Explicit Formula for the Generating Series of Diagonal 3D Rook Paths

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Joint article in preparation with Alin Bostan, Mark van Hoeij, and Lucien Pech

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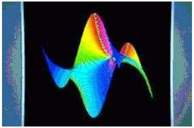

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Diagonal 3D Rook Paths

Problem

Determine the number a_n of paths from $(0,0,0)$ to (n,n,n) that use positive multiples of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

A144045 in Sloane's EIS (since 2008)

$1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, 4223303759148, \dots$

Conjecture by Erickson, Fernando, and Tran (201?)

$$\begin{aligned}2n^2(n-1)a_n - (n-1)(121n^2-91n-6)a_{n-1} \\ - (n-2)(475n^2-2512n+2829)a_{n-2} \\ + 18(n-3)(97n^2-519n+702)a_{n-3} \\ - 1152(n-3)(n-4)^2 a_{n-4} = 0, \qquad \text{for } n \geq 4.\end{aligned}$$

Our Contributions

1. Proof of the 4th-order recurrence. There is even 3rd-order recurrence:

$$\begin{aligned}192n^2(35n+88)(n+1)a_n \\ -(n+1)(11305n^3+59889n^2+100586n+54864)a_{n+1} \\ +(n+2)(4655n^3+30114n^2+63493n+43362)a_{n+2} \\ -2(n+2)(35n+53)(n+3)^2a_{n+3}=0, \qquad \text{for } n\geq 0.\end{aligned}$$

2. Explicit form for the enumerative generating series:

$$G(x)=1+6\cdot\int_0^x \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\,dw.$$

3. New, simpler proof of the asymptotic formula

$$a_n \sim \frac{9\sqrt{3}}{40\pi}\cdot\frac{64^n}{n}.$$

Remark: The 2D Case is Easy

A051708: 1, 2, 14, 106, 838, 6802, 56190, 470010, ...
(In the EIS since 1999.)

Generating series is algebraic:

$$\frac{1-x}{2\sqrt{1-10x+9x^2}}+\frac{1}{2}.$$

From Combinatorics to Algebra

Generating series for the number $r_{i,j,k}$ of paths that end at (i, j, k)

$$f(s,t,u) := \left(1 - \sum_{n\geq 1} s^n - \sum_{n\geq 1} t^n - \sum_{n\geq 1} u^n\right)^{-1}$$
$$= \frac{(1-s)(1-t)(1-u)}{1-2(s+t+u)+3(st+tu+us)-4stu} \in \mathbb{Q}[[s,t,u]].$$

Diagonal representation

$$G(x) = \operatorname{Diag}(f) = \sum_{n\geq 0} r_{n,n,n} x^n \in \mathbb{Q}[[x]].$$

Key Equation

Lemma (Lipshitz, 1988)

After setting

$$F := \frac{1}{st} f\left(s, \frac{t}{s}, \frac{x}{t}\right),$$

if $P(x, \partial_x)$ *satisfies*

$$P(F) = \frac{\partial S}{\partial s} + \frac{\partial T}{\partial t}$$

for some $S, T \in \mathbb{Q}(x, s, t)$, *then* $P\big(\mathrm{Diag}(f)\big) = 0$.

Proof: In a $\mathbb{Q}[x, s, t]\langle \partial_x, \partial_s, \partial_t \rangle$-module of suitable formal power series,

$$P(\mathrm{Diag}(f)) = P([s^{-1}t^{-1}]F) = [s^{-1}t^{-1}]P(F) = 0.$$

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Remaining problem

Compute such P, S, and T.

Lipshitz's Approach: Counting Dimensions

Method

1. Find annihilator $L(x, \partial_x, \partial_s, \partial_t)$ of F.
2. Write $L = P(x, \partial_x) + \partial_s A(x, \partial_x, \partial_s, \partial_t) + \partial_t B(x, \partial_x, \partial_s, \partial_t)$ (modifying L to ensure $P \neq 0$ if needed).
3. Return P, $S := A(F)$, and $T := B(F)$.

Note: $P(F), S, T \in \mathbb{Q}[x, s, t, q^{-1}]$ if $F = p/q$.

Existence of L

$x^i \partial_x^j \partial_s^k \partial_t^\ell (F)$	$\rightarrow$	$q^{-(N+1)} x^i s^j t^k$
$0 \leq i + j + k + \ell \leq N$		$0 \leq i \leq 2N + 1,\ 0 \leq j, k \leq 3N + 2$
$\binom{N+4}{4} \sim N^4/24$		$18(N+1)^3 \sim 18N^3$

$N \geq 425$ ensures a relation, but linear algebra in dimension $> 10^9$!
Refined bound (better filtration + predicting zeros): still $> 1.6 \cdot 10^6$!!

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Existence of L

$$\begin{array}{ccc} x^i \partial_x^j \partial_s^k \partial_t^\ell (F) & \to & q^{-(N+1)} x^i s^j t^k \\ 0 \le i + j + k + \ell \le N & & 0 \le i \le 2N+1,\ 0 \le j, k \le 3N+2 \\ \binom{N+4}{4} \sim N^4/24 & & 18(N+1)^3 \sim 18N^3 \end{array}$$

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Algorithm is useless in practice.

Doron's Influence to Computer Algebra: Creative Telescoping for Summation and Integration

Larger and larger classes of inputs

1. hypergeometric, q-hypergeometric, bibasic, . . . (DZ, Koornwinder, Paule, Riese, Schorn, 1990–)
2. hyperexponential (Almkvist–DZ, 1990)
3. D-finite (FC, 2000; Koutschan, 2010)
4. Abel-type (Majewicz, 1996); Stirling-type (Kauers, 2007); Bernoulli and Euler type (Chen–Sun, 2008)
5. beyond D-finite (FC–Kauers–Salvy, 2009)
6. in difference fields (Schneider, 200?–)
7. symmetric functions (FC–Mishna–Salvy, 2002, 2005)

Complexity point of view

Polynomial time algorithm for bivariate rational functions (Bostan, Chen, FC, Li, 2010)

Rooks: A Case for Multivariate Creative Telescoping

Pech's Maple implementation of Chyzak's generalized algorithm to D-finite functions (2000) finds:

$$P = P_2\partial_x, \quad S = \