Zeros of the Riemann zeta function on the critical line

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RH and critical zeros

- Riemann Hypothesis (RH) says all (nontrivial) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ are *critical*, that is, $\beta = \frac{1}{2}$

- in absence of proof of RH, natural to ask if one can show many, rather than all, zeros are critical

- define

$$N(T) = \#\left\{\beta + i\gamma : \zeta(\beta + i\gamma) = 0, 0 < \beta < 1, 0 < \gamma \leq T\right\},$$
$$N_0(T) = \#\left\{\frac{1}{2} + i\gamma : \zeta(\frac{1}{2} + i\gamma) = 0, 0 < \gamma \leq T\right\}.$$

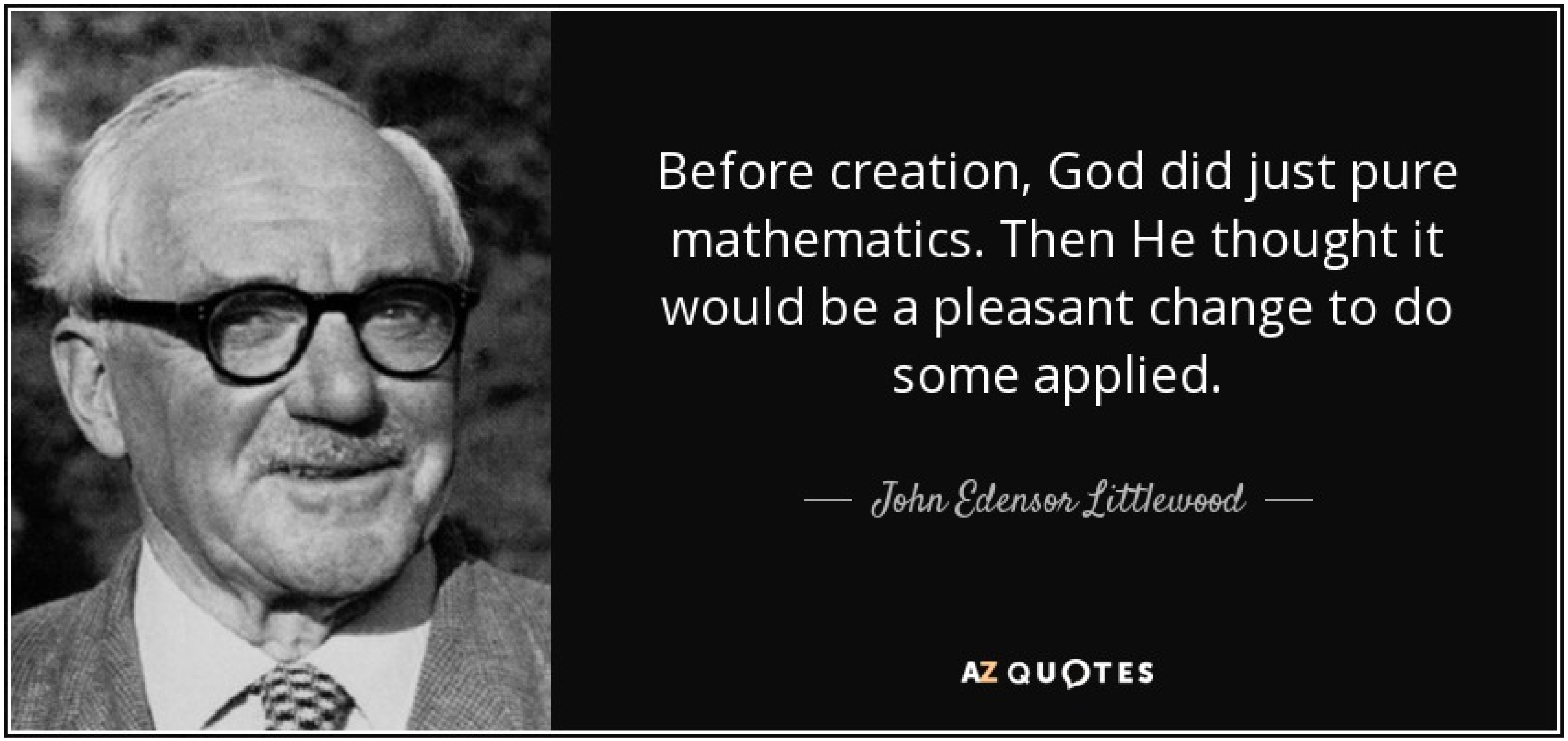
- RH is equivalent to $N(T) = N_0(T)$

History (I)

