss's theta series

Two truncated identities of Gauss

There are two other classical theta identities, usually attributed to Gauss:

$$1 + 2\sum_{n=1}^{\infty}(-q)^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

and

$$\sum_{n=0}^{\infty}(-q)^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.$$

Motivated by Theorem 2.2, V.J.W. Guo and J. Zeng considered these identities and proved new truncated forms of these.

Guo, V.J.W., Zeng, J.:
Two truncated identities of Gauss,
J. Combin. Theory Ser. A, **120** (2013) 700–707.

Two truncated identities of Gauss

First Guo and Zeng note that the reciprocal of the infinite product in the first theta identity of Gauss

$$1+2\sum_{n=1}^{\infty}(-q)^{n^2}=\frac{(q;q)_\infty}{(-q;q)_\infty}$$

is the generating function for the overpartitions of n

$$\frac{(-q;q)_\infty}{(q;q)_\infty}=\sum_{n=0}^{\infty}\overline{p}(n)q^n.$$

Recall that an overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, $\overline{p}(3)=8$ because there are 8 possible overpartitions of 3:

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

Two truncated identities of Gauss

THEOREM 3.1. [Guo-Zeng, 2013]

For $|q| < 1$ and $k \geqslant 1$, there holds

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-q)^{j^2}\right)$$
$$= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q;q)_k(-1;q)_{n-k}q^{(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

As consequence of Theorems 3.1, Guo and Zeng derived the following family of linear inequalities for $\overline{p}(n)$.

Two truncated identities of Gauss

Corollary 3.2. [Guo-Zeng, 2013]

For $n, k > 0$,

$$(-1)^k \left(\overline{p}(n) + 2 \sum_{j=1}^{k} (-1)^j \overline{p}(n - j^2) \right) \geqslant 0,$$

with strict inequality if $n \geqslant (k+1)^2$. For example,

$$\overline{p}(n) - 2\overline{p}(n-1) \leqslant 0,$$
$$\overline{p}(n) - 2\overline{p}(n-1) + 2\overline{p}(n-4) \geqslant 0,$$
$$\overline{p}(n) - 2\overline{p}(n-1) + 2\overline{p}(n-4) - 2\overline{p}(n-9) \leqslant 0, \text{ and}$$
$$\overline{p}(n) - 2\overline{p}(n-1) + 2\overline{p}(n-4) - 2\overline{p}(n-9) + 2\overline{p}(n-16) \leqslant 0.$$

Two truncated identities of Gauss

Next Guo and Zeng note that the reciprocal of the infinite product in the second theta identity of Gauss

$$\sum_{n=0}^{\infty}(-q)^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.$$

is the generating function for the number of partitions of n in which odd parts are not repeated.

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} pod(n)q^n,$$

Two truncated identities of Gauss

THEOREM 3.3. [Guo-Zeng, 2013]

For $|q| < 1$ and $k \geqslant 1$, there holds

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2}$$
$$= 1 + (-1)^{k-1}\sum_{n=k}^{\infty}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{2(k+1)n-k}}{(q^2;q^2)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2}.$$

As consequence of Theorems 3.3, Guo and Zeng derived the following family of linear inequalities for $pod(n)$.

Two truncated identities of Gauss

Corollary 3.4. [Guo-Zeng, 2013]

For $n, k > 0$,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}pod\big(n-j(j+1)/2\big)\geqslant 0,$$

with strict inequality if $n \geqslant (2k+1)k$. For example,

$$\begin{aligned}
&pod(n)-pod(n-1)\geqslant 0,\\
&pod(n)-pod(n-1)-pod(n-3)+pod(n-6)\leqslant 0, \text{ and}\\
&pod(n)-pod(n-1)-pod(n-3)+pod(n-6)\\
&\qquad\qquad +pod(n-10)-pod(n-15)\geqslant 0.
\end{aligned}$$

Two truncated identities of Gauss

Recently, Andrews and Merca have revealed that Theorems 3.1 and 3.3 are essentially corollaries of the Rogers-Fine identity:

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n \tau^n}{(\beta;q)_n} = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n (\alpha\tau q/\beta;q)_n \beta^n \tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n (\tau;q)_{n+1}}.$$

Andrews, G. E., Merca, M.:
Truncated Theta Series and a Problem of Guo and Zeng,
J. Combin. Theory Ser. A, **154** (2018) 610–619.

Two truncated identities of Gauss

In this context, Andrews and Merca provided the following revisions of Theorems 3.1 and 3.3.

THEOREM 3.5. [Andrews-M, 2018]

For $|q| < 1$ and $k \geqslant 1$, there holds

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1+2(-1)^k\frac{(-q;q)_k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(j+k+1)}(-q^{j+k+2};q)_\infty}{(1-q^{j+k+1})(q^{j+k+2};q)_\infty}.$$

Two truncated identities of Gauss

THEOREM 3.6. [Andrews-M, 2018]

For $|q| < 1$ and $k \geqslant 1$, there holds

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2}$$
$$= 1 + (-1)^{k-1}\frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty}.$$

By Theorems 3.5 and 3.6, Andrews and Merca deduced the following two partition theoretic interpretations.

Two truncated identities of Gauss

Corollary 3.7. [Andrews-M, 2018]

For $n, k \geqslant 1$,

$$(-1)^k \left(\overline{p}(n) + 2 \sum_{j=1}^{k} (-1)^j \overline{p}(n - j^2) \right) = \overline{M}_k(n),$$

where $\overline{M}_k(n)$ is the number of overpartitions of n in which the first part larger than k appears at least $k+1$ times.

