# An elementary proof of the Kronecker-Hurwitz class number relation

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Transient Transcendence in Transylvania
Brașov, 13-17 May 2019

# Outline

- Proof (by picture) of a combinatorial refinement of K.-H. relation (1860s)
- Motivation: the proof was found as by-product of a new algebraic proof of the trace formula for Hecke operators on modular forms (Eichler-Selberg 1950s), following an idea of Zagier from 1990
- Another proof of K.-H. relation
- Cohomological interpretation of the trace formula

# The Kronecker-Hurwitz class number

For $D > 0$, $H(D)$ equals the number of $\mathrm{PSL}_2(\mathbb{Z})$-equivalence classes of positive definite integral binary quadratic forms of discriminant $-D$, with those classes that contain a multiple of $x^2 + y^2$ or of $x^2 - xy + y^2$ counted with multiplicity $1/2$ or $1/3$, respectively. Set $H(0) = -1/12$.

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Action:

$$Q \circ \gamma(x, y) = Q(ax + by, cx + dy) \text{ for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z}).$$

# The Kronecker-Hurwitz relation

| $D$ | 0 | 3 | 4 | 7 | 8 | 11 | 12 | 15 | 16 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $H(D)$ | $-\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{4}{3}$ | 2 | $\frac{3}{2}$ | 1 | 2 |

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**Theorem (Kronecker (1860), Gierster (1880), Hurwitz (1885))**

*For any $n \geqslant 1$ we have*

$$\sum_{t^2 \leqslant 4n} H(4n - t^2) = \sum_{\substack{n=ad \\ a,\, d>0}} \max(a, d) .$$

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**Example.** $n = 5$:
$H(20)+2H(19)+2H(16)+2H(11)+2H(4) = 2+2+3+2+1 = 5+5$

# Notation

$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}), \quad \mathcal{M}_n = \{M \in M_2(\mathbb{Z}) : \det M = n\}/\{\pm 1\}$$

If $n$ is square-free, $\mathcal{M}_n = \Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \Gamma$ is a double coset, otherwise a finite union of double cosets.

# Dictionary matrices–quadratic forms

We have a $\Gamma$-equivariant bijection

$$M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longleftrightarrow Q_M(x, y) = cx^2 + (d - a)xy - by^2$$

between integral matrices of determinant $n$ and trace $t$ and quadratic forms of discriminant $t^2 - 4n$.

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- Conjugacy classes $\longleftrightarrow$ Γ-equivalence classes
- Fixed points $Mz = z \longleftrightarrow$ Roots $Q_M(z, 1) = 0$.
- The fixed point of $\gamma M \gamma^{-1}$ is $\gamma z_M$.
- A matrix $M \in \mathcal{M}_n$ is elliptic iff $\mathrm{Tr}(M)^2 - 4n < 0$ iff it has a unique fixed point $z_M$ in the upper half-plane $\mathfrak{H}$.

$$\begin{aligned} \mathrm{Stab}\, M :=& \{\gamma \in \Gamma : \gamma M \gamma^{-1} = M\} = \{\gamma \in \Gamma : Q_M \circ \gamma = Q_M\} \\ =& \{\gamma \in \Gamma : \gamma z_M = z_M\} \in \{1, 2, 3\} \end{aligned}$$

# Reformulation of K.H-relation

$2H(4n-t^2)$= the (weighted) number of elliptic $\Gamma$-conjugacy classes of trace $\pm t \neq 0$ in $\mathcal{M}_n$.

Let $\chi$ be a modified characteristic function of the fund. domain

$$\mathcal{F} \;=\; \{z \in \mathfrak{H} : -1/2 \leqslant \mathrm{Re}(z) \leqslant 1/2,\; |z| \geqslant 1\}$$

for the action of $\Gamma$ on $\mathfrak{H}$, such that $\chi(z)$ is $1/2\pi$ times the angle subtended by $\mathcal{F}$ at $z$.

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The K.-H. relation can be restated as follows

$$\sum_{\substack{M \in \mathcal{M}_n \\ M \text{ elliptic}}} \chi(z_M) \;=\; \sum_{\substack{n=ad \\ a,\,d>0}} \max(a,d) + \begin{cases} 1/6 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

# A refinement of the K.-H. relation

We have a disjoint decomposition into right cosets

$$\mathcal{M}_n = \bigcup_{\substack{n=ad \\ 0\leqslant b<a}} \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \Gamma \,.$$

## Theorem (P. -Zagier 2016)

*For each right coset* $K = \left(\begin{smallmatrix} a_K & * \\ 0 & d_K \end{smallmatrix}\right) \Gamma \subset \mathcal{M}_n$ *with* $a_K, d_K > 0$*:*

$$\sum_{\substack{M \in K \\ M \text{ elliptic}}} \chi(z_M) = \begin{cases} 2 & \text{if } a_K > d_K\,, \\ 1 + \frac{1}{6} \cdot \delta_{K=\sqrt{n}\,\Gamma} & \text{if } a_K = d_K\,, \\ 0 & \text{if } a_K < d_K\,. \end{cases}$$

Generically, there is a 1 to 2 correspondence between right cosets $K \subset \mathcal{M}_n$ with $a_K > d_K$ and elliptic conjugacy classes in $\mathcal{M}_n$.