- Hardy (1914) showed $N_0(T) \to \infty$ as $T \to \infty$
- Hardy and Littlewood later gave the quantitative bounds $N_0(T) \gg T^{\frac{3}{4}-\epsilon}$ (1917) and $N_0(T) \gg T$ (1921)
- since $N(T) \asymp T \log T$, these results only give that "0%" of the zeros of $\zeta(s)$ are critical

G. H. Hardy

J. E. Littlewood



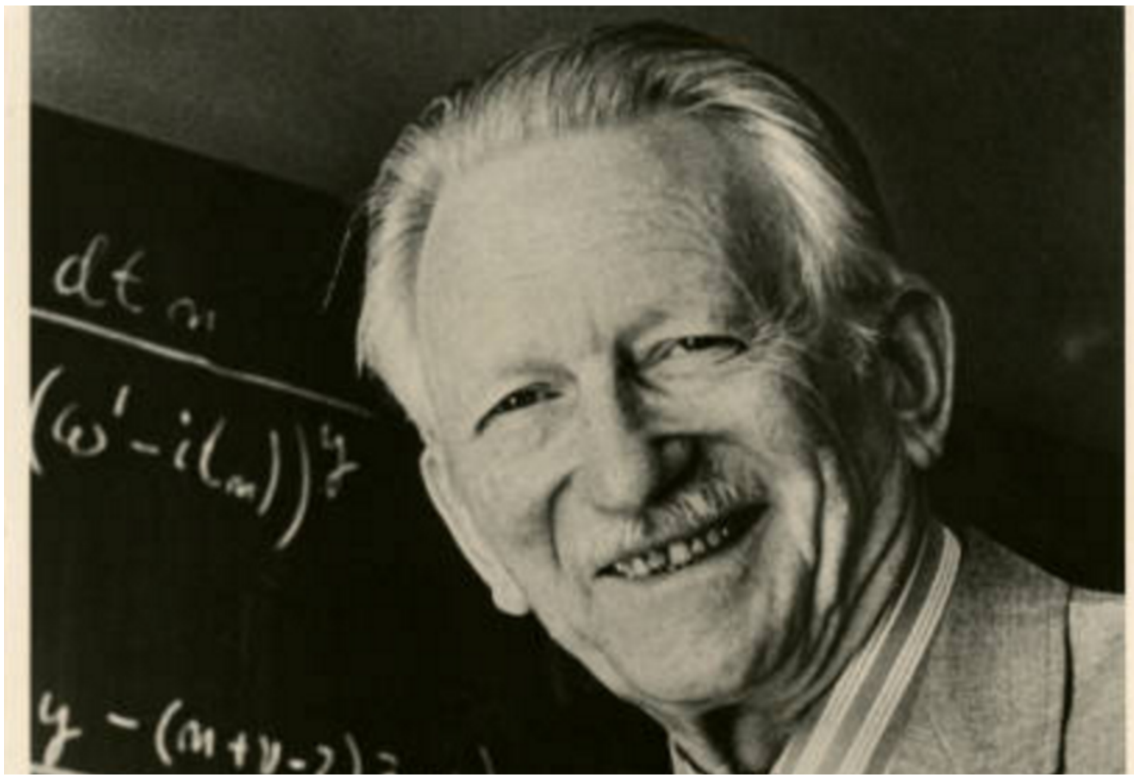
History (II)

- Selberg (1942) proved $N_0(T) \gg N(T)$, therefore *positive proportion* of zeros are critical

