The Tangent Method for the determination of Arctic Curves

Andrea Sportiello
work in collaboration with F. Colomo

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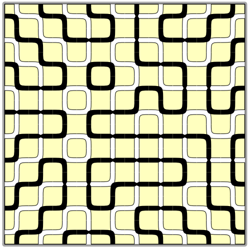
Alternating Sign Matrices

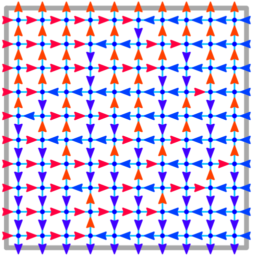
Four famous problems
in *Statistical Mechanics, Integrability and Combinatorics*
are related by simple bijections:

Fully-Packed Loops (FPL),
the 6 Vertex Model with DWBC (6VM),
perfect 3-colourings of the grid (3-Col),
and Alternating-Sign Matrices (ASM).

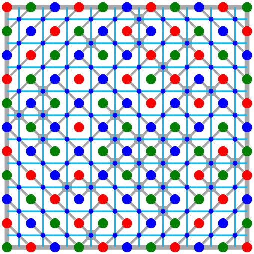
Alternating Sign Matrices



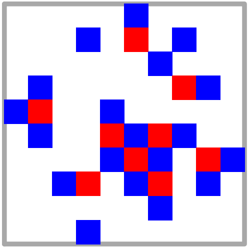
FPL



6VM

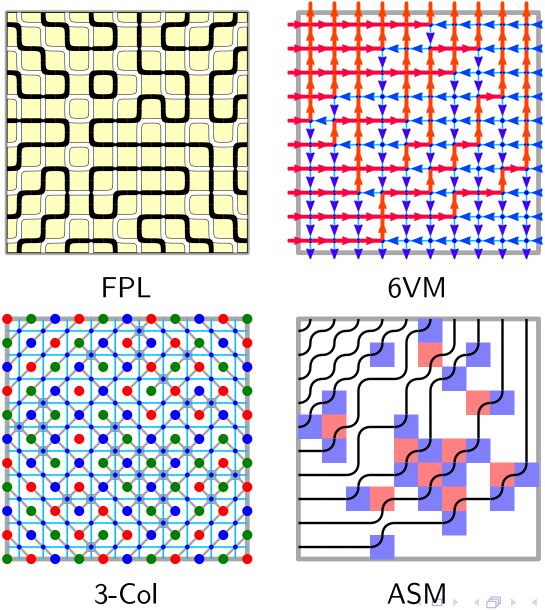


3-Col



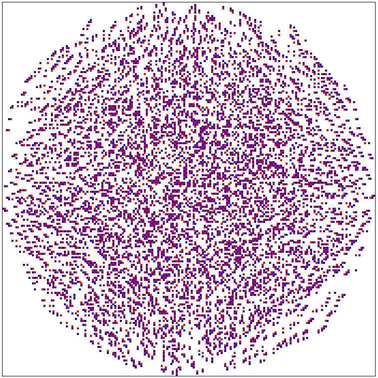
ASM

Alternating Sign Matrices



Asymptotic shape of ASM's

In large ASM's you see the emergence of a limit shape



This is analogous (but different) to the Arctic Circle for domino tilings of the Aztec Diamond.

The analytic determination of this curve is our subject today.

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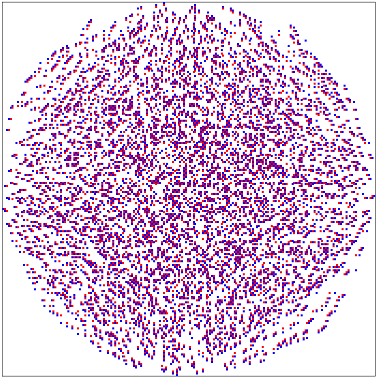

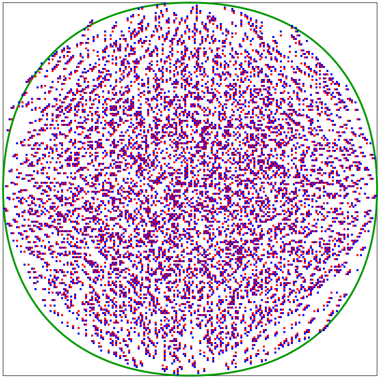

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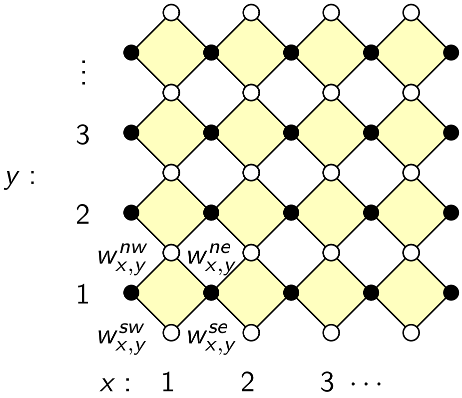
A reminder:
Arctic curves at free-fermionic points

weighted "Domino Tilings of the Aztec Diamond"
(a planar-graph dimer-covering problem, thus a fermionic system. . .)



