Degeneration of Frobenius structures

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Part I: Definition and observations

$$K \supset \mathbb{C}(t) \quad \text{differential field} , \qquad A, B \in \mathbb{C}(t)^{r\times r}$$

$$(1)\ \left\{ \frac{dU}{dt} = AU \right\} \qquad\qquad (2)\ \left\{ \frac{dV}{dt} = BV \right\}$$

Differential system (1) is equivalent to (2) over K if there exists an invertible matrix $H \in K^{r\times r}$ such that

$$\frac{dH}{dt} = BH - HA.$$

Observe:

- U is a solution to (1) $\Rightarrow$ $V = HU$ is a solution to (2)
- U, V fundamental solution matrices for (1) and (2) $\Rightarrow$ $H = V\Lambda U^{-1}$ where Λ is a constant marix:

$$\frac{d}{dt}(V^{-1}HU) = -(V^{-1}B)HU + V^{-1}(BH - HA)U + V^{-1}H(AU) = 0$$

Example: local study / regular singularities

take $A \in \mathbb{Q}(t)^{r\times r}$ with no pole at $t = 0$
$K = \mathcal{O}[t^{-1}]$ field of germs of meromorphic functions

there is $\Gamma \in \mathbb{C}^{r\times r}$ (unique up to conjugation) such that

$$\left\{ t\frac{dU}{dt} = AU \right\} \sim^K \left\{ t\frac{dV}{dt} = \Gamma V \right\}$$

- eigenvalues of Γ are called local exponents at $t = 0$
- $V = t^\Gamma$ is a fundamental solution matrix for the second system; $M_0 = \exp(2\pi i \Gamma)$ its monodromy around $t = 0$
- differential systems are equivalent over K if and only if their local monodromies M_0 are conjugate

Frobenius structure

fix p prime

Definition. A differential system with $A \in \mathbb{Q}(t)^{r\times r}$ admits a Frobenius structure of period $h = 1, 2, \ldots$ if

$$(1)\ \left\{ t\frac{dV}{dt} = p^h A(t^{p^h}) V \right\} \quad \sim^{E_p} \quad (2)\ \left\{ t\frac{dU}{dt} = A(t) U \right\}$$

$$E_p = \text{ completion of } \mathbb{C}_p(t) \text{ w.r.t. the Gauss norm}$$

(field of p-adic analytic elements)

$$|\sum a_i t^i|_{Gauss} = \max_i |a_i|_p$$

$$t\frac{d\Phi(t)}{dt} = A(t)\Phi(t) - p^h \Phi(t) A(t^{p^h}), \quad \Phi \in E_p^{r\times r}$$

- U is a solution to (2) $\Rightarrow$ $V(t) = U(t^{p^h})$ is a solution to (1)
- let U be a fundamental solution matrix
  $\exists \Lambda \in \mathbb{C}_p^{r\times r}$ such that $\Phi(t) = U(t)\Lambda U(t^{p^h})^{-1} \in E_p^{r\times r}$

Example: $\frac{dU}{dt} = \frac{1}{2}\frac{1}{1-t}U, \qquad U(t) = \frac{1}{\sqrt{1-t}}$

$p \neq 2 \qquad \Phi(t) = U(t)U(t^p)^{-1} \in^? E_p$

$$\sqrt{\frac{1-t^p}{1-t}} = (1-t)^{\frac{p-1}{2}}\left(\frac{1-t^p}{(1-t)^p}\right)^{1/2} \qquad g(t) = \frac{1-t^p-(1-t)^p}{p} \in \mathbb{Z}[t]$$

$$= (1-t)^{\frac{p-1}{2}}\left(1 + p\frac{g(t)}{(1-t)^p}\right)^{1/2}$$

$$= (1-t)^{\frac{p-1}{2}}\sum_{k\geqslant 0}\binom{1/2}{k}p^k\frac{g(t)^k}{(1-t)^{pk}}$$

$$\in \widehat{\mathbb{Z}[t,(1-t)^{-1}]} \subset E_p$$

Notation: $\widehat{R} = \varprojlim R/p^sR$ is the p-adic completion

$$(1) \qquad t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab$$

$$a, b, c-a, c-b \notin \mathbb{Z} \quad (\Leftrightarrow \text{ irreducible monodromy on } \mathbb{P}^1 \backslash \{0,1,\infty\})$$

