Finite Automata, Automatic Sets, and Difference Equations

Michael F. Singer
(joint work with Reinhard Schäfke)

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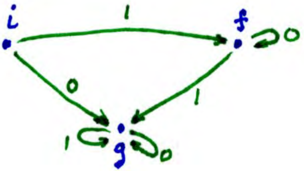
Finite Automata

- Σ - finite alphabet, Σ^* - words on Σ
- Σ-automaton = finite, labelled, digraph. Vertices = *states*
  - $\exists$ distinguished state =*initial state*
  - Some states are *final states*
  - For any state q and $\sigma \in \Sigma$, $\exists!$ arrow labelled σ leaving q

  ex. $\Sigma = \{0, 1\}$, states = $\{i, f, g\}$, initial = $\{i\}$, final = $\{f\}$



- $w \in \Sigma^*$ is recognizable by a Σ-autom. if this autom. reads the word and ends in a final state. A set $S \subset \Sigma^*$ is recognizable if $\exists$ a Σ-autom. whose set of recognizable words is S.

  ex.100 (yes), 110 (no)

Finite Automata

- Neural Nets
- Formal Languages
- Complexity of Numbers
  - E. Borel: Are irrational algebraic numbers normal?
  - Hartmanis/Stearns: Do there exist real time computable irrational algebraic numbers?
  - Loxton/van der Poorten: Can b-ary expansions of irrational algebraic numbers be generated by a finite automaton?
    NO- Adamczewski/Bugeaud.

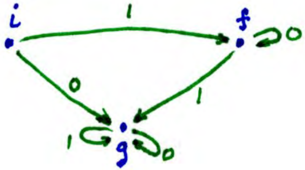
How powerful are finite automata?

Automatic Sets

$n \in \mathbb{N} \Rightarrow [n]_k$ is the base k representation of n.

A set $S \subset \mathbb{N}$ is k-automatic (k-recognizable) if the set $\{[n]_k \mid n \in S\}$ is Σ-recognizable, $\Sigma = \{0, 1, \ldots, k-1\}$.

The set of powers of 2 is 2-automatic.



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The set of powers of 2 is 2-automatic. Is it 3-automatic?

Thm (Cobham, 1969). For $k, l \geq 2$ multiplicatively independent, a subset $S \subset \mathbb{N}$ is k- and l-automatic then it is ultimately periodic.

S ultimately periodic = $\exists\ c, d$ s.t. for all $x > c$, $x \in S \Leftrightarrow x + d \in S$.

Difference Equations

Prop. $S \subset \mathbb{N}$ is k-automatic $\Rightarrow y(x) = \sum_{n \in S} x^n$ satisfies a k-Mahler equation

$$L(y(x)) = y(x^{k^m}) + a_{m-1}(x) y(x^{k^{m-1}}) + \ldots + a_0(x) y(x) = 0, \quad a_i(x) \in \mathbb{C}(x).$$

ex. $S = \{2^i \mid i = 0, 1, \ldots\} \Rightarrow y(x) = \sum_{i=0}^{\infty} x^{2^i}$ satisfies

$$y(x^4) - (x^2 + 1) y(x^2) + x^2 y(x) = 0.$$

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $y(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only it is a rational function.

Cobham's Theorem and Mahler Equations

Thm (Cobham, 1969). For k, l, multiplicatively independent, a subset $S \subset \mathbb{N}$ is k- and l-automatic if and only if it is ultimately periodic.

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $y(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only if it is a rational function.

A-B Thm $\Rightarrow$ C Thm:

- $S \Rightarrow y(x) = \sum_{n\in\mathbb{N}} \alpha_n x^n, \;\; \alpha_n = 0, 1, \; \alpha_n = 1 \Leftrightarrow n \in S$
- k-automatic $\Rightarrow \sum a_i(x) y(x^{k^i}) = 0$
- l-automatic $\Rightarrow \sum b_i(x) y(x^{l^i}) = 0$
- A-B Thm $\Rightarrow y(x) = \frac{p(x)}{q(x)}$
- $q(x)(\sum \alpha_n x^n) = p(x)$
  $\Rightarrow A_0\alpha_{N+i} + A_1\alpha_{N+i-1} + .. + A_N\alpha_i = 0, i >> 0$
- $\alpha_{N+i} = -\frac{1}{A_0}(A_1\alpha_{N+i-1} + .. + A_N\alpha_i)$ and $\alpha_i = 0, 1 \Rightarrow$ ultimately periodic

A-B use the C Thm. to prove the A-B Thm!