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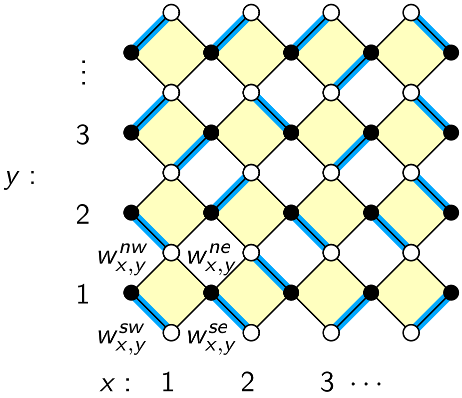
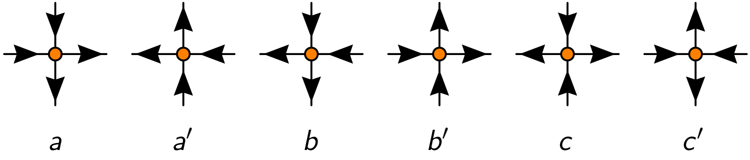


Domino Tilings of the Aztec Diamond ➽ ASM at $\omega = 2$

Consider the customary 6-Vertex Model weights...



then consider the following map: (note: $\Delta = \frac{aa' + bb' - cc'}{2\sqrt{aa'bb'}} = 0$)

$w^{sw}_{x,y}$	$w^{ne}_{x,y}$	$w^{se}_{x,y}$	$w^{nw}_{x,y}$	1	$w^{se}_{x,y} w^{nw}_{x,y} + w^{sw}_{x,y} w^{ne}_{x,y}$
a	a'	b	b'	c	c'

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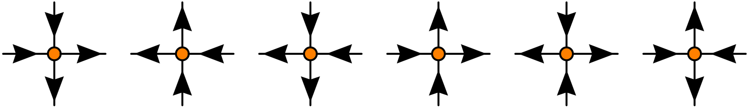

a $\quad a'$ $\quad b$ $\quad b'$ $\quad c$ $\quad c'$

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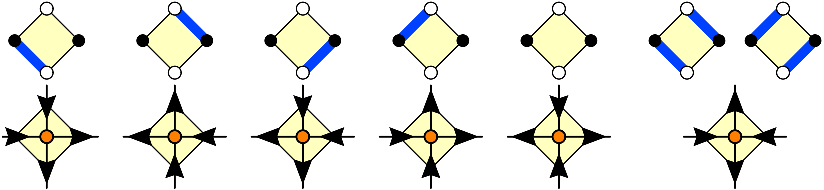
$w^{sw}_{x,y}$ $\quad w^{ne}_{x,y}$ $\quad w^{se}_{x,y}$ $\quad w^{nw}_{x,y}$ $\quad 1$ $\quad w^{se}_{x,y} w^{nw}_{x,y} + w^{sw}_{x,y} w^{ne}_{x,y}$



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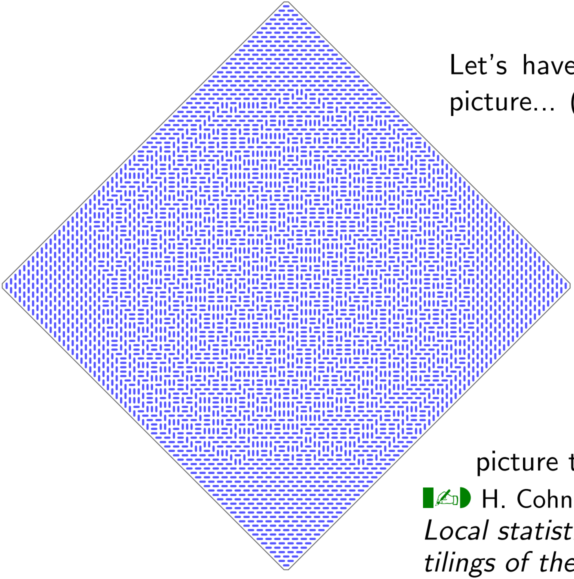
Domino Tilings of the Aztec Diamond: a bigger picture

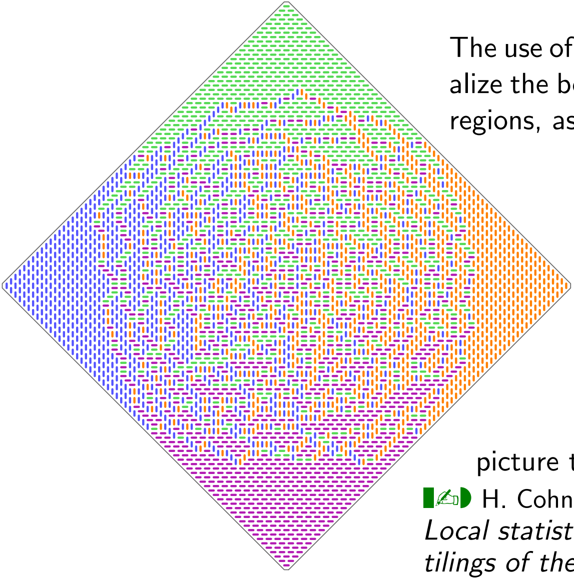


Let's have a look at a bigger picture... (here $L = 64$)

picture taken from:

H. Cohn, N. Elkies and J. Propp, *Local statistics for random domino tilings of the Aztec diamond*, 1995

Domino Tilings of the Aztec Diamond: a bigger picture



The use of colours allow to visualize the boundary of the frozen regions, as well as the NILP's

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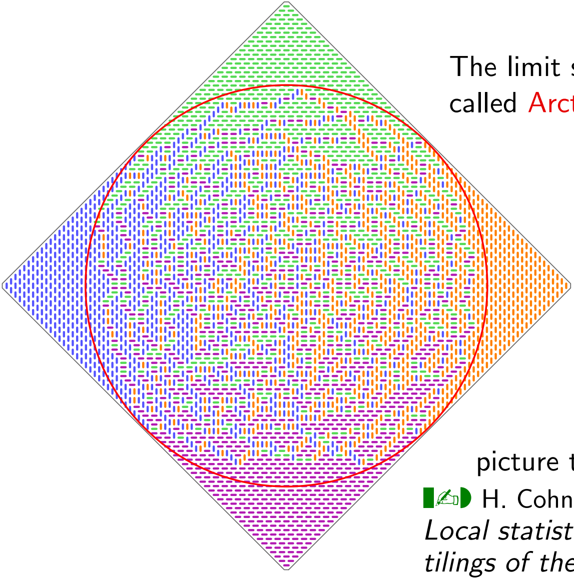

The limit shape, that they called Arctic curve, is a circle.