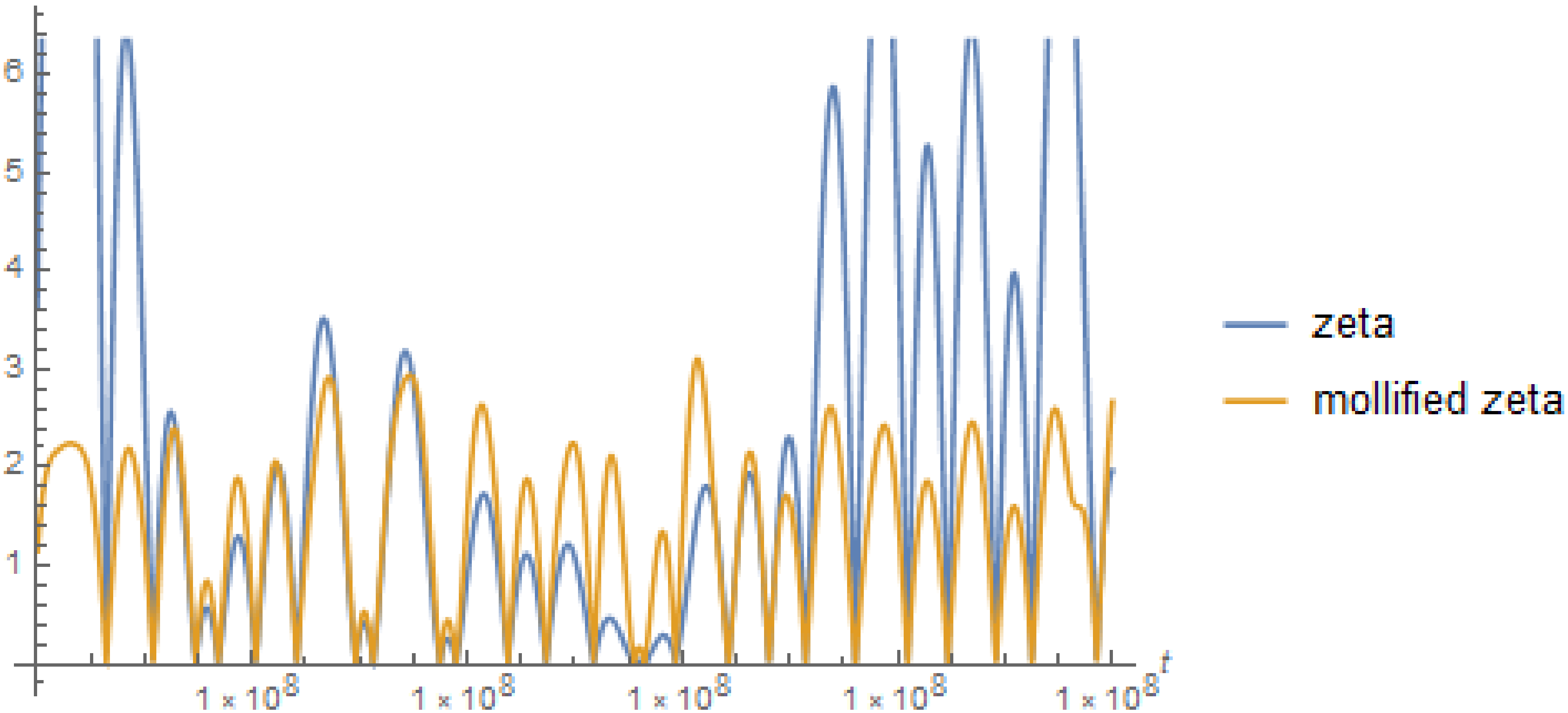
- key idea in proof is introduction of a *mollifier*, which serves to dampen large values of ζ

- the occasional large values of $\zeta(s)$ on the critical line are the source of Hardy/Littlewood's weaker result $N_0(T) \gg T$

Selberg



Mollification in action



—$|\zeta(\frac{1}{2}+it)|$ and mollified zeta for $10^8 \leq t \leq 10^8 + 10$; ζ has a large value of ≈ 18.5 on this interval

History (III)

- Levinson (1974) invented a new method for detecting critical zeros, found 33% of zeros are on critical line

- Conrey (1989) introduced several refinements, including Kloosterman sums, and obtained 40.77%

- Bui, Conrey, Young (2011) and Feng (2012) obtained 41.05% and 41.07% by introducing new mollifiers

Theorems

Theorem (Pratt, Robles (2017)). *More than 41.49% of the zeros of* $\zeta(s)$ *are critical.*

Theorem (Pratt, Robles, Z., Zeindler (2018)). *More than* 5/12 *of the zeros of* $\zeta(s)$ *are on the critical line.*

How do you make a mollifier? (I)

- want mollifier to approximate $\zeta(\frac{1}{2}+it)^{-1}$ (t is at height $\asymp T$)