Two truncated identities of Gauss

Corollary 3.8. [Andrews-M, 2018]

For $n, k \geqslant 1$,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}pod\big(n-j(j+1)/2\big)=MP_k(n),$$

where $MP_k(n)$ is the number of partitions of n in which the first part larger than $2k-1$ is odd and appears exactly k times. All other odd parts appear at most once.

Partitions into odd parts

Partitions into odd parts

In analogy with Theorems 3.1 and 3.5, we have the following result.

THEOREM 3.9. [M, 2019]

For the positive integers k and r, we have:

$$(-q;q)_\infty \left(1+2\sum_{j=1}^{k}(-1)^j q^{r\cdot j^2}\right) = \frac{(-q;q)_\infty (q^r;q^r)_\infty}{(-q^r;q^r)_\infty}$$

$$+2(-1)^k q^{r(k+1)^2}\frac{(q^r;q^{2r})_\infty}{(q;q^2)_\infty}\sum_{j=0}^{\infty}\frac{q^{(2k+2j+3)rj}}{(q^{2r};q^{2r})_j (q^r;q^{2r})_{k+j+1}}.$$

Partitions into odd parts

Related to Theorem 3.9, we recall that $(-q,q)_\infty$ is the generating function for $Q(n)$ which counts the partitions of n into odd parts, i.e.,

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty.$$

In order to simplify the expressions, we denote the nth triangular number by

$$T_n = n(n+1)/2$$

and the nth generalized pentagonal number by

$$G_n = T_n - T_{\lfloor n/2 \rfloor} = \frac{1}{2}\left\lceil \frac{n}{2} \right\rceil \left(3\left\lceil \frac{n}{2} \right\rceil + (-1)^n \right),$$

for any nonnegative integer n.

Partitions into odd parts

The following two theta identities

$$\frac{(-q;q)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}=\sum_{j=0}^{\infty}(-1)^{T_{\lfloor j/2\rfloor}}q^{G_j},$$

$$\frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}=\sum_{j=0}^{\infty}q^{G_j},$$

and the positivity of the sum

$$\sum_{j=0}^{\infty}\frac{q^{(2k+2j+3)rj}}{(q^{2r};q^{2r})_j(q^r;q^{2r})_{k+j+1}}$$

in Theorem 3.9 allows us to deduce the three families of linear inequalities for the partition function $Q(n)$.

Partitions into odd parts

Corollary 3.10. [M, 2019]

For $m, n \geqslant 0$, $k \geqslant 1$,

$$(a)\ (-1)^k \left(Q(n) + 2\sum_{j=1}^{k} (-1)^j Q(n-j^2) - (-1)^{T_m} \delta_{n,G_m} \right) \geqslant 0,$$

with strict inequality if and only if $n \geqslant (k+1)^2$.

$$(b)\ (-1)^k \left(Q(n) + 2\sum_{j=1}^{k} (-1)^j Q(n-2j^2) - (-1)^{T_{\lfloor m/2 \rfloor}} \delta_{n,G_m} \right) \geqslant 0,$$

with strict inequality if and only if $n \geqslant 2(k+1)^2$.

$$(c)\ (-1)^k \left(Q(n) + 2\sum_{j=1}^{k} (-1)^j Q(n-3j^2) - \delta_{n,G_m} \right) \geqslant 0,$$

with strict inequality if and only if $n \geqslant 3(k+1)^2$.

Overpartitions into odd parts

Overpartitions into odd parts

We consider overpartitions into odd parts and shall prove similar results. Let $\overline{p_o}(n)$ be the number of overpartitions into odd parts. Then its generating function is

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

In analogy with the truncated identities in Theorems 3.1 and 3.5, we have two symmetrical results on $\overline{p_o}(n)$.

M. Merca, C. Wang, A.J. Yee:
A truncated theta identity of Gauss and overpartitions into odd parts,
Annals of Combinatorics, accepted, to appear.

Overpartitions into odd parts

THEOREM 3.14. [M-Wang-Yee, 2019]

For a positive integer k,

$$(i)\quad \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right)=1+2\sum_{j=1}^{\infty}(-1)^j q^{2j^2}$$

$$+2(-1)^k q^{(k+1)^2}(-q;q^2)_\infty\sum_{j=0}^{\infty}\frac{q^{j(2k+2j+3)}}{(q^2;q^2)_j(q;q^2)_{k+j+1}},$$

$$(ii)\quad \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{2j^2}\right)=1+2\sum_{j=1}^{\infty}q^{j^2}$$

$$+2(-1)^k q^{2(k+1)^2}(-q;q^2)_\infty^2\sum_{j=0}^{\infty}\frac{q^{2j(2k+2j+3)}}{(q^4;q^4)_j(q^2;q^4)_{k+j+1}}.$$

Overpartitions into odd parts

We consider $a(n) = (-1)^{\left\lfloor \sqrt{n/2} \right\rfloor} \cdot 2\delta_{n,2\left\lfloor \sqrt{n/2} \right\rfloor^2}$.

Corollary 3.15. [M-Wang-Yee, 2019]

For $k, n > 0$,

$$(i) \quad (-1)^k \left(\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n - j^2) - a(n) \right) \geqslant 0,$$

with strict inequality if $n \geqslant (k+1)^2$.

$$(ii) \quad (-1)^k \left(\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n - 2j^2) - 2\delta_{n,\left\lfloor \sqrt{n} \right\rfloor^2} \right) \geqslant 0,$$

with strict inequality if $n \geqslant 2(k+1)^2$.