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The Colomo–Pronko formula

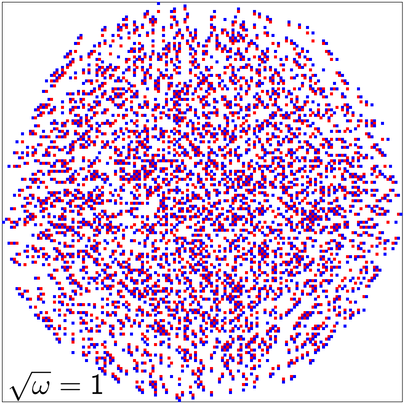
What about integrable systems out of fermionic points?

All of this is beautiful, but planar dimer coverings are fermionic...

As we know, ω-enumerations of ASM are Yang-Baxter integrable, with a fermionic point at $\omega = 2$ (domino tilings of the Aztec Diamond)

Numerical simulations (through CFTP) seem to show that the arctic curve varies smoothly with ω, at least within certain ranges...

...but what is know theoretically?



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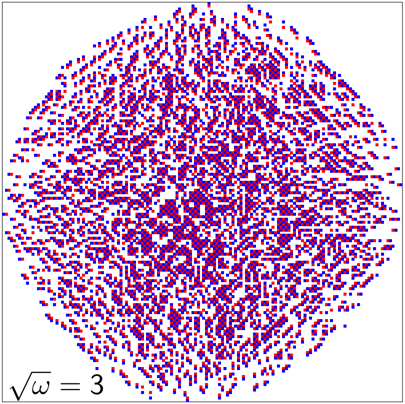

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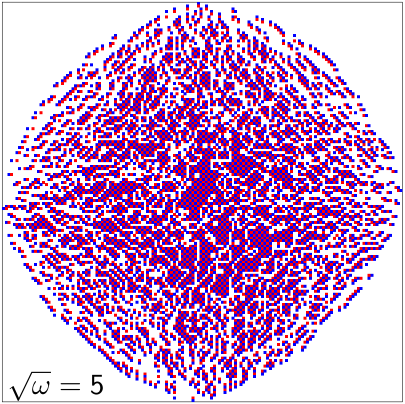

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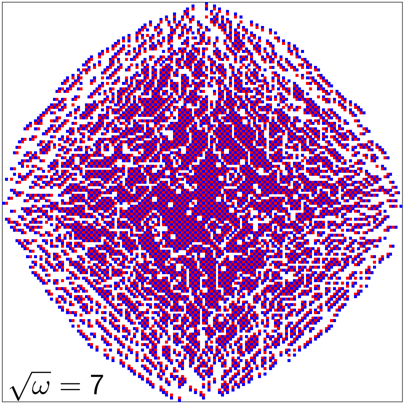

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The Colomo–Pronko formula

What is know theoretically? almost nothing up to recent times... Then Colomo and Pronko, from 2008 to 2010, came with a series of papers in which they:

- found a multicontour formula for a suitable quantity (emptiness formation probability). . .
- . . . from which, in analogy with certain random matrix models, derived a conjectural threshold condition. . .
- . . . which gives the (conjectural) Arctic Curve, first at $\omega = 1$. . .
- . . . then at generic ω, in terms of the refined enumerations $A_\omega(n; r)$, of which they calculated the asymptotics for $\omega \leq 4$ (where the corresponding 6-Vertex Model is "disordered"). . . . . .
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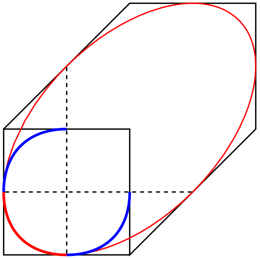
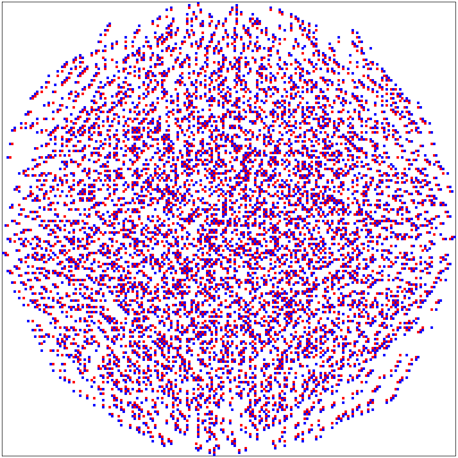
The Colomo–Pronko formula: $\omega = 1$

Picture and formula for $\omega = 1$:

The South-West arc satisfies
$x(1-x) + y(1-y) + xy = 1/4$
$x, y \in [0, 1/2]$

(just a "$+xy$" modification w.r.t. a circle)