Theorem (Dwork) Differential equation (1) with $a, b, c \in \mathbb{Q} \cap \mathbb{Z}_p$ admits a Frobenius structure of period

$$h = \min\{m : (p^m-1)a, (p^m-1)b, (p^m-1)c \in \mathbb{Z}\}.$$

Remark: multiplying (1) by t we obtain the operator

$$\theta(\theta+c-1) - t(\theta+a)(\theta+b), \quad \theta = t\frac{d}{dt},$$

so the local exponents are

0	∞	1
0	a	0
$1-c$	b	$c-a-b$

and h in the theorem is such that the eigenvalues of local monodromies M_0, M_1 and M_∞ are (p^h-1)st roots of unity.

Remark (continued): a Frobenius structure is a solution to the 1st order differential system

$$t\frac{d\Phi(t)}{dt} = A(t)\Phi(t) - p^h\Phi(t)A(t^{p^h}).$$

If there is a meromorphic solution $\Phi \in K^{r\times r}$, $K = \mathcal{O}[t^{-1}]$ then

$$M_0 \sim M_0^{p^h} \Rightarrow \text{ eigenvalues are } (p^h - 1)\text{st roots of } 1$$

It is often the case that $\Phi \in (E_p \cap K)^{r\times r}$.
Similar considerations apply at other singular points.

Example: $\theta^2 - t(\theta + \frac{1}{3})(\theta + \frac{2}{3}) \qquad A(t) = \begin{pmatrix} 0 & 1 \\ \frac{t}{1-t} & -\frac{2}{9}\frac{t}{1-t} \end{pmatrix}$

$$u_0(t) = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, 1; t) = 1 + \tfrac{2}{3^2}t + \tfrac{10}{3^4}t^2 + \tfrac{560}{3^8}t^3 + \ldots \in \mathbb{Z}[[3^{-3}t]]$$
$$u_1(t) = \log(t)u_0(t) + \tfrac{5}{9}t + \tfrac{19}{54}t^2 + \ldots = \log(t)u_0(t) + u_1^{an}(t)$$

$$U(t) = \begin{pmatrix} u_0 & u_1 \\ \theta u_0 & \theta u_1 \end{pmatrix} = \begin{pmatrix} u_0 & u_1^{an} \\ \theta u_0 & \theta u_1^{an} + u_0 \end{pmatrix} \begin{pmatrix} 1 & \log(t) \\ 0 & 1 \end{pmatrix}, \; \theta(U) = AU$$
$$U^{an}(t) := \begin{pmatrix} u_0 & u_1^{an} \\ \theta u_0 & \theta u_1^{an} + u_0 \end{pmatrix} = U\Big|_{\text{“}\log(t)=0\text{”}} \in \mathbb{Q}[[t]]^{2\times 2}, \; U^{an}(0) = Id$$

Look for Λ such that $\Phi(t) = U(t)\Lambda U(t^p)^{-1} \in E_p^{2\times 2}$. Observe:

$$\begin{pmatrix} 1 & \log(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} 1 & -p\log(t) \\ 0 & 1 \end{pmatrix} = \text{const} \quad \Leftrightarrow \quad \begin{cases} \lambda_{21} = 0 \\ \lambda_{22} = p\lambda_{11} \end{cases}$$

$\Rightarrow$ analytic at $t = 0$ solutions to $\theta(\Phi) = A(t)\Phi(t) - p\Phi(t)A(t^p)$ are

$$\Phi(t) = U(t) \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & p\lambda_{11} \end{pmatrix} U(t^p)^{-1} = U^{an}(t) \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & p\lambda_{11} \end{pmatrix} U^{an}(t^p)^{-1}$$

Claim: for $p \neq 2, 3$, there exists $\alpha \in \mathbb{Z}_p$ such that

$$\Phi(t) = U^{an}(t) \begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix} U^{an}(t^p)^{-1} \in \widehat{\mathbb{Z}[t, \tfrac{1}{3(1-t)}]}^{2\times 2}.$$

Here $\widehat{\mathbb{Z}[t, \tfrac{1}{3(1-t)}]} \subset E_p$ is the p-adic completion.