Partitions with nonnegative crank

Partitions with nonnegative crank

For a partition λ we define:

- $\ell(\lambda)$ to be the largest part of λ,
- $\omega(\lambda)$ to be the number of 1's in λ,
- $\mu(\lambda)$ to be the number of parts of λ larger then $\omega(\lambda)$.

In 1988, Andrews and Garvan defined the crank of an integer partition as follows:

$$c(\lambda) = \begin{cases} \ell(\lambda), & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda), & \text{if } \omega(\lambda) > 0. \end{cases}$$

Partitions with nonnegative crank

$$c(\lambda) = \begin{cases} \ell(\lambda), & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda), & \text{if } \omega(\lambda) > 0. \end{cases}$$

Partition λ	Largest part $\ell(\lambda)$	Number of 1's $\omega(\lambda)$	Number of parts larger than $\omega(\lambda)$ $\mu(\lambda)$	Crank $c(\lambda)$
5	5	0	1	5
4+1	4	1	1	0
3+2	3	0	2	3
3+1+1	3	2	1	-1
2+2+1	1	1	2	1
2+1+1+1	3	3	0	-3
1+1+1+1+1	1	5	0	-5

Partitions with nonnegative crank

We denote by $C(n)$ the number of partition of n with nonnegative crank.

THEOREM 3.16. [Uncu, 2018]

The generating function for partitions with nonnegative crank is

$$\sum_{n=0}^{\infty} C(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

A.K. Uncu:
Weighted Rogers-Ramanujan partitions and Dyson crank,
Ramanujan J **46**(2) (2018) 579–591.

Partitions with nonnegative crank

In 2011, Andrews remarked that the following theta identity which involve the generating function for the number of partitions with nonnegative crank

$$\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n^2}$$

is effectively equivalent to an identity of Auluck published in 1951.

G. E. Andrews:
Concave compositions,
Electron. J. Combin., **18** (2011) #P6.

F. C. Auluck:
On some new types of partitions associated with generalized Ferrers graphs,
Proc. Cambridge Phil. Soc., **47** (1951), 679–686.

Partitions with nonnegative crank

We have the following truncated form of the Auluck identity.

THEOREM 3.17. [M, 2019]

For $k \geqslant 1$,

$$\frac{1}{(q;q)_\infty}\sum_{n=0}^{k-1}(-1)^n q^{n(n+1)/2}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n^2}+(-1)^{k-1}q^{k(k+1)/2}\sum_{n=0}^{\infty}\frac{q^{n(n+k+1)}}{(q;q)_n(q;q)_{n+k}}.$$

In this way, we derive a new infinite family of linear inequalities for $p(n)$.

Partitions with nonnegative crank

Corollary 3.18. [M, 2019]

For $n \geqslant 0$, $k \geqslant 1$,

$$(-1)^{k-1}\left(\sum_{j=0}^{k-1}(-1)^j p\big(n-j(j+1)/2\big) - C(n)\right) \geqslant 0,$$

with strict inequality if $n \geqslant k(k+1)/2$. For example,

$$\begin{aligned}
&p(n) \geqslant C(n),\\
&p(n) - p(n-1) \leqslant C(n),\\
&p(n) - p(n-1) + p(n-3) \geqslant C(n), \text{ and}\\
&p(n) - p(n-1) + p(n-3) - p(n-6) \leqslant C(n).
\end{aligned}$$

Garden of Eden partitions

Garden of Eden partitions

In 2007, B. Hopkins and J. A. Sellers provided a formula that counts the number of partitions of n that have rank -2 or less. Following the terminology of combinatorial game theory, they call these Garden of Eden partitions. These partitions arise naturally in analyzing the game *Bulgarian solitaire*.

Hopkins and Sellers obtained

$$\sum_{n=0}^{\infty} ge(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{3n(n+1)/2},$$

where $ge(n)$ counts the Garden of Eden partitions of n.

B. Hopkins, J. A. Sellers,:
Exact enumeration of Garden of Eden partitions, ,
Integers: Elec. J. of Comb. Number Th. **7**(2) (2007) A19.

Garden of Eden partitions

This generating function allows us to provide a new infinite family of linear inequalities for the partition function $p(n)$.

THEOREM 3.19. [M, 2019]

For $n \geqslant 0$, $k \geqslant 1$,

$$(-1)^{k-1}\left(\sum_{j=1}^{k}(-1)^{j-1}p(n-3j(j+1)/2)-ge(n)\right)\geqslant 0,$$

with strict inequality if $n \geqslant 3(k+1)(k+2)/2$. For example,

$$p(n-3)\geqslant ge(n),$$
$$p(n-3)-p(n-9)\leqslant ge(n),$$
$$p(n-3)-p(n-9)+p(n-18)\geqslant ge(n), \text{ and}$$
$$p(n-3)-p(n-9)+p(n-18)-p(n-30)\leqslant ge(n).$$

Truncated Jacobi triple product series

Truncated Jacobi triple product series

At the end of our paper,

G. E. Andrews, M. Merca:
The truncated pentagonal number theorem,
J. Combin. Theory Ser. A, **119** (2012) 1639–1643.

we posed the following conjecture.