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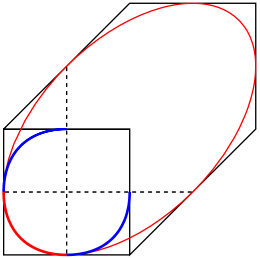
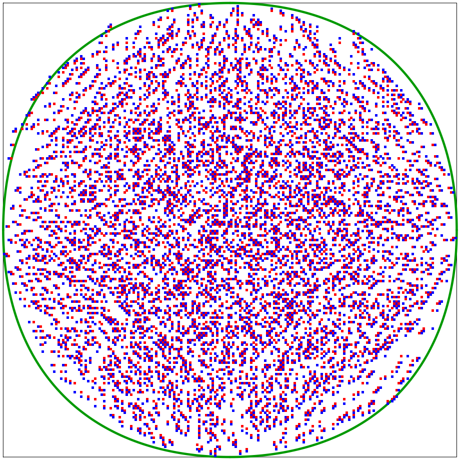
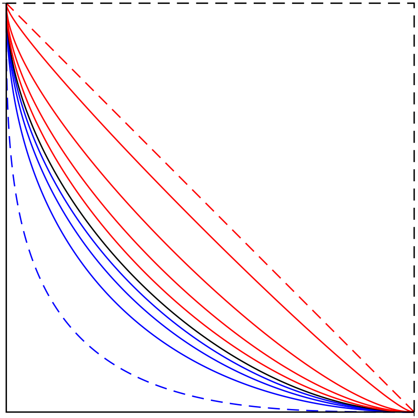




The Colomo–Pronko–Zinn-Justin formula: $0 < \omega < +\infty$

Theoretical prediction (for the South-West arc) drawn for values:

	ω
	$(+\infty)$
	202
	22
	10
antiferro	6
———	4
disordered	3
domino til.	2
ASM	1
	(0)



Refined enumeration of ASM's

We call $A_\omega(n)$ the counting polynomial associated to ω-weighted ASM of size n:

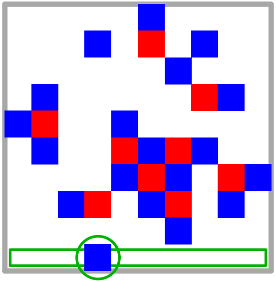
$$A_\omega(n) = \sum_{A \in \mathcal{A}_n} \omega^{\#\{-1 \text{ in } A\}}$$

Thus, e.g., $A_1(n) = \prod_{0 \le j \le n-1} \frac{(3j+1)!}{(n+j)!}$, the total number of size-n ASM

Call $A_\omega(n, r)$ the counting polynomial associated to ω-weighted ASM of size n, such that the only $+1$ in the bottom row is at the r-th column

Thus, e.g.,

$$\frac{A_1(n+1, r+1)}{A_1(n+1)} = \frac{\binom{n+r}{n}\binom{2n-r}{n}}{\binom{3n+1}{n}}$$



example at $n = 10$, $r = 4$

The Colomo–Pronko formula: generic ω

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$, in parametric form $x = x(z)$, $y = y(z)$ on the interval $z \in [1, +\infty)$, is the solution of the system of equations

$$F(z; x, y) = 0\,; \qquad \frac{\partial}{\partial z} F(z; x, y) = 0\,.$$

The function $F(z; x, y)$, that depends on x and y linearly, is

$$F(z; x, y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \psi(z)$$

$$\psi(z) := \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln \left(\sum_{r=1}^{n} A_\omega(n, r) z^{r-1} \right)\,.$$

$\mathcal{C}(x, y)$ is algebraic (with small degree) only at discrete special values of ω (including the famous $1, 2, 3$ cases), namely, parametrising $\omega = 2 - 2\cos\theta$, when $\theta/\pi \in \mathbb{Q}$ (with small denominator)

How are these results derived?

Call $h_n(z) = \sum_{r=1}^{n} A_\omega(n,r) z^{r-1}$

Define the Emptiness Formation Probability, $EFP(n;r,s)$: the probability that in the top-left $s \times r$ rectangle of the $n \times n$ ASM there are no ± 1 elements.

By the "inverse scattering method", it can be determined that the $EFP(n;r,s)$ is related to $h_n(z)$, through a multi-contour integral formula

$$h_{n,s}(z_1,\dots,z_s) := \frac{1}{\boldsymbol{\Delta}(z)} \det\left(z_j^{k-1}(z_j-1)^{s-k} h_{n-k+1}(z_j)\right)_{j,k}$$

$$EFP(n;r,s) = \oint_0 \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_0 \frac{\mathrm{d}z_s}{2\pi i} \prod_j \frac{((\omega-1)z_j+1)^{s-j}}{z_j^r (z_j-1)^{s-j+1}}$$

$$\times \prod_{j<k} \frac{z_j - z_k}{t\, z_j z_k + (\omega - 1 - t) z_j + 1}\ h_{n,s}(z_1,\dots,z_s)$$

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When (r,s) crosses the Arctic Curve, $EFP(n;r,s)$ is expected to show a 0-1 threshold transition, that can be studied through saddle-point methods, helped by previous techniques developed for a certain Random Matrix Model (Triple Penner Model)

How to derive this?

…in a few words, something very complicated already for the square domain

And something relying deeply on "miracles" of integrability methods, that have no guarantee to occur in other domains

Furthermore, already for $\omega = 1$, the curve is not $\mathcal{C}_\infty$ at the points of contact with the boundary of the domain, and is not even piecewise algebraic at generic ω

This is at difference from the beautiful, and essentially complete, theory for bipartite dimer models, developed by Kenyon and Okounkov (which thus contains ASM's at $\omega = 2$)

How can we hope for an analogue of Kenyon–Okounkov results on the whole phase diagram of the 6-Vertex Model?