A-B Theorem and Similar Results

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $F(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only if it is a rational function.

Thm (Ramis, 1992). $F(x) \in C[[x]]$ satisfies a linear differential equation

$$L_1(F(x)) = \frac{d^n}{dx^n}(F(x)) + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(F(x)) + \ldots + a_0(x)F(x) = 0$$

and a linear q-difference equation (q not a root of 1)

$$L_2(F(x)) = F(q^m x) + b_{m-1}(x)F(q^{m-1}x) + \ldots + b_0(x)F(x) = 0$$

with $a_i(x), b_i(x) \in C(x)$, if and only if it is a rational function.

Functions Satisfying Two Linear Differential/Difference Equations

$$L_1(y) = \partial_1^n(y) + a_{n-1}\partial_1^{n-1}(y) + \ldots + a_1\partial_1(y) + a_0 y = 0$$
$$L_2(y) = \partial_2^m(y) + b_{m-1}\partial_2^{m-1}(y) + \ldots + b_1\partial_2(y) + b_0 y = 0 \quad a_i, b_i \in C(x)$$

y	∂_1	∂_2	Conclusion
$y \in C((x))$	$\frac{\partial}{\partial x}$	$\partial_2(x) = qx, q \neq$ root of 1	$y \in C(x)$ Ramis, 1992
$y \in C((x))$	$\frac{\partial}{\partial x}$	$\partial_2(x) = x^p,\ p \in \mathbb{Z}_{\geq 2}$	$y \in C(x)$ Bézivin, 1994
$y \in C((\frac{1}{x}))$	$\frac{\partial}{\partial x}$	$\partial(x) = x+1$	$y \in C(x)$
$y \in C((x))$	$\partial_1(x) = q_1 x$	$\partial_2(x) = q_2 x$, q_1, q_2 indep.	$y \in C(x)$ Bézivin-Boutabba, 1992
$y \in C((x))$	$\partial_1(x) = x^{p_1}$	$\partial_2(x) = x^{p_2}$, p_1, p_2 indep.	$y \in C(x)$ Adamczewski-Bell, 2013
$y \in C((\frac{1}{x}))$	$\partial_1(x) = x+1$	$\partial_2(x) = x+\alpha, \alpha \notin \mathbb{Q}$	$y \in C(x)$
y Merom.	$\frac{\partial}{\partial x}$	$\partial(x) = x+1$	$y = \sum r_j(x)e^{\alpha_j x}$, $r_j \in C(x)$ Bézivin-Gramain*, 1996
y Merom.	$\partial_1(x) = x+1$	$\partial_2(x) = x+\alpha, \alpha \in \mathbb{R}\backslash\mathbb{Q}$	$y = \sum r_j(x)e^{\alpha_j x}$, $r_j \in C(x)$ Bézivin-Gramain*, 1996
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We have a general approach that allows us to prove and generalize all these results

A-B Theorem and Systems of Difference Equations

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $F(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only if it is a rational function.

Thm (Schäfke-Singer, 2016) *Consistent Mahler systems*

$$\begin{array}{rcl} Y(x^k) = A_1(x)Y(x) & & Y(x^l) = A_2(x)Y(x) \\ A_1(x), A_2(x) & \in & GL_n(\mathbb{C}(x)) \end{array}$$

are equivalent to constant Mahler systems

$$\begin{array}{rcl} Z(x^k) = B_1 Z(x) & & Z(x^l) = B_2 Z(x) \\ B_1, B_2 & \in & GL_n(\mathbb{C}), \end{array}$$

that is, there is a change of variables $Y = GZ$ taking the first systems to the second.

A-B Theorem and Systems of Difference Equations

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $F(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only if it is a rational function.