Conjecture 2. [Andrews-M, 2012]

For $1 \leqslant S < R/2$ and $k \geqslant 1$,

$$\frac{(-1)^{k-1}}{(q^S, q^{R-S}, q^R; q^R)_\infty} \sum_{j=0}^{k-1} (-1)^j q^{j(j+1)R/2-jS} \left(1 - q^{(2j+1)S}\right)$$

has nonnegative coefficients.

Truncated Jacobi triple product series

In 2015, this conjecture was proved independently by R. Mao and A. J. Yee.

The proof of Mao uses q-series manipulations while the proof of Yee is based on a combinatorial argument.

R. Mao:
Proofs of two conjectures on truncated series,
J. Comb. Th., Ser. A **130** (2015) 15–25.

A. J. Yee:
Truncated Jacobi triple product theorems,
J. Comb. Th., Ser. A **130** (2015) 1–14.

Truncated Jacobi triple product series

Very recently, Wang and Yee reprove this ex-conjecture by providing an explicit series form with nonnegative coefficients.

C. Wang, A.J. Yee,:
Truncated Jacobi's triple product series,
J. Comb. Th., Ser. A **166** (2019) 382–392.

Wang and Yee considered the Jacobi triple product identity,

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = (q, z, q/z; q)_\infty,$$

and provided the following truncated forms of this identity.

Truncated Jacobi triple product series

THEOREM 4.1. [Wang-Yee, 2019]

For $k \geqslant 0$,

$$\frac{1}{(q, z, q/z; q)_\infty} \sum_{n=-k}^{k+1} (-1)^n z^n q^{n(n-1)/2} = 1 + (-1)^k q^{k(k+1)/2} \times$$

$$\times \sum_{n=k+1}^{\infty} \left(\sum_{i+j+h+l=n} \frac{q^{(k+1)j+hl} z^{h-l}}{(q;q)_i (q;q)_j (q;q)_h (q;q)_l} \right) q^n \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Truncated Jacobi triple product series

Related to the truncated Jacobi triple product series, we experimentally discovered a stronger conjecture.

Conjecture 4. [M, 2019]

For $1 \leqslant S < R$ and $k \geqslant 1$, the theta series

$$\frac{(-1)^k}{(q^S, q^{R-S}; q^R)_\infty} \sum_{j=k}^{\infty} (-1)^j q^{j(j+1)R/2-jS} \left(1 - q^{(2j+1)S}\right),$$

has nonnegative coefficients.

M. Merca:
Truncated Theta Series and Rogers-Ramanujan Functions,
Exp. Math. (2019)
https://doi.org/10.1080/10586458.2018.1542642.

Truncated Watson quintuple product series

The Watson quintuple product identity

The Watson quintuple product identity

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(z^{-3n} - z^{3n+1})$$
$$= (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty$$

can be considered as the next product-series identity after the Jacobi triple product identity.

$$q \to q^R$$

$$z \to q^S$$

The Watson quintuple product identity

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)R/2}(q^{-3nS} - q^{(3n+1)S}) = (q^S, q^{R-S}, q^R; q^R)_\infty (q^{R-2S}, q^{R+2S}; q^{2R})_\infty .$$

Inspired by ex-Conjecture 2, Chan, Ho and Mao examined two truncated series of this identity and discovered that similar properties hold.

S.H. Chan, T.P.N. Ho, R. Mao:
Truncated series from the quintuple product identity,
J. Number Theory **169** (2016) 420–438.

The Watson quintuple product identity

THEOREM 5.1. [Chan-Ho-Mao, 2016]

(i) For $1 \leqslant S < R/2$ and $k \geqslant 0$,

$$\frac{\sum_{n=-k}^{k} q^{n(3n+1)R/2}(q^{-3nS} - q^{(3n+1)S})}{(q^S, q^{R-S}, q^R; q^R)_\infty (q^{R-2S}, q^{R+2S}; q^{2R})_\infty} - 1$$

has nonnegative coefficients.

(ii) For $1 \leqslant S < R/2$ and $k \geqslant 1$,

$$\frac{\sum_{n=-k}^{k-1} q^{n(3n+1)R/2}(q^{-3nS} - q^{(3n+1)S})}{(q^S, q^{R-S}, q^R; q^R)_\infty (q^{R-2S}, q^{R+2S}; q^{2R})_\infty} - 1$$

has nonpositive coefficients.

The Watson quintuple product identity

Related to Theorem 5.1, we experimentally discovered two stronger results.

Conjecture 5. [M, 2019]

For $1 \leqslant S < R/2$ and $k \geqslant 0$, the theta series

$$(q^R;q^R)_\infty (q^{R-2S},q^{R+2S};q^{2R})_\infty \times \\ \times \left(\frac{\sum_{n=-k}^{k} q^{n(3n+1)R/2}(q^{-3nS}-q^{(3n+1)S})}{(q^S,q^{R-S},q^R;q^R)_\infty (q^{R-2S},q^{R+2S};q^{2R})_\infty} - 1 \right)$$

has nonnegative coefficients.

The Watson quintuple product identity

Conjecture 6. [M, 2019]

For $1 \leqslant S < R/2$ and $k \geqslant 1$, the theta series

$$(q^R;q^R)_\infty(q^{R-2S},q^{R+2S};q^{2R})_\infty \times \\ \times \left(\frac{\sum_{n=-k}^{k-1} q^{n(3n+1)R/2}(q^{-3nS}-q^{(3n+1)S})}{(q^S,q^{R-S},q^R;q^R)_\infty(q^{R-2S},q^{R+2S};q^{2R})_\infty} - 1 \right)$$

has nonpositive coefficients.