A reminder on the basic theory of Plane Curves

A *curve* $\mathcal{C}$ will be represented either by the *Cartesian equation* $A(x, y) = 0$, or the *parametric equations* $x = f(t)$, $y = g(t)$. It is constituted by the concatenation of a finite number of *arcs*. An arc is a portion of the curve for which a "smooth" parametric presentation exists.

A curve is *algebraic* if the defining Cartesian equation $A(x, y) = 0$ is algebraic, otherwise it is *trascendental*.

A double point s.t. the two arcs passing through P have the same tangent is a *cusp*. A cusp is *of the first kind* if P is an endpoint of both arcs, and there is an arc of $\mathcal{C}$ on each side of the tangent, and *of the second kind* if P is an endpoint of both arcs, and the two arcs lie on the same side of the tangent,

A reminder on the basic theory of Plane Curves

The *envelope* $\mathcal{E}$ of a one-parameter family of curves $\{\mathcal{C}_z\}_{z \in I}$ is the curve, minimal under inclusion, that is tangent to every curve of the family.

If the equation of the family $\{\mathcal{C}_z\}$ is given in Cartesian coordinates by $U(z; x, y) = 0$, the non-singular points (x, y) of the envelope $\mathcal{E}$ are the solutions of the system of equations

$$U(z; x, y) = 0\,; \qquad \frac{\mathrm{d}}{\mathrm{d}z} U(z; x, y) = 0\,.$$

We call *caustic* the envelope of a family of straight lines. In this case U is linear in x and y:

$$U(z; x, y) = x\,A(z) + y\,B(z) + C(z)$$

The Colomo–Pronko formula at generic ω – reloaded

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For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$ is the caustic of the family of lines, for z in the interval $z \in [1, +\infty)$,

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The Colomo–Pronko formula at generic ω – reloaded

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$ is the caustic of the family of lines, for z in the interval $z \in [1, +\infty)$,

$$F(z; x, y) = \frac{1}{z}(x - 1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \psi(z)\,.$$

But this has not been derived geometrically!

The source of "transcendence" in the formula...

What's the algebraic nature of the Colomo–Pronko formula?
Let us pass to the trigonometric/hyperbolic parametrisation of the vertex weights (for the disordered/antiferro regimes)

disordered	antiferro
$\eta \in [0, \pi/2]$	$\eta > 0$
$a(\lambda) = \sin(\eta - \lambda)$	$a(\lambda) = \sinh(\eta - \lambda)$
$b(\lambda) = \sin(\eta + \lambda)$	$b(\lambda) = \sinh(\eta + \lambda)$
$c = -\sin(2\eta)$	$c = \sinh(2\eta)$

$$-\eta < \lambda < \eta$$
$$\xi \in [0, \eta - \lambda]$$
$$\phi(\xi) = \frac{c}{a(\xi) b(\xi)}$$
$$\alpha := \frac{\pi}{2\eta}$$

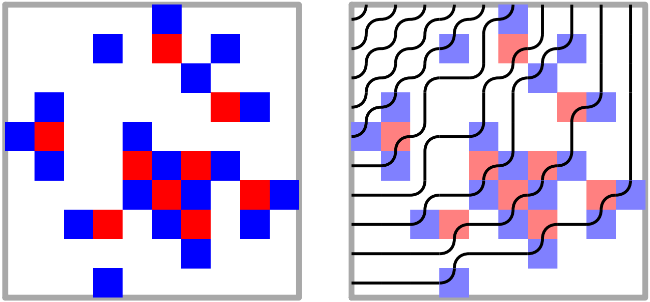
family of lines $\quad f(\xi; x, y) = x\,\phi(\xi + \lambda) + y\,\phi(\xi - \eta) - \Psi_{\mathrm{D/AF}}(\xi)$

$$\Psi_{\mathrm{D}}(\xi) = \cot\xi - \cot(\xi + \lambda - \eta) - \psi_{\mathrm{D}}(\xi) + \psi_{\mathrm{D}}(\xi + \lambda + \eta)$$
$$\Psi_{\mathrm{AF}}(\xi) = \coth\xi - \coth(\xi + \lambda - \eta) - \psi_{\mathrm{AF}}(\xi) + \psi_{\mathrm{AF}}(\xi + \lambda + \eta)$$
$$\psi_{\mathrm{D}}(\xi) = \alpha \cot(\alpha\xi) \qquad \psi_{\mathrm{AF}}(\xi) = \alpha\,(\ln\vartheta_1)'(\alpha\xi)$$

The tangent method

A reminder on interacting NILP

Recall that an ASM can be seen (in 4 different ways) as a configuration of interacting non-intersecting lattice paths (NILP), which are in fact non-interacting when $\omega = 2$.