- α can be found experimentally: e.g. $p = 5$

  $\Phi(t) = \Phi_0(t) + \alpha\Phi_1(t), \qquad \Phi_i \in (\mathbb{Q} \cap \mathbb{Z}_p)[[t]]^{2\times 2}$

  $$\Phi_0(t) \equiv \begin{pmatrix} 1+3t & 3t \\ 0 & 0 \end{pmatrix} \mod 5, \quad \deg(\Phi_1(t) \mod 5) \approx \infty$$

  $\deg(\Phi_0(t) \mod 5^2) \approx \infty$, but

  $$\deg\left(\Phi_0(t) + i \cdot 5 \cdot \Phi_1(t) \mod 5^2\right) = \begin{cases} 3, i = 2 \\ \infty, i = 0, 1, 3, 4 \end{cases}$$

  $\Rightarrow \alpha = 2 \cdot 5 + O(5^2)$

$$\deg\left(\Phi_0(t) + 2\cdot 5\cdot \Phi_1(t) \mod 5^2\right) = 3$$

$$\deg\left(\Phi_0(t) + (2\cdot 5 + i\cdot 5^2)\cdot \Phi_1(t) \mod 5^3\right) \approx \infty,\ i = 0,1,2,3,4$$

$$\deg\left((\Phi_0(t) + (2\cdot 5 + i\cdot 5^2)\cdot \Phi_1(t))(1-t)^5 \mod 5^3\right) = \begin{cases} 8, i = 4 \\ \infty, i = 0,1,2,3 \end{cases}$$

$$\alpha = 2\cdot 5 + 4\cdot 5^2 + O(5^3)$$

$$\dots$$

$$\deg\left((\Phi_0(t) + \alpha\cdot \Phi_1(t))(1-t)^{5(m-2)} \mod 5^m\right) = 5(m-2) + 3$$

$$\begin{aligned}\alpha &= 2\cdot 5 + 4\cdot 5^2 + 3\cdot 5^3 + 4\cdot 5^4 + 3\cdot 5^5 + 2\cdot 5^6 + 3\cdot 5^7 + 4\cdot 5^8 + 5^9 + \dots \\ &= -12\cdot \log_5(3)\end{aligned}$$

For all p we tried, this experiment gives a unique α

$$\Phi(0) = \begin{pmatrix} 1 & -3(p-1)\log_p(3) \\ 0 & p \end{pmatrix}$$

Part II: constructing Frobenius structures over small rings

Running example: $L = \theta^2 - t(\theta + \frac{1}{3})(\theta + \frac{2}{3})$, $\theta = t\frac{d}{dt}$.

Observe that

$$\frac{1}{(2\pi i)^2} \oint\oint \frac{1}{1 - t\left(x_1 + x_2 + \frac{1}{x_1 x_2}\right)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} = \sum_{n \geqslant 0} \frac{(3n)!}{(n!)^3} t^{3n}$$

$$= {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, 1; 27t^3\right)$$

is a period function of the family of toric hypersurfaces

$$t\left(x_1 + x_2 + \frac{1}{x_1 x_2}\right) = 1$$

Cohomology and differential forms (Griffiths, Batyrev)

$f(\mathbf{x}) \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, R char 0 domain, e.g. $R = \mathbb{Z}[t]$
$Z_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$ hypersurface

$\Delta \subset \mathbb{R}^n$ Newton polytope of $f(\mathbf{x})$

$$\Omega_f^n := \left\{ \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} \;\middle|\; \begin{array}{l} m \geqslant 1, h \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \\ supp(h) \subset m\Delta \end{array} \right\}$$

$$\Omega_f^{n-1} := \left\{ \sum_{i=1}^{n} \frac{h_i(\mathbf{x})}{f(\mathbf{x})^{m_i}} \frac{dx_1}{x_1} \ldots \frac{\check{dx_i}}{x_i} \ldots \frac{dx_n}{x_n} \;\middle|\; supp(h_i) \subset m_i\Delta \right\}$$

$W_f := \Omega_f^n / d\Omega_f^{n-1} \cong R^r$
(when f is Δ-*regular*, and after a possible localization of R)

$r = n! \cdot vol(\Delta) - 1$

$$0 \to H^n_{dR}(\mathbb{T}^n) \to H^n_{dR}(\mathbb{T}^n \backslash Z_f) \to PH^{n-1}_{dR}(Z_f) \to 0$$

$$\|$$

(primitive cohomology) $\quad W_f = \Omega^n_f / d\Omega^{n-1}_f$

$f(\mathbf{x})$ is called Δ-*regular* when for any face $\tau \subset \Delta$ of positive dimension equations

$$f|_\tau(\mathbf{x}) = \frac{\partial f}{\partial x_1}|_\tau(\mathbf{x}) = \ldots = \frac{\partial f}{\partial x_n}(\mathbf{x})|_\tau = 0$$

have no common solution in $(\overline{K}^\times)^n$, $K = Frac(R)$.