Linear partitions inequalities and PTE problem

Linear partitions inequalities and PTE problem

Recall that the Prouhet-Tarry-Escott problem asks for two distinct multisets of integers $\{x_1, x_2, \ldots, x_r\}$ and $\{y_1, y_2, \ldots, y_r\}$ such that

$$\sum_{i=1}^{r} x_i^j = \sum_{i=1}^{r} y_i^j, \quad \text{for all} \quad j = 1, 2, \ldots, k,$$

and

$$\sum_{i=1}^{r} x_i^{k+1} \neq \sum_{i=1}^{r} y_i^{k+1},$$

where k is a positive integer.

If $k = r - 1$, such a solution is called ideal.

Linear partitions inequalities and PTE problem

In what follows, we consider k a nonnegative integer and we write

$$\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$$

to denote a solution to the Prouhet-Tarry-Escott problem if k is positive or to denote the case

$$x_1 + x_2 + \cdots + x_r \neq y_1 + y_2 + \cdots + y_r$$

if $k = 0$.

Any solution of the Prouhet-Tarry-Escott problem and its partition inequality are related by the following result.

Linear partitions inequalities and PTE problem

THEOREM 6.1. [M-Katriel, 2019]

Let $\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$. There is a nonnegative integer N such that for $n \geqslant N$, the expression

$$\sum_{i=1}^{r} p(n + x_i) - \sum_{i=1}^{r} p(n + y_i)$$

has the same sign as the expression

$$\sum_{i=1}^{r} x_i^{k+1} - \sum_{i=1}^{r} y_i^{k+1}.$$

Linear partitions inequalities and PTE problem

Proof. The first forward difference of the function f is defined by

$$\Delta f(x) = f(x+1) - f(x).$$

Iterating, we obtain the k-th order forward difference:

$$\Delta^k f(x) = \Delta(\Delta^{k-1} f(x)) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+i).$$

Considering Newton's forward difference formula, Euler's partition function $p(n)$ can be expressed in terms of the k-th order forward difference as follows:

$$p(n+\xi) = \sum_{j=0}^{\infty} \frac{\Delta^j p(n)}{j!} (\xi)_j,$$

where

$$(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1)$$

is the falling factorial with $(\xi)_0 = 1$.

Linear partitions inequalities and PTE problem

Then we can write:

$$\sum_{i=1}^{r}\left(p(n+x_i)-p(n+y_i)\right)=\sum_{j=0}^{\infty}\frac{\Delta^j p(n)}{j!}\sum_{i=1}^{r}\left((x_i)_j-(y_i)_j\right).$$

In this paper

A. M. Odlyzko:
Differences of the partition function,
Acta Arith. **49** (1988), 237–254.

the author proves that for each fixed j there is a positive integer $n_0(j)$ such that $\Delta^j p(n) > 0$ for $n \geqslant n_0(j)$.

Linear partitions inequalities and PTE problem

Without loss of generality, we consider that $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ are nonnegative.
By $\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\}$, we deduce that

$$(i)\quad \sum_{i=1}^{r}(x_i)_j = \sum_{i=1}^{r}(y_i)_j, \qquad \text{for all} \quad j = 0, 1, \ldots, k;$$

$$(ii)\quad \sum_{i=1}^{r}(x_i)_j \neq \sum_{i=1}^{r}(y_i)_j, \qquad \text{for} \quad j > k;$$

$$(iii)\quad \sum_{i=1}^{r}\big((x_i)_{k+1} - (y_i)_{k+1}\big) = \sum_{i=1}^{r}(x_i^{k+1} - y_i^{k+1}).$$

Linear partitions inequalities and PTE problem

Let $M = \max(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r)$. For $j > M$, it is clear that

$$\sum_{i=1}^{r} \left((x_i)_j - (y_i)_j\right) = 0.$$

Thus we obtain

$$\sum_{i=1}^{r} \left(p(n + x_i) - p(n + y_i)\right) = \sum_{j=k+1}^{M} \frac{\Delta^j p(n)}{j!} \sum_{i=1}^{r} \left((x_i)_j - (y_i)_j\right).$$

In addition, for $k + 1 \leqslant j \leqslant M$ it is not difficult to prove that the expression

$$\sum_{i=1}^{r} \left((x_i)_j - (y_i)_j\right) \quad \text{has the same sign as} \quad \sum_{i=1}^{r} (x_i^{k+1} - y_i^{k+1}).$$

Linear partitions inequalities and PTE problem

M. Merca, J. Katriel:
A general method for proving the non-trivial linear homogeneous partition inequalities,
Ramanujan J (2019) https://doi.org/10.1007/s11139-018-0125-5.

Linear partitions inequalities and PTE problem

Example (1)

In the first paper with G. E. Andrews , for any positive integer k, we proved the following inequality,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{\lceil j/2\rceil}p(n-G_j)\geqslant 0,$$

where $G_k=\frac{1}{2}\lceil k/2\rceil\left(3\lceil k/2\rceil+(-1)^k\right)$ is the kth generalized pentagonal number.

We point out that this inequality can be derived as a special case of Theorem 6.1 because

$$\left\{G_j\,\Big|_{j\equiv 0,3 \pmod 4}^{0\leqslant j\leqslant 2r-1}\right\}\overset{0}{=}\left\{G_j\,\Big|_{j\equiv 1,2 \pmod 4}^{0\leqslant j\leqslant 2r-1}\right\}$$

Linear partitions inequalities and PTE problem

Example (2)

In the second paper with G. E. Andrews, we proposed the following conjecture: for n odd,

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{[j/2]}p(n-T_j)\leqslant(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{[j/2]}p(n-G_j),$$

where $T_n=n(n+1)/2$ is the nth triangular number.