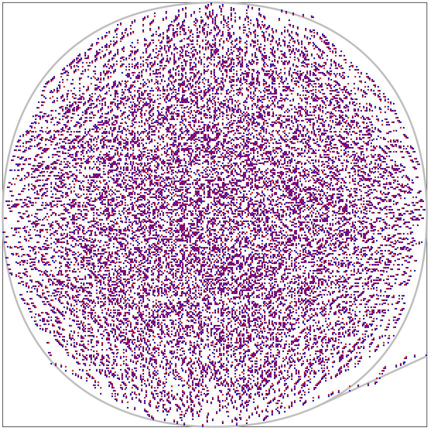
The refinement position is the point at which the most external path leaves the boundary

The structure of a typical refined ASM

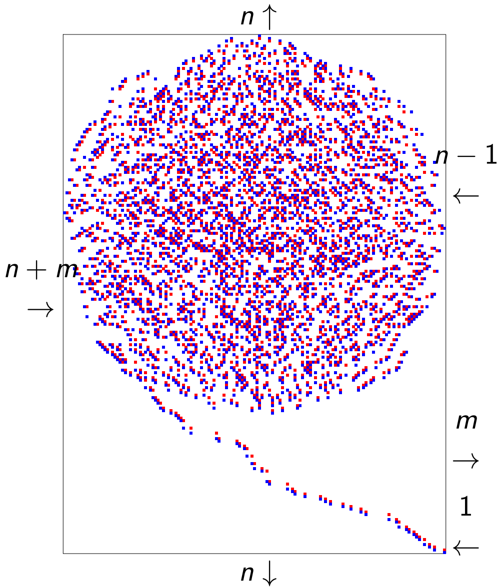
After some thinking, you get convinced that a typical large *ASM*, of size n refined at r, must look like a typical ASM, plus a straight line connecting $(0, r)$ to the Arctic Curve, and tangent to the Arctic Curve

Indeed, this is what you see in a simulation...

$n = 300, r = 250$



The Geometric Tangent Method in a picture



In this geometry, there is no reason for the isolated line to change direction at row n. Then:

IF the arctic curve exists

IF it does not depend on m

IF the path leaves the curve tangentially

THEN from the method we get a caustic parametrisation of the curve

Can we make the Tangent Method fully rigorous?

The Tangent Method exists in a further declination, which comes with with a good and a bad news.

The bad news is that now you need the doubly-refined enumeration, $A^{(1,2)}(n; r, s)$

The good news is that this method can be made rigorous, and determines the arctic curve at size n, up to a $\mathcal{O}(\sqrt{n})$ band of uncertainty.

For simplicity, I discuss this second method only for the $\omega = 1$ square-domain case.

Prolog: Emptiness formation probability of anything. . .

For X a (deterministic or random) object (let's call it a probe), define $E_n(X)$ as the probability that $X \cap B = \varnothing$, where B is the set of positions of ± 1's in a random ASM of size n (i.e., positions of c-vertices in the 6VM)

Examples of X:

- $E_n^{\text{point}}(r, s)$, a single cell at coordinate (r, s) (1-point function in the bulk);
- $E_n^{\text{rect}}(r, s)$, a $r \times s$ rectangle in a corner of the domain (the Colomo–Pronko EFP);
- $E_n^{\text{line}}(r, s)$, a straigth segment from $(r, 0)$ to $(0, s)$;
- $E_n^{\text{RW}}(r, s)$, a directed random walk from $(r, 0)$ to $(0, s)$;

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Examples of X:

- $E_n^{\text{point}}(r, s)$, a single cell at coordinate (r, s) (1-point function in the bulk); too difficult to evaluate
- $E_n^{\text{rect}}(r, s)$, a $r \times s$ rectangle in a corner of the domain (the Colomo–Pronko EFP); viable, but still messy
- $E_n^{\text{line}}(r, s)$, a straigth segment from $(r, 0)$ to $(0, s)$; clean definition, but also quite difficult to evaluate
- $E_n^{\text{RW}}(r, s)$, a directed random walk from $(r, 0)$ to $(0, s)$; **easy** to evaluate, and can be **related to $E_n^{\text{line}}(r, s)$**!

A simple but crucial remark

Here we have our simple but crucial remark:

$$A^{(1,2)}(n+1;r+1,s+1) = A(n)\binom{r+s}{r}E_n^{\mathrm{RW}}(r,s)$$

The knowledge of $A^{(1,2)}(n;r,s)$ (the “row-column” doubly-refined enumeration) is not so explicit as $A^{(1)}(n;r)$, but is well under control (see e.g. Yu. Stroganov, *A new way to deal with Izergin-Korepin determinant at root of unity*)

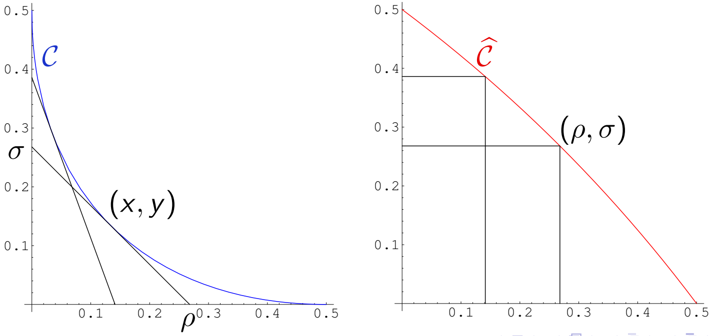
$$A^{(1,2)}(n;r,s+1) + A^{(1,2)}(n;r+1,s) - A^{(1,2)}(n;r+1,s+1) = A^{(1,3)}(n;r,s)$$

$$A^{(1,3)}(n;r,s) - A^{(1,3)}(n;r-1,s-1) = A(n-1)^{-1} \Big[A(n-1,r-1)\big(A(n,s)-A(n,s-1)\big) + (r \leftrightarrow s)\Big]$$

From the curve to its "caustic transform"