Thm (Schäfke-Singer, 2016) Assume k, l are multiplicatively independent and the system

$$Y(x^k) = A_1(x)Y(x) \qquad Y(x^l) = A_2(x)Y(x) \tag{1}$$

with $A_1, A_2 \in \mathrm{GL}_n(C(x))$ is consistent. Then there exists $G(x) \in \mathrm{GL}_n(K), K = C(x^{1/s}),\ s \in \mathbb{N}$, such that the substitution $Y = G(x)Z$ transforms (1) to

$$Z(x^k) = B_1 Z(x) \qquad Z(x^l) = B_2 Z(x) \tag{2}$$

with $B_1, B_2 \in \mathrm{GL}_n(C)$.

Consistent: $x \mapsto x^l, x^k$ commute $\Rightarrow Y((x^k)^l) = Y((x^l)^k) \Rightarrow$

$$A_2(x^k)A_1(x) = A_1(x^l)A_2(x)$$

A-B Theorem and Systems of Difference Equations

Thm (Adamczewski-Bell, 2013). For k, l multiplicatively independent, $F(x) \in C[[x]]$ satisfies both a k- and l-Mahler equation if and only if it is a rational function.

Thm (Schäfke-Singer, 2016) *Consistent Mahler systems are equivalent to constant Mahler systems.*

Sc-Si $\Rightarrow$ A-B

- $F(x)$ is a component of a solution $Y(x)$ of a *consistent* system $Y(x^k) = A_1(x)Y(x) \quad Y(x^l) = A_2(x)Y(x)$.
- S-S $\Rightarrow$ $Y(x) = G(x)Z(x)$, $G \in \mathrm{GL}_n(C(x^{\frac{1}{s}}))$, Z satisfies constant system.
- Const. sys. have only const. solns. $\Rightarrow Z \in \mathrm{GL}_n(C) \Rightarrow Y \in \mathrm{GL}_n(C(x^{\frac{1}{s}}))$
- $F \in C(x^{\frac{1}{s}}) \cap C((x)) = C(x)$.

Ramis's Theorem and Systems of Difference Eqns

Thm (Ramis, 1992). $F(x) \in C[[x]]$ satisfies a linear differential equation and a linear q-difference equation then $F(x)$ is a rational function.

Thm (Schäfke-Singer, 2016) Assume q is not a root of 1 and the system

$$\frac{dY}{dx} = A_1(x)Y(x) \qquad Y(qx) = A_2(x)Y(x) \tag{3}$$

with $A_1, A_2 \in \mathrm{GL}_n(C(x))$ is consistent. Then there exists $G(x) \in \mathrm{GL}_n(C(x))$ such that the substitution $Y = G(x)Z$ transforms (3) to

$$\frac{dZ}{dx} = \frac{B_1}{x}Z(x) \qquad Z(qx) = B_2 Z(x) \tag{4}$$

with $\underline{B_1, B_2 \in \mathrm{GL}_n(C)}$.

Consistent: $x\frac{d}{dx}$ commutes with $x \mapsto qx \Rightarrow \frac{dA_2}{dx} + A_2 A_1 = qA_1(qx)A_2$

Consistent Systems of Differential/Difference Eqns

MetaTheorem:

- Consistent systems have simple singuarities
- Systems with simple singularities are equivalent to simple systems.

A-B, Ramis's, … Theorem follows because

- Simple systems have simple solutions

Final Comments

Similar result for

$$\partial Y(x) = A(x)Y(x), \quad \sigma Y(x) = B(x)Y(x)$$

with $\partial = \frac{d}{dx}$, and $\sigma(x) = x + a$, or $\sigma(x) = qx$ or $\sigma(x) = x^p$ and systems of two linear difference equations

$$\sigma_1 Y(x) = A(x)Y(x), \quad \sigma_2 Y(x) = B(x)Y(x)$$

with (σ_1, σ_2) a sufficiently independent pair of shift operators, pair of q-dilation operators or pair of Mahler operators. This yields A-B results for these operators.

R. Schäfke, M.F. Singer, *Consistent systems of linear differential and difference equations*, arXiv:1605.02616, 46 Pages, to appear in JEMS.

R. Schäfke, M.F. Singer, *Mahler equations and rationality*, arXiv:1605.08830, 7 pages.