This inequality is a very special case of Theorem 6.1:

$$\left\{T_j,\,G_{j+(-1)^j}\,\Big|_{j\equiv0,3\pmod 4}^{0\leqslant j\leqslant 2r-1}\right\}\stackrel{1}{=}\left\{G_j,\,T_{j+(-1)^j}\,\Big|_{j\equiv0,3\pmod 4}^{0\leqslant j\leqslant 2r-1}\right\}.$$

Linear partitions inequalities and PTE problem

Given a solution to the Prouhet-Tarry-Escott problem, we can generate an infinite family of solutions. That is, if

$$\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\},$$

then

$$\{Mx_1 + N, \ldots, Mx_r + N\} \stackrel{k}{=} \{My_1 + N, \ldots, My_r + N\}.$$

Thus, without loss of generality, we can consider the nontrivial solution

$$\{x_1, x_2, \ldots, x_r\} \stackrel{k}{=} \{y_1, y_2, \ldots, y_r\},$$

with

$$0 = x_1 \geqslant x_2 \geqslant \ldots \geqslant x_r \qquad \text{and} \qquad 0 > y_1 \geqslant y_2 \geqslant \ldots \geqslant y_r.$$

We have the following connection between the ideal solutions of the Prouhet-Tarry-Escott problem and the non-trivial linear homogeneous partition inequalities.

Linear partitions inequalities and PTE problem

THEOREM 6.2. [M-Katriel, 2019]

Let $\{x_1, x_2, \ldots, x_r\} \stackrel{r-1}{=} \{y_1, y_2, \ldots, y_r\}$ be an ideal solution of the Prouhet-Tarry-Escott problem with

$$0 = x_1 \geqslant x_2 \geqslant \ldots \geqslant x_r \qquad \text{and} \qquad 0 > y_1 \geqslant y_2 \geqslant \ldots \geqslant y_r.$$

There is a nonnegative integer N such that for $n \geqslant N$, the coefficients of q^n in

$$\frac{1}{(q;q)_\infty} \sum_{i=1}^{r} (q^{-x_i} - q^{-y_i})$$

are all positive, i.e., for $n \geqslant N$,

$$\sum_{i=1}^{r} p(n + x_i) - \sum_{i=1}^{r} p(n + y_i) > 0.$$

Linear partitions inequalities and PTE problem

Considering the following lemma, Theorem 6.2 can be derived as a specialization of Theorem 6.1.

Lemma 6.3.

Let $\{x_1, x_2, \ldots, x_r\} \stackrel{r-1}{=} \{y_1, y_2, \ldots, y_r\}$ be an ideal solution for the Prouhet-Tarry-Escott problem with

$$0 = x_1 \leqslant x_2 \leqslant \ldots \leqslant x_r \quad \text{and} \quad 0 < y_1 \leqslant y_2 \leqslant \ldots \leqslant y_r.$$

Then

$$\sum_{i=1}^{r} (x_i^r - y_i^r) = (-1)^r r \prod_{i=1}^{r} y_i.$$

Linear partitions inequalities and PTE problem

To prove this result, we use several tools from symmetric functions theory. We work with formal symmetric functions and we will use the standard notation for the classical families of symmetric functions: e_k for the k-th elementary symmetric function

$$e_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant n} \xi_{n_1} \xi_{n_2} \ldots \xi_{n_k}$$

and p_k for the k-th power sum symmetric function

$$p_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{j=1}^{n} \xi_j^k.$$

Linear partitions inequalities and PTE problem

The following relation is well known as Newton's identity:

$$k e_k(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j}(\xi_1, \xi_2, \ldots, \xi_n) p_j(\xi_1, \xi_2, \ldots, \xi_n).$$

Having

$$p_j(x_1, x_2, \ldots, x_r) = p_j(y_1, y_2, \ldots, y_r), \qquad \text{for all } j = 1, 2, \ldots, r-1,$$

we deduce that

$$e_j(x_1, x_2, \ldots, x_r) = e_j(y_1, y_2, \ldots, y_r), \qquad \text{for all } j = 1, 2, \ldots, r-1$$

and we obtain

$$\begin{aligned} & r\big(e_r(x_1, x_2, \ldots, x_r) - e_r(y_1, y_2, \ldots, y_r)\big) \\ & \qquad = (-1)^{r-1}\big(p_r(x_1, x_2, \ldots, x_r) - p_r(y_1, y_2, \ldots, y_r)\big). \end{aligned}$$

Taking into account that $e_r(x_1, x_2, \ldots, x_r) = 0$, the proof is finished.

Linear partitions inequalities and PTE problem

Conjecture 9. [M-Katriel, 2019]

Let $\{x_1, x_2, \ldots, x_r\} \overset{r-1}{=} \{y_1, y_2, \ldots, y_r\}$ be an ideal solution of the Prouhet-Tarry-Escott problem with

$$0 = x_1 \geqslant x_2 \geqslant \ldots \geqslant x_r \quad \text{and} \quad 0 > y_1 \geqslant y_2 \geqslant \ldots \geqslant y_r.$$

The coefficients of q^n in

$$\frac{1}{(1-q)^r} \sum_{i=1}^{r} (q^{-x_i} - q^{-y_i})$$

are all nonnegative.