We want to find (the bottom-left corner of) the $\omega = 1$ arctic curve $\mathcal{C}$, which satisfies $x(1-x) + y(1-y) + xy = 1/4$

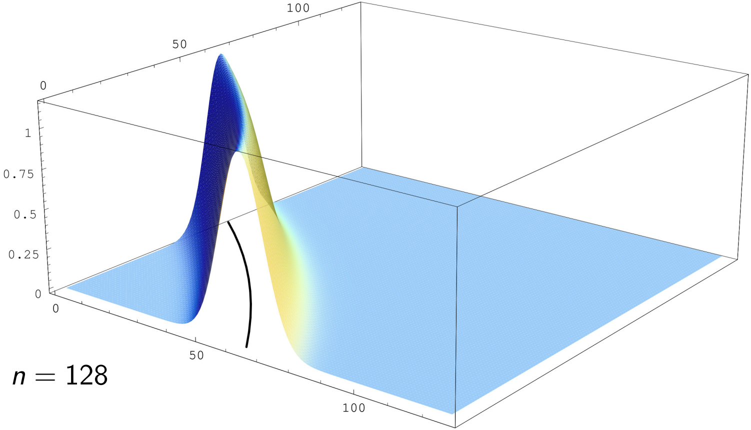
However, as our goal is to find it through the limit $n \to \infty$ of $E_n^{\text{line}}(\rho n, \sigma n)$, we shall equivalently represent it on the (ρ, σ) plane, where it gives $(\rho, \sigma)_\theta = \left(\frac{1-\sqrt{3}\tan\theta}{2}, \frac{1-\sqrt{3}\tan(\frac{\pi}{6}-\theta)}{2}\right)$, for $\theta \in [0, \frac{\pi}{6}]$



Let's have a look at $E_n^{\mathrm{RW}}(r,s)$

Let's have a look at $E_n^{\mathrm{RW}}(r,s)$, that shall converge to a step function

It is nicer to look at $-\sqrt{n}\partial_{(1,1)}E_n^{\mathrm{RW}}(r,s)$, that shall appear as a (rescaled) distribution concentrated on our curve.



$n = 128$

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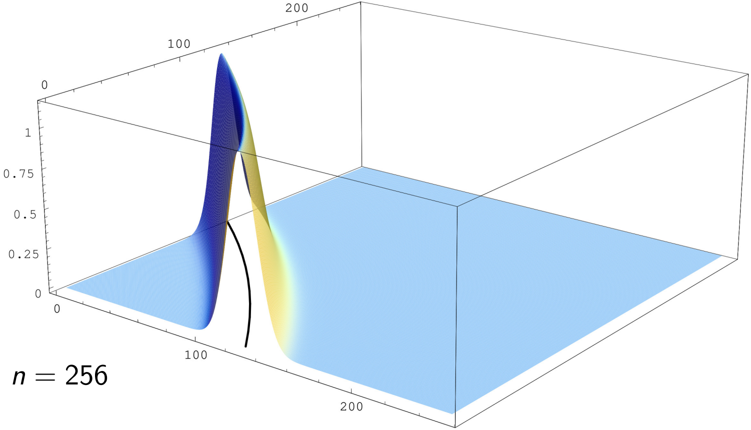

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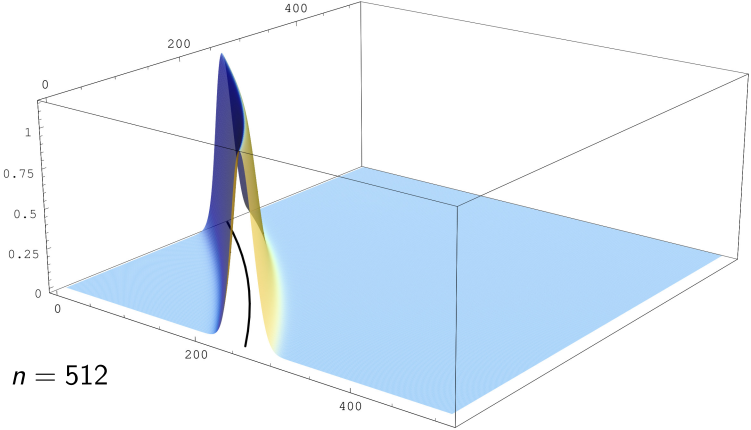

A nice accident

In fact, although $A^{(1,2)}(n; r, s)$, the row-column 2ref enumeration is well under control, $A^{(1,3)}(n; r, s)$, the row-row 2ref enumeration is a bit easier

By a lucky accident, at $\omega = 1$ we have

$$A^{(1,2)}(n; r, s+1) + A^{(1,2)}(n; r+1, s) - A^{(1,2)}(n; r+1, s+1) = A^{(1,3)}(n; r, s)$$

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$$\frac{A^{(1,2)}(n;r,s+1)+A^{(1,2)}(n;r+1,s)-A^{(1,2)}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}} = \frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

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$$\frac{r}{r+s}\frac{A^{(1,2)}(n;r,s+1)}{A(n-1)\binom{r+s-1}{r-1}} + \frac{s}{r+s}\frac{A^{(1,2)}(n;r+1,s)}{A(n-1)\binom{r+s-1}{r}} - \frac{A^{(1,2)}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}} = \frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

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$$-\frac{r\partial_r^- + s\partial_s^-}{r+s} E_{n-1}^{\mathrm{RW}}(r,s) = \frac{A^{(1,3)}(n; r, s)}{A(n-1)\binom{r+s}{r}}$$