# A further refinement

Let $\chi^-$ be a characteristic function of the half-fundamental domain

$$\mathcal{F}^- = \{z \in \mathcal{F} : \mathrm{Re}(z) \leqslant 0\},$$

(defined like $\chi$), and define $\alpha : \mathrm{PGL}_2^+(\mathbb{R}) \to \mathbb{Q}$ by

$$\alpha(M) = \begin{cases} \chi^-(z_M) & \text{if } M \text{ is elliptic with fixed point } z_M \in \mathfrak{H}, \\ -\dfrac{1}{12} & \text{if } M \text{ is scalar,} \\ 0 & \text{otherwise.} \end{cases}$$

Note: $\alpha$ is well-defined, $\alpha(M) = \alpha(\lambda M)$.

# A further refinement

**Theorem**

*For* $M = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ *with* $y > 0$, *we have*

$$\sum_{\gamma \in \Gamma} \alpha(M\gamma) = \begin{cases} 1 & \text{if } y > 1\,, \\ 1/2 & \text{if } y = 1\,, \\ 0 & \text{if } y < 1\,. \end{cases}$$

There is a "weighted bijection" between half the elliptic conjugacy classes in $\mathcal{M}_n$, and those right cosets $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\Gamma$ with $a \geqslant d$.

# A tesselation of a Euclidean half-plane

We have to count the number of $\gamma \in \Gamma$ such $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\gamma$ has fixed point in $\mathcal{F}^-$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$, and define:

$$\begin{aligned} \Delta(\gamma) &= \{(x,y) \in \mathbb{R}^2 : \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\gamma \text{ has fixed point in } \mathcal{F}^-\} \\ &= \{(x,y) \in \mathbb{R}^2 : 0 \leqslant d - cx - ay \leqslant c \leqslant -dx - by\} \end{aligned}.$$

The triangle $\Delta(\gamma)$ is contained in the Euclidean half-plane

$$\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y \geqslant 1\},$$

which proves the case $y < 1$ of the previous theorem.

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$(y = c(-dx - by) + d^2 - d(d - cx - ay) \geqslant c^2 + d^2 - c|d| \geqslant 1)$

# A tesselation of a Euclidean half-plane

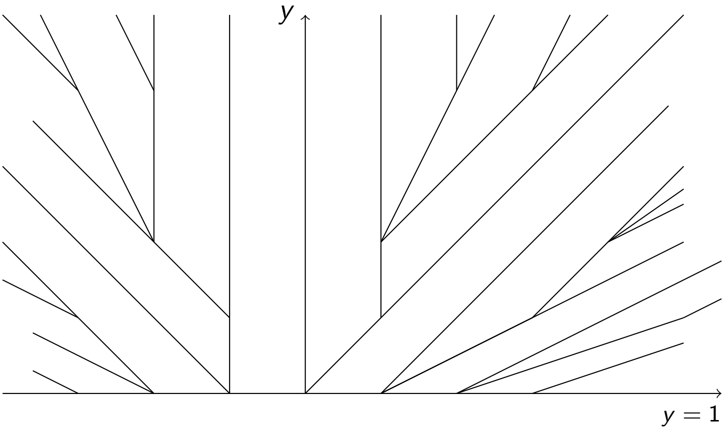
**Theorem**

*Let* $\Gamma_\infty = \{\gamma \in \Gamma : \gamma\infty = \infty\}$. *We have a tesselation*

$$\mathcal{H} = \bigcup_{\gamma \in \Gamma \setminus \Gamma_\infty} \Delta(\gamma)$$

*of the half-plane* $\mathcal{H}$ *into semi-infinite triangles with disjoint interiors.*

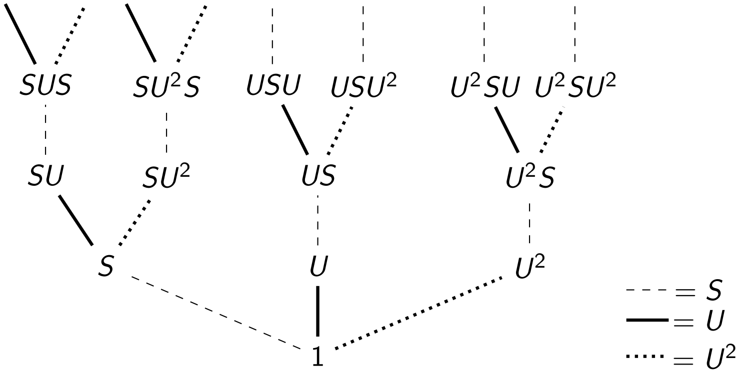
# A tesselation of a Euclidean half-plane




The region $\mathcal{H}$ and a few triangles $\Delta(\gamma)$.