- can approximate $\zeta(\frac{1}{2}+it)$ by Dirichlet polynomial

$$\zeta(\frac{1}{2}+it) \approx \sum_{n \le T} \frac{1}{n^{\frac{1}{2}+it}}$$

even inside critical strip, so good choice for mollifier is

$$M(\frac{1}{2}+it) \approx \sum_{m \le T^\theta} \frac{\mu(m)}{m^{\frac{1}{2}+it}}$$

How do you make a mollifier? (II)

- $M(\frac{1}{2}+it) \approx \sum_{m \le T^\theta} \frac{\mu(m)}{m^{\frac{1}{2}+it}}$

- $0 < \theta < 1$ is fixed number; refer to θ as *length* of the mollifier

- heuristically, larger values of θ provide better mollification

- Conrey's 40.77%: comes from increasing length to $\theta = \frac{4}{7}$, up from Levinson's $\theta = \frac{1}{2}$

Feng's mollifier

- in Levinson's method, want to mollify $\zeta(s) + \frac{\zeta'(s)}{\log T}$, not $\zeta(s)$

- Feng chose a mollifier of the form

$$M_F\left(\frac{1}{2} + it\right) \approx \sum_{0 \le k \le K} \frac{1}{(\log T)^k} \sum_{n \le T^\theta} \frac{\mu(n)(\mu * \Lambda^{*k})(n)}{n^{\frac{1}{2}+it}}$$

- presence of factor $\mu(n)$ simplifies main term analysis, but introduces problems in error term analysis

- we remove this $\mu(n)$ and study the resulting main terms and error terms

Error terms

- if m and q are coprime integers, define $\overline{m}$ by $m\overline{m} \equiv 1 \pmod{q}$

- error terms look like

$$\sum_{a \le A} \nu(a) \sum_{\substack{u \le U \\ v \le V \\ (u,v)=1}} \sum (\mu * \Lambda^{*k})(u) r(v) e\left(-a\frac{\overline{u}}{v}\right)$$

- to get θ as large as $\frac{4}{7}$ we must exploit structure of $\mu * \Lambda^{*k}$

Combinatorial decompositions

- use combinatorial identities to decompose $(\mu * \Lambda^{*k})(n)$ into *Type I* and *Type II* pieces

- Type I: $(\alpha * f)(n)$, where α is "rough", but only supported on small integers, and f a smooth function

- Type II: $(\alpha * \beta)(n)$, where α, β both rough, but supported on integers that are not too small and not too large

Type I sums

- arrange error term as

$$\sum_{a\leq A}\nu(a)\sum_{v\leq V}r(v)\sum_{w\leq W}\alpha(w)\sum_{n\leq U/w}f(n)e\left(-a\frac{\overline{nw}}{v}\right)$$

- since f is smooth, n sum is incomplete Kloosterman sum

- use Pólya-Vinogradov, or completion, technique, to bound the sum on n

- ultimately relies on Weil's proof of the Riemann Hypothesis for curves over finite fields

Type II sums

- arrange error term as

$$\sum_{g \asymp G} \sum_{v \leq V} |\alpha(g)||r(v)| \left| \sum_{a \leq A} \sum_{h \asymp H} \nu(a)\beta(h)e\left(-a\frac{\overline{gh}}{v}\right) \right|$$

- estimates of Deshouillers and Iwaniec on cancellation in sums of Kloosterman sums

- spectral theory of automorphic forms

Main terms

- we work throughout with mollifiers that approximate the inverse of

$$\zeta(s) + \frac{\zeta'(s)}{\log T} + \cdots + \frac{\zeta^{(d)}(s)}{(\log T)^d}$$

  for $d \geq 1$ arbitrary

- main term analysis is extremely difficult, since in general coefficients of mollifier are not multiplicative

- key identity is

$$\log x = -\frac{\partial}{\partial \gamma} \frac{1}{x^\gamma}\bigg|_{\gamma=0} = -\frac{1}{2\pi i} \oint \frac{1}{x^z} \frac{dz}{z^2}$$