New identities involving the Rogers-Ramanujan functions

Some open problems

For $|q| < 1$, the Rogers-Ramanujan functions are defined by

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.$$

Some open problems

For $|q| < 1$, the Rogers-Ramanujan functions are defined by

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.$$

These functions satisfy the famous Rogers-Ramanujan identities.

THEOREM 7.1.

For $|q| < 1$,

1. $G(q) = \dfrac{1}{(q, q^4; q^5)_\infty}$;
2. $H(q) = \dfrac{1}{(q^2, q^3; q^5)_\infty}$.

Some open problems

Due to MacMahon, we have the following combinatorial version of the Rogers-Ramanujan identities.

THEOREM 7.2.

Let n be a non-negative integer.

1. The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts congruent to $\{1, 4\}$ mod 5.

Some open problems

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1. The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts congruent to $\{1, 4\}$ mod 5.
2. The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts congruent to $\{2, 3\}$ mod 5.

Some open problems

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2. The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts congruent to $\{2, 3\}$ mod 5.

Let

$$G(q) = \sum_{n=0}^{\infty} g(n)q^n \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty} h(n)q^n.$$

Some open problems

There is a substantial amount of numerical evidence to state the following conjectures.

Conjecture 10. [M, 2019]

(i) For $k > 0$, the series

$$\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty}\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}$$

has nonnegative coefficients.

(ii For $k > 1$, the series

$$\frac{(q;q)_\infty}{(q^2,q^3;q^5)_\infty}\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}$$

has nonnegative coefficients.

Some open problems

Assuming these conjectures, we can say that for $k > 1$ the partition functions $p(n)$, $g(n)$ and $h(n)$ share a common infinite family of linear inequalities:

$$(-1)^k \sum_{j=k}^{\infty} (-1)^j \Big(\rho\big(n - j(3j+1)/2\big) - \rho\big(n - j(3j+5)/2 - 1\big) \Big) \geqslant 0,$$

where ρ is any of the partition functions p, g and h.

New identities involving the Rogers-Ramanujan functions

Jacobi's triple product identity

Considering Jacobi's triple product identity, we experimentally discover six identities involving the Rogers-Ramanujan functions $G(q)$ and $H(q)$. Assuming these identities, we can easily derive new linear recurrence relations for the partition functions $g(n)$ and $h(n)$. Other infinite families of linear homogeneous inequalities can be easily derived for $g(n)$ and $h(n)$.

Jacobi's triple product identity

IDENTITY 1. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 4, 7, 9, 17, 20, 26, 43, 62, 87, 99, 106, \ldots\}$ be the set of all nonnegative integers m such that $840m + 361$ is a perfect square. Then

$$\frac{(q, q^6, q^7; q^7)_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{t(n)} q^{a_n},$$

where

$$t(n) = \begin{cases} 0, & \text{if } n \equiv \{0,1,3,5,10,12,14,15\} \bmod 16 \ , \\ 1, & \text{otherwise.} \end{cases}$$

Jacobi's triple product identity

IDENTITY 2. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 2, 12, 13, 31, 35, 41, 64, 72, 78, 116, \ldots\}$ be the set of all nonnegative integers m such that $840m + 529$ is a perfect square. Then

$$\frac{(q, q^6, q^7; q^7)_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{t(n)} q^{a_n},$$

where

$$t(n) = \begin{cases} 0, & \text{if } n \equiv \{0, 2, 3, 6, 9, 12, 13, 15\} \bmod 16, \\ 1, & \text{otherwise.} \end{cases}$$

Jacobi's triple product identity

IDENTITY 3. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 4, 12, 14, 27, 38, 47, 58, 69, 86, 115, \ldots\}$ be the set of all nonnegative integers m such that $840m + 121$ is a perfect square. Then

$$\frac{(q^2, q^5, q^7; q^7)_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/8]} q^{a_n}.$$

Jacobi's triple product identity

IDENTITY 4. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 3, 5, 6, 22, 24, 29, 44, 61, 82, 91, 95, 143, \dots\}$ be the set of all nonnegative integers m such that $840m + 289$ is a perfect square. Then

$$\frac{(q^2, q^5, q^7; q^7)_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{t(n)} q^{a_n},$$

where

$$t(n) = \begin{cases} 0, & \text{if } n \equiv \{0, 1, 3, 5, 10, 12, 14, 15\} \bmod 16, \\ 1, & \text{otherwise.} \end{cases}$$

Jacobi's triple product identity

IDENTITY 5. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 2, 6, 23, 34, 52, 53, 68, 75, 94, 145, \ldots\}$ be the set of all nonnegative integers m such that $840m + 1$ is a perfect square. Then

$$\frac{(q^3, q^4, q^7; q^7)_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/8]} q^{a_n}.$$

Jacobi's triple product identity

IDENTITY 6. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 2, 8, 11, 15, 19, 33, 46, 59, 86, 102, \ldots\}$ be the set of all nonnegative integers m such that $840m + 169$ is a perfect square. Then

$$\frac{(q^3, q^4, q^7; q^7)_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{t(n)} q^{a_n},$$

where

$$t(n) = \begin{cases} 0, & \text{if } n \equiv \{0, 1, 2, 4, 11, 13, 14, 15\} \bmod 16, \\ 1, & \text{otherwise.} \end{cases}$$

Jacobi's triple product identity

Moreover, there is a substantial amount of numerical evidence to state the following conjecture which allows us to derive six common infinite families of linear homogeneous inequalities for the partition functions $g(n)$ and $h(n)$.