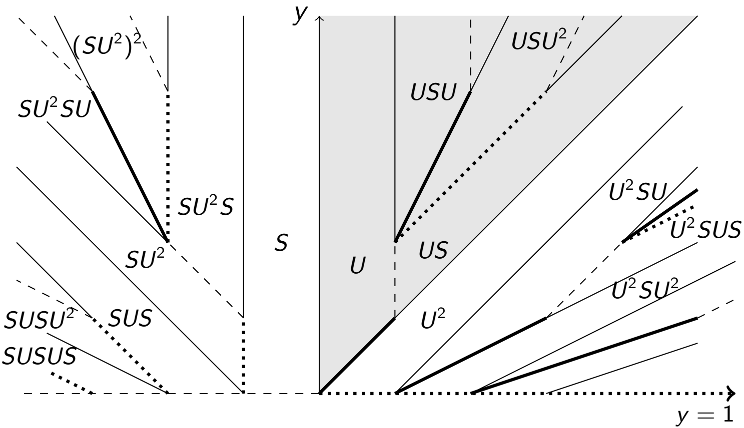
# Proof

The group $\Gamma$ is a free product of its two subgroups generated by the elements $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ of orders 2 and 3.




A tree associated to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$: the vertices are labeled by the elements of $\Gamma$ and the edges by the generators $S$, $U$ and $U^2$ as shown.

# Proof




The region $\mathcal{R}$ (shaded) is covered by triangles of words starting in $U$. The finite side of $\Delta(\gamma)$ has been labeled by the final letter of $\gamma$ as a word in $S, U, U^2$.

# Proof

We show $\mathcal{R} := \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant y - 1\} = \bigcup_{\gamma \in \mathcal{T}} \Delta(\gamma)$ ,
where $\mathcal{T} \subset \Gamma$ is the set of words starting in $U$:

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We show $\mathcal{R} := \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant y - 1\} = \bigcup_{\gamma \in \mathcal{T}} \Delta(\gamma)$ ,
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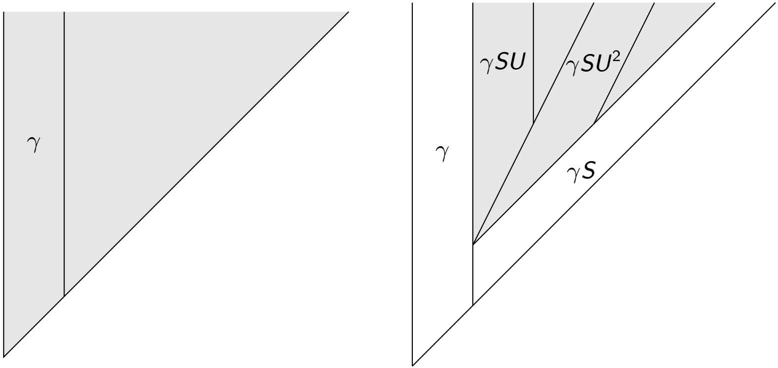


The cone $\mathcal{C}(\gamma)$ (left, shaded) decomposes into two triangles and two smaller cones. ∎

# Traces of Hecke operators

The group $\Gamma$ acts both on the left and on the right on the $\mathbb{Q}$-vector space $\mathcal{R}_n = \mathbb{Q}[\mathcal{M}_n]$.

- "Hecke operator acting on modular forms":

$$T_n^\infty = \sum_{\substack{n=ad \\ 0\leqslant b\leqslant d}} \left(\begin{smallmatrix} a & b \\ o & d \end{smallmatrix}\right) \in \mathcal{R}_n.$$

- "Hecke operator acting on period polynomials": There exists an element $\widetilde{T}_n \in \mathbb{Q}[\mathcal{M}_n]$ such that

$$(1-S)\widetilde{T}_n \; - \; T_n^\infty(1-S) \; \in \; (1-T)\mathcal{R}_n \qquad \text{(A)}$$

$$\left\{ \begin{array}{rcl} \widetilde{T}_n(1+S) & \in & (1+U+U^2)\mathcal{R}_n \\ \widetilde{T}_n(1+U+U^2) & \in & (1+S)\mathcal{R}_n \end{array} \right. \qquad \text{(B)}$$

