Rational-Functions Telescopers: Blending Creative Telescoping with Hermite Reduction

Alin Bostan[1], Shaoshi Chen[1,2], Frédéric Chyzak[1], and Ziming Li[2]
[1]Algorithms project, INRIA Paris-Rocquencourt, France.
[2]Key Lab of Mathematics Mechanization, AMSS, Beijing, China.

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The long-term goal initiated in this work is to obtain fast algorithms and implementations for definite integration in the framework of (differential) creative telescoping introduced in [1]. Our approach bases on complexity analysis, by obtaining tight degree bounds on the various differential operators and polynomials involved in the method and its variants. To make the problem more tractable, we restrict in this work to the integration of *rational* functions. Indeed, by considering a more constrained class of inputs, we are able to blend the general method of creative telescoping with the well-known Hermite reduction [3]. The rational class already has many applications, for instance in combinatorics, where many non-trivial problems are encoded as diagonals of rational formal power series, themselves expressible as integrals.

Given a rational function $f \in K(x,y)$ (in characteristic zero), the core of (differential) creative telescoping consists in obtaining a linear differential operator L in $K(x)\langle D_x \rangle$ and a rational function $g \in K(x,y)$ satisfying $L(f) = D_y(g)$. The operator L is then called a *telescoper* for f and g a *certificate.* A telescoper for f is said to be *minimal* if it is of minimal order over all telescopers for f.

The classical way to compute minimal telescopers [1] is to apply a differential analogue of Gosper's indefinite summation algorithm, which reduces the problem to solving an auxiliary linear differential equation for rational-function solutions; a nice feature of the algorithm is a direct calculation of the denominator of the solutions and of a factor of their numerators, leading to better speed. An algorithm later developed by Geddes and Le [4] performs Hermite reduction on f to get an additive decomposition of the form

$$f = D_y(a) + \sum_{i=1}^{m} \frac{u_i}{v_i}, \quad \text{where } u_i, v_i \in K(x)[y] \text{ and } v_i \text{ is squarefree.}$$

Then the algorithm in [1] is applied to each u_i/v_i to get a minimal telescoper L_i. The least common left multiple of $L_1, \dots, L_m$ is then proved to be the minimal telescoper for f.

As a first contribution in this poster, we present a new, provably faster algorithm for computing minimal telescopers for rational functions. Instead of a single use of Hermite reduction, we obtain a normal form of each $D_x^i(f)$ by an application of Hermite reduction:

$$D_x^i(f) = D_y(g_i) + \frac{w_i}{w}, \tag{1}$$

where w divides the squarefree part of the denominator of f. If $e_0, \ldots, e_\rho \in K(x)$ are not all zero and such that $\sum_{j=0}^{\rho} e_j w_j = 0$, then the operator $\sum_{j=0}^{\rho} e_j D_x^j$ is a telescoper for f. The first nontrivial linear relation obtained in this way yields a minimal telescoper for f. For $i \geq 1$, (1) is obtained by applying Hermite reduction to the derivative of w_{i-1}/w with respect to x, which amounts to only one-step reduction.

As a second contribution, we derive complexity estimates for these methods (see table below), showing that our approach is faster, although it can produce an output of degree in x larger than with the classical [1]. This is a new instance of the philosophy, promoted in [2], of relaxing minimality to achieve better complexity. In the same vein, we analysed the bidegrees of outputs generated by other promising approaches, although at this point the correctness of the expected algorithms is not proven: Lipshitz' work on diagonals [5] can be rephrased into an existence theorem for telescopers, with quantifiably small size; the approach followed in the recent work on algebraic functions [2] leads to even smaller sizes.

A third contribution is a fast Maple implementation, which uses a carefully-coded original Hermite reduction algorithm, the special form of w_i/w in (1), and usual modular techniques (probabilistic rank estimate) to determine when to invoke the solver for linear algebraic equations. First experimental results indicate that our implementation can outperform Maple's library routine.

	Method	Bidegree in (x, D_x) of L	Complexity
Minimal Telescoper	Hermite reduction (new)	$(\mathcal{O}(d_x d_y^3), \mathcal{O}(d_y))$	$\tilde{\mathcal{O}}(d_x d_y^6)$
	Almkvist and Zeilberger	$(\mathcal{O}(d_x d_y^2), \mathcal{O}(d_y))$	$\tilde{\mathcal{O}}(d_x d_y^{2\omega+2})$
	Geddes and Le	$(\mathcal{O}(m d_x d_y^4), \mathcal{O}(d_y))$	$\tilde{\mathcal{O}}(d_x d^{2\omega+3})$
Non-minimal Telescoper	Lipshitz elimination	$(\mathcal{O}(d_x^2 + d_y^2), \mathcal{O}(d_x^2 + d_y^2))$	no algo (yet)
	Cubic-Size [2]	$(\mathcal{O}(d_x d_y), \mathcal{O}(d_y))$	no algo (yet)

(d_x, d_y) is the bidegree of the input f; the softO notation $\tilde{\mathcal{O}}(\,)$ indicates that polylogarithmic factors are neglected; ω is the exponent of matrix multiplication. Bounds on the bidegrees are also available for the certificates g.

References

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