Example: $f(\mathbf{x}) = t(x_1 + x_2 + \frac{1}{x_1 x_2}) - 1$, $R = \mathbb{Z}[t, \frac{1}{3(1-27t^3)}]$
$W_f \cong R^2$

Connection

$W_f = \Omega_f^n / d\Omega_f^{n-1}$ is a differential R-module:
for any derivation θ of $R \rightsquigarrow \theta : W_f \to W_f$

E.g.

$$\frac{d}{dt}\left[\frac{1}{tg(\mathbf{x})-1}\frac{d\mathbf{x}}{\mathbf{x}}\right] = \left[\frac{1}{tg(\mathbf{x})-1}\frac{d\mathbf{x}}{\mathbf{x}}\right] = \left[\frac{g(\mathbf{x})}{(tg(\mathbf{x})-1)^2}\frac{d\mathbf{x}}{\mathbf{x}}\right]$$

Cartier operator

$\sigma : \widehat{R} \to \widehat{R}$ ring endomorphism s.t. $\sigma(r) - r^p \in p\widehat{R}$
E.g. $R = \mathbb{Z}[t, s(t)^{-1}]$, $\sigma(r(t)) = r(t^p)$ (need $p \nmid s(t)$)

The Cartier operator (see F. Beukers, M.V. "Dwork crystals I")

$$\mathcal{C}_p : \Omega_f^n \to \widehat{\Omega}_{f^\sigma}^n = \varprojlim \Omega_{f^\sigma}^n / p^s \Omega_{f^\sigma}^n$$

$$\frac{h(\mathbf{x})}{f(\mathbf{x})^m} \frac{d\mathbf{x}}{\mathbf{x}} = \sum a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \frac{d\mathbf{x}}{\mathbf{x}} \mapsto \sum a_{p\mathbf{u}} \mathbf{x}^{\mathbf{u}} \frac{d\mathbf{x}}{\mathbf{x}} = \varprojlim \frac{h_s(\mathbf{x})}{f^\sigma(\mathbf{x})^{k(s)}} \frac{d\mathbf{x}}{\mathbf{x}}$$

descends to a homomorphism of differential modules

$$\mathcal{C}_p : W_f \otimes_R \widehat{R} \to W_{f^\sigma} \otimes_{\sigma(R)} \widehat{R} \quad (\widehat{R}\text{-linear})$$

$$\theta \circ \mathcal{C}_p = \mathcal{C}_p \circ \theta \quad \text{for any derivation } \theta \text{ on } R$$

Frobenius structure

Let us write the commutation relation

$$\theta \circ \mathcal{C}_p = \mathcal{C}_p \circ \theta$$

using matrices:

$$\begin{aligned} &N \in R^{r\times r} \text{ matrix of } \theta \text{ on } W_f, \\ &N' \in \sigma(R)^{r\times r} \text{ matrix of } \theta \text{ on } W_{f^\sigma} \\ &C \in \widehat{R}^{r\times r} \text{ matrix of } \mathcal{C}_p : W_f \otimes_R \widehat{R} \to W_{f^\sigma} \otimes_{\sigma(R)} \widehat{R} \end{aligned}$$

$\Rightarrow$

$$\theta(C) + N'C = CN$$

If $N' = p\sigma(N)$, then $\Phi = C^\top$ is a Frobenius structure for $A = N^\top$.

Case $f(\mathbf{x}) = tg(\mathbf{x}) - 1$, $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Theorem. Assume:

- $\mathbf{0}$ is a unique internal integral point of Δ
- $f(\mathbf{x}) = tg(\mathbf{x}) - 1$ is Δ-regular for generic t
- there exists a localization $R = \mathbb{Z}[t, s(t)^{-1}]$ s.t.

$$W_f \cong \oplus_{i=0}^{r-1} R\, \theta^i \omega, \qquad \omega = \left[\frac{1}{f(\mathbf{x})}\frac{d\mathbf{x}}{\mathbf{x}}\right] \in W_f.$$

Consider $L = \theta^r + a_1(t)\theta^{r-1} + \ldots + a_r(t)$ such that $L\omega = 0$.