Main term mess... (I)

$$\mathcal{S}_d = \frac{1}{(2\pi i)^{\ell_1}}\oint\cdots\oint\frac{1}{(2\pi i)^{\ell_2}}\oint\cdots\oint\cdots\frac{1}{(2\pi i)^{\ell_d}}\oint\cdots\oint(-1)^{1\times\ell_1}(-1)^{2\times\ell_2}\cdots(-1)^{d\times\ell_d}$$

$$\times\frac{1}{(2\pi i)^{\bar{\ell}_1}}\oint\cdots\oint\frac{1}{(2\pi i)^{\bar{\ell}_2}}\oint\cdots\oint\cdots\frac{1}{(2\pi i)^{\bar{\ell}_d}}\oint\cdots\oint(-1)^{1\times\bar{\ell}_1}(-1)^{2\times\bar{\ell}_2}\cdots(-1)^{d\times\bar{\ell}_d}$$

$$\times\sum_{\substack{d_0,d_{1,1},\cdots,d_{1,\ell_1},d_{2,1},\cdots,d_{2,\ell_1}\cdots d_{d,1}\cdots d_{d,\ell_1}\\ e_0,e_{1,1},\cdots,e_{1,\ell_2},e_{2,1},\cdots,e_{2,\ell_2}\cdots e_{d,1}\cdots e_{d,\ell_2}}}\cdots\sum\mu^{\star L_d+1}(d_0)\mu^{\star\bar{L}_d+1}(e_0)$$

$$\times\frac{(d_0(\prod_{1\le i\le\ell_1}d_{1,i})(\prod_{1\le i\le\ell_2}d_{2,i})\cdots(\prod_{1\le i\le\ell_d}d_{d,i}),e_0(\prod_{1\le j\le\bar{\ell}_1}e_{1,j})(\prod_{1\le j\le\bar{\ell}_2}e_{2,j})\cdots(\prod_{1\le j\le\bar{\ell}_d}e_{d,j}))^{\alpha+\beta}}{[d_0(\prod_{1\le i\le\ell_1}d_{1,i})(\prod_{1\le i\le\ell_2}d_{2,i})\cdots(\prod_{1\le i\le\ell_d}d_{d,i}),e_0(\prod_{1\le j\le\bar{\ell}_1}e_{1,j})(\prod_{1\le j\le\bar{\ell}_2}e_{2,j})\cdots(\prod_{1\le j\le\bar{\ell}_d}e_{d,j})]}$$

$$\times\frac{1}{d_0^{\alpha+s}(\prod_{1\le i\le\ell_1}d_{1,i}^{\alpha+s+z_{1,i}})(\prod_{1\le i\le\ell_2}d_{2,i}^{\alpha+s+z_{2,i}})\cdots(\prod_{1\le i\le\ell_d}d_{d,i}^{\alpha+s+z_{d,i}})\log^{\sum_{r=1}^{d}r\ell_r}N}$$

$$\times\frac{1}{e_0^{\beta+u}(\prod_{1\le j\le\bar{\ell}_1}e_{1,i}^{\beta+u+w_{1,j}})(\prod_{1\le j\le\bar{\ell}_2}e_{2,j}^{\beta+u+w_{2,j}})\cdots(\prod_{1\le j\le\bar{\ell}_d}e_{d,j}^{\beta+u+w_{d,j}})\log^{\sum_{\bar{r}=1}^{d}\bar{r}\bar{\ell}_{\bar{r}}}N}$$

$$\times\frac{dz_{1,1}}{z_{1,1}^{1+1}}\cdots\frac{dz_{1,\ell_1}}{z_{1,\ell_1}^{1+1}}\frac{dz_{2,1}}{z_{2,1}^{2+1}}\cdots\frac{dz_{2,\ell_2}}{z_{2,\ell_2}^{2+1}}\cdots\frac{dz_{d,1}}{z_{d,1}^{d+1}}\cdots\frac{dz_{d,\ell_d}}{z_{d,\ell_d}^{d+1}}\frac{dw_{1,1}}{w_{1,1}^{1+1}}\cdots\frac{dw_{1,\bar{\ell}_1}}{w_{1,\bar{\ell}_1}^{1+1}}\frac{dw_{2,1}}{w_{2,1}^{2+1}}\cdots\frac{dw_{2,\bar{\ell}_2}}{w_{2,\bar{\ell}_2}^{2+1}}\cdots\frac{dw_{d,1}}{w_{d,1}^{d+1}}\cdots\frac{dz_{w,\bar{\ell}_d}}{z_{w,\bar{\ell}_d}^{d+1}}.$$

Main term mess... (II)

- multiplicativity $\Rightarrow$ product of zeta functions and an arithmetic factor A

- but then you have to take the derivatives...

- the symmetries in A make many of the derivatives vanish, which is very helpful

Thank You!