- Example: $\widetilde{T}_1 = 1 - \frac{1}{2}(1+S) - \frac{1}{3}(1+U+U^2)$ sat. (A), (B)

# Traces of Hecke operators

For $M \in \mathcal{M}_n$ let $\Delta(M) = \operatorname{Tr}^2(M) - 4n$ and define

$$w(M) = \begin{cases} -1/|\operatorname{Stab} M| & \text{if } \Delta(M) < 0 \\ 1 & \text{if } \Delta(M) = u^2 > 0, u \in \mathbb{Z} \\ 1/6 & \text{if } M \text{ scalar} \\ 0 & \text{otherwise.} \end{cases}$$

Then $w(M)$ is a conjugacy class invariant and if $t^2 - 4n \leqslant 0$:

$$\sum_{\substack{X \subset \mathcal{M}_n \\ \operatorname{Tr}(X) = \pm t}} w(X) = \begin{cases} -2H(4n - t^2) & \text{if } t \neq 0 \,, \\ -H(4n - t^2) & \text{if } t = 0. \end{cases}$$

One can use this to extend the Kronecker-Hurwitz class number: $H(D) = -u/2$ if $D = -u^2 < 0$.

# Traces of Hecke operators

For $\widetilde{T} = \sum c_M M \in \mathcal{R}_n$, $\mathcal{S} \subset \mathcal{M}_n$ let $\deg_{\mathcal{S}}(\widetilde{T}) := \sum_{M \in \mathcal{S}} c_M \in \mathbb{Q}$.

**Theorem (P.-Zagier 2018)**

*Let $n$ be a positive integer, and let $\widetilde{T}_n \in \mathcal{R}_n$ satisfy both* (A) *and* (B)*.*

1. *For any right $\Gamma$-coset $K \subset \mathcal{M}_n$ we have $\deg_K \widetilde{T}_n = -1$.*
2. *For any $\Gamma$-conjugacy class $X$ we have*

$$\deg_X \widetilde{T}_n = w(X) \ .$$

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The theorem easily implies a formula for the trace of Hecke operators on modular forms for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. With more work, it proves trace formulas for congruence subgroups of $\Gamma$ as well!

Computing deg $\widetilde{T}_n$ by the two parts of the theorem yields another proof of the following version of the the Kronecker-Hurwitz formula:

$$-\sum_{X \subset \mathcal{M}_n} w(X) \;=\; |\Gamma \backslash \mathcal{M}_n| = \sum_{d|n} d \;,$$

where the sum is over all conjugacy classes $X$.

# Eichler-Selberg trace formula

Let $S_k(\Gamma) \subset M_k(\Gamma)$ be the space of cusp forms, resp. modular forms of weight $k$ for $\Gamma$. For even $k \geqslant 4$ we have

$$\begin{aligned} \mathrm{Tr}(T_n, M_k(\Gamma)) + \mathrm{Tr}(T_n, S_k(\Gamma)) &= \sum_{X \subset \mathcal{M}_n} w(X) p_{k-2}(\mathrm{Tr}(X), n) \\ &= -\sum_{t \in \mathbb{Z}} p_{k-2}(t, n) H(4n - t^2) \end{aligned}$$

where the sum is over conjugacy classes $X$ and $p_k(t, n)$ is the Gegenbauer polynomial, defined by

$$(1 - tX + nX^2)^{-1} = \sum_{\mathrm{w}=0}^{\infty} p_{\mathrm{w}}(t, n) X^{\mathrm{w}}.$$

# Cohomological formulation

For even $k \geqslant 2$, let $V_{k-2} \simeq \mathsf{Sym}^{k-2}\,\mathbb{C}^2$ be the $\mathrm{GL}_2(\mathbb{C})$-module of homogenous polynomials in two variables of degree $k-2$. Then

$$\sum_i (-1)^i \operatorname{Tr}(T_n, H^i(\Gamma, V_{k-2})) = -\sum_{X \subset \mathcal{M}_n} w(X) \operatorname{Tr}(M_X, V_{k-2})$$

where $M_X$ is a representative of the conjugacy class $X$.

**Exercise:** $\operatorname{Tr}(M, V_{\mathrm{w}}) = p_{\mathrm{w}}(\operatorname{Tr} M, \det M)$.

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**Remarks:**

- If $k = 2$ only $i = 0$ contributes, yielding the class number relation, while if $k \geqslant 4$ only $i = 1$ contributes, yielding the trace formula.

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- **Theorem (P. 2018)** This shape of the trace formula generalizes to arbitrary congruence subgroups.
- The ultimate generalization is the "Topological trace formula" of Goresky and MacPherson, computing Lefschetz numbers of Hecke correspondences on very general reductive groups.