When $p \nmid s(t)$, there exists a Frobenius structure for the differential operator L with coefficients in $\widehat{R}$.

Remark: this generalizes to the case $W_f^G \cong \oplus_{i=0}^{r'-1} R\theta^i\omega$ (invariants of a group action), which gives many more examples.

Case $f(\mathbf{x}) = tg(\mathbf{x}) - 1$, $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Remark: $L\omega = 0 \Rightarrow$ differential operator L annihilates the period function

$$u_0(t) = \frac{1}{(2\pi i)^n} \oint \frac{1}{1 - tg(\mathbf{x})} \frac{d\mathbf{x}}{\mathbf{x}} = \sum_{n \geqslant 0} c_n t^n \in \mathbb{Z}[[t]]$$

$$c_n = \text{ constant term of } g(\mathbf{x})^n$$

Examples: assumptions are satisfied for

$$g(\mathbf{x}) = x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n}$$

$$s(t) = (n+1)\big(1 - (n+1)t^{n+1}\big) \quad rk(W_f) = n$$

$$\Rightarrow \quad L = \theta^n - ((n+1)t)^{n+1}(\theta+1)(\theta+2)\ldots(\theta+n)$$

admits a Frobenius structure over $\mathbb{Z}[\widehat{t, s(t)}^{-1}]$

Degeneration at $t = 0$

Solutions to L:

$$u_0(t) \in \mathbb{Z}[[t]]$$
$$u_1(t) = \log(t)u_0(t) + u_1^{an}(t), \quad u_1^{an}(t) \in t\mathbb{Q}[[t]]$$
$$u_2(t) = \frac{\log(t)^2}{2}u_0(t) + \log(t)u_1^{an}(t) + u_2^{an}(t), \quad u_2^{an}(t) \in t\mathbb{Q}[[t]]$$

$\ldots$ (case of maximally unipotent local monodromy)

$$U = (\theta^i u_j)_{i,j=0}^{r-1} \quad U^{an} = U|_{\text{" }\log(t)=0\text{"}} \in \mathbb{Q}[[t]]^{r\times r} \quad U^{an}(0) = Id_r$$

Corollary. There exist $\alpha_1, \ldots, \alpha_{r-1} \in \mathbb{Z}_p$ such that

$$\Phi(t) = U^{an}(t) \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \ldots & \alpha_r \\ 0 & p & p\alpha_1 & \ldots & p\alpha_{r-1} \\ 0 & 0 & p^2 & \ldots & p^2\alpha_{r-2} \\ \ldots & & & \ldots & \end{pmatrix} U^{an}(t^p)^{-1} \in \widehat{R}^{r\times r}.$$

Back to experiments

$$L = \theta^2 - (3t)^3(\theta+1)(\theta+2) \quad \Phi(0) = diag(1, p)$$

$$L = \theta^3 - (4t)^4(\theta+1)(\theta+2) \quad \Phi(0) = diag(1, p, p^2)$$

$$L = \theta^4 - (5t)^5(\theta+1)(\theta+2)(\theta+3)(\theta+4)$$

$$\Phi(0) = \begin{pmatrix} 1 & 0 & 0 & \alpha_3 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^3 \end{pmatrix} \quad \begin{aligned} \alpha_3 &= \frac{4p^3}{25}\left(\Gamma_p^{(3)}(0) - \Gamma'_p(0)^3\right) \\ &= -\frac{8p^3}{25}\zeta_p(3) \end{aligned}$$

(proved by Ilya Shapiro, "Frobenius map for quintic threefolds" & "Frobenius map and the p-adic gamma function"!)

Mirror symmetry

This talk is inspired by the note of Duco van Straten "CY-operators and L-functions".

Together with Xenia de la Ossa and Philip Candelas, they conjecture that for all *Calabi–Yau differential operators* order 4

$$\alpha_1 = \alpha_2 = 0, \quad \alpha_3 = \lambda \cdot p^3 \cdot \zeta_p(3)$$

with $\lambda \in \mathbb{Q}$ being independent of p.

They also give a conjectural expression for λ in terms of invariants of the mirror manifold!