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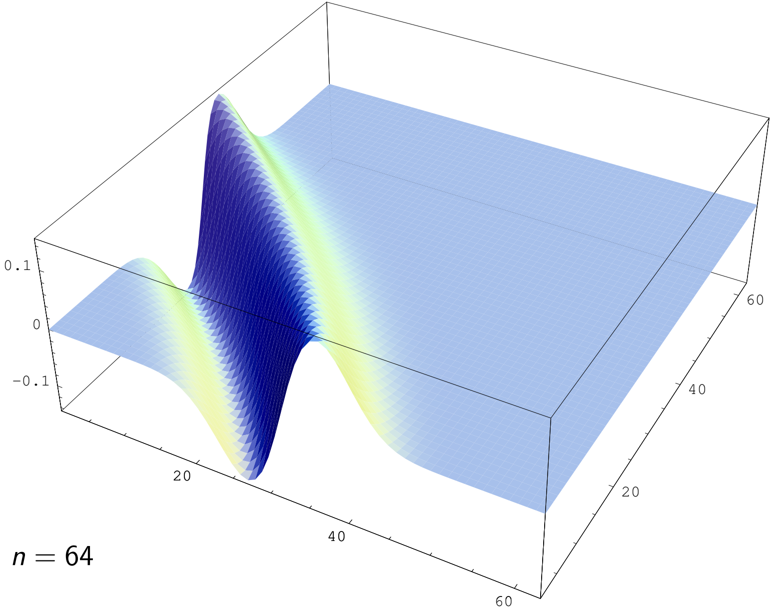
Thus $\frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$ is sensibly larger than 0 only on the transform of the arctic curve, and its gradient along the $(1,1)$ direction shall change sign on this curve

The astonishingly tiny finite-size corrections



$n = 64$

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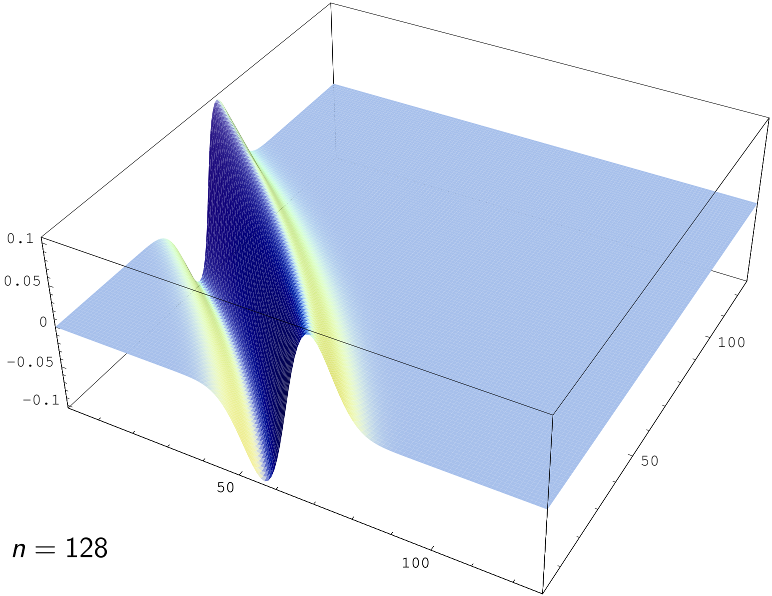

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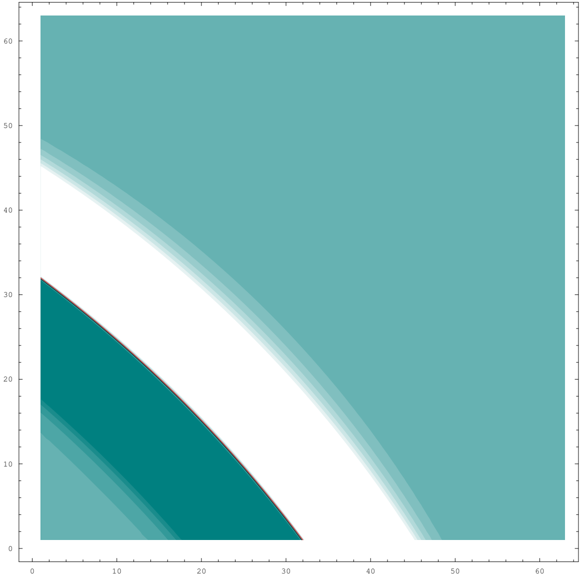

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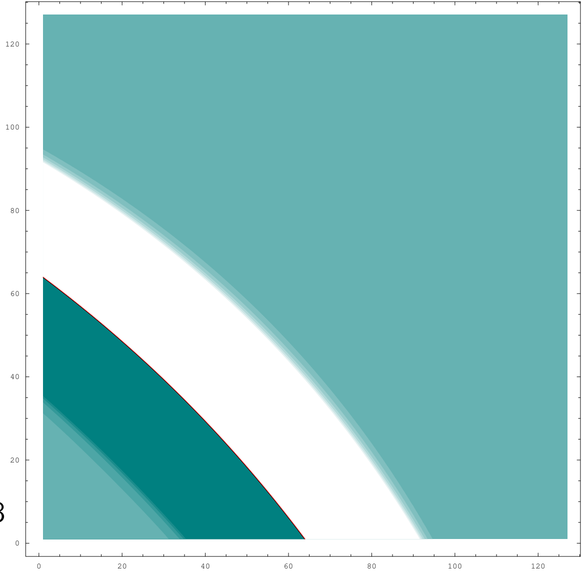

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