$$\int x^\ell C_m^{(\mu)}(x) C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,\mathrm{d}x \quad [\text{shift } \ell, m, n, \mu, \nu], \tag{16}$$

$$\int (x+a)^{\gamma+\lambda-1}(a-x)^{\beta-1} C_m^{(\gamma)}(x/a) C_n^{(\lambda)}(x/a)\,\mathrm{d}x, \qquad [\text{diff. } a, \text{shift } n, m, \beta, \gamma, \lambda]. \tag{17}$$

6 CONCLUSION

A closer look at our algorithm reveals several aspects of the complexity of creative telescoping. To simplify the discussion, we restrict to the bivariate case and measure the *arithmetic complexity*, obtained by counting arithmetic operations in $\mathbb{Q}$. We look for bounds in terms of the *input size* (order and degree of the operators at hand).

In this setting, *the complexity of computing $\mathcal{T}_f$ is not bounded polynomially* (whatever the algorithm). Consider for instance, the integral representation of Hermite polynomials

$$H_n(t) = \frac{2^n}{i\sqrt{\pi}} \int_{-i\infty}^{i\infty} (t+x)^n e^{x^2}\,\mathrm{d}x.$$

If one computes a telescoper over $\mathbb{Q}(n,t)$, then our algorithm produces the classical differential equation $y'' + 2ny = 2ty'$. However, if n is a given positive integer then the minimal telescoper is the first-order factor $H_n(t)\partial_t - H_n'(t)$, with coefficients of degree n. Its size *is exponential in the bit size* of the input. Thus, *no algorithm computing the minimal telescoper can run in polynomial complexity.*

However, in the frequent cases like this one where the set S of singularities discussed in Corollary 4.8 is bounded polynomially in terms of the size of the input, then the dimension of the quotient and therefore *the order of the telescopers is bounded polynomially* as a consequence of Adolphson's result (Proposition 3.13). The non-polynomial cost of minimality thus resides only in the degree of the coefficients. Note that in the differential case, polynomial time computation of non-minimal telescopers is also achieved by well-known methods in holonomy theory, e.g., [34, proof of Lemma 3].

In our algorithm, the non-polynomial complexity arises first in the computation of the exceptional set Exc_M and next in the reductions by the elements of this set. Removing this part of the computation and using the weak Hermite reduction yields a weak form of the algorithm that does not find minimal telescopers but runs in polynomial complexity, if the set S has polynomial size.

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