Conjecture 14. [M, 2019]

For $S \in \{1,2,3,4,5,6\}$ and $k \geqslant 1$,

$$\frac{(-1)^k}{(q,q^4;q^5)_\infty} \sum_{j=k}^{\infty} (-1)^j q^{7j(j+1)/2-jS} \left(1-q^{(2j+1)S}\right)$$

and

$$\frac{(-1)^k}{(q^2,q^3;q^5)_\infty} \sum_{j=k}^{\infty} (-1)^j q^{7j(j+1)/2-jS} \left(1-q^{(2j+1)S}\right)$$

have nonnegative coefficients.

New identities involving the Rogers-Ramanujan functions

Watson's quintuple product identity

In this section, we present fourteen identities for the Rogers-Ramanujan functions $G(q)$ and $H(q)$. We have discovered these identities in Maple considering Watson's quintuple product identity.

Watson's quintuple product identity

IDENTITY 7. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 4, 7, 10, 21, 26, 33, 59, 61, 95, 108, \dots\}$ be the set of all nonnegative integers m such that $240m + 1$ is a perfect square. Then

$$\frac{(q, q^7, q^8; q^8)_\infty (q^6, q^{10}; q^{16})_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/4]} q^{a_n}.$$

IDENTITY 8. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 2, 9, 22, 39, 44, 53, 67, 78, 85, 116, \dots\}$ be the set of all nonnegative integers m such that $240m + 49$ is a perfect square. Then

$$\frac{(q, q^7, q^8; q^8)_\infty (q^6, q^{10}; q^{16})_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{\lfloor 5n/4 \rfloor} q^{a_n}.$$

Watson's quintuple product identity

IDENTITY 9. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 8, 13, 17, 24, 45, 56, 64, 77, 112, \dots\}$ be the set of all nonnegative integers m such that $15m+1$ is a perfect square. Then

$$\frac{(q^2, q^6, q^8; q^8)_\infty (q^4, q^{12}; q^{16})_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/4]} q^{a_n}.$$

IDENTITY 10. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 3, 4, 11, 19, 32, 35, 52, 68, 91, 96, 123, \dots\}$ be the set of all nonnegative integers m such that $15m+4$ is a perfect square. Then

$$\frac{(q^2, q^6, q^8; q^8)_\infty (q^4, q^{12}; q^{16})_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/4]} q^{a_n}.$$

Watson's quintuple product identity

IDENTITY 11. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 1, 3, 14, 15, 34, 42, 49, 71, 80, 92, 133, \ldots\}$ be the set of all nonnegative integers m such that $240m + 121$ is a perfect square. Then

$$\frac{(q^3, q^5, q^8; q^8)_\infty (q^2, q^{14}; q^{16})_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{[n/4]} q^{a_n}.$$

IDENTITY 12. [M, 2019]

Let $\{a_n\}_{n\geqslant 0} = \{0, 5, 7, 11, 18, 24, 28, 47, 73, 102, \ldots\}$ be the set of all nonnegative integers m such that $240m + 169$ is a perfect square. Then

$$\frac{(q^3, q^5, q^8; q^8)_\infty (q^2, q^{14}; q^{16})_\infty}{(q^2, q^3; q^5)_\infty} = \sum_{n=0}^{\infty} (-1)^{\lfloor 5n/4 \rfloor} q^{a_n}.$$

Watson's quintuple product identity

IDENTITIY 13. [M, 2019]

$$\frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q, q^4; q^5)_\infty} = \sum_{n=-\infty}^{\infty} q^{n(5n+1)}.$$

IDENTITY 14. [M, 2019]

$$\frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q^2, q^3; q^5)_\infty}$$

$$= \sum_{n=0}^{\infty} \left(q^{3n(3n+1)} + q^{(3n+1)(3n+2)} + q^{(3n+2)(3n+3)} - q^{5n(n+1)+1} \right).$$

Watson's quintuple product identity

IDENTITY 15. [M, 2019]

$$\frac{(q^2,q^8,q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty}{(q,q^4;q^5)_\infty} = 1+\sum_{n=1}^{\infty}\left(q^{n^2}+q^{5n^2}\right).$$

IDENTITY 16. [M, 2019]

$$\frac{(q^2,q^8,q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty}{(q^2,q^3;q^5)_\infty} = \sum_{n=-\infty}^{\infty} q^{n(5n+2)}.$$

Watson's quintuple product identity

IDENTITY 17. [M, 2019]

$$\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q,q^4;q^5)_\infty}$$
$$=\sum_{n=0}^{\infty}\left(q^{3n(3n+1)}+q^{(3n+1)(3n+2)}+q^{(3n+2)(3n+3)}+q^{5n(n+1)+1}\right).$$

IDENTITY 18. [M, 2019]

$$\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q^2,q^3;q^5)_\infty}=\sum_{n=-\infty}^{\infty}q^{n(5n+3)}.$$

Watson's quintuple product identity

IDENTITY 19. [M, 2019]

$$\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q,q^4;q^5)_\infty} = \sum_{n=-\infty}^{\infty} q^{n(5n+4)}.$$

IDENTITY 20. [M, 2019]

$$\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^2,q^3;q^5)_\infty} = \sum_{n=1}^{\infty} \left(q^{n^2-1} - q^{5n^2-1}\right).$$

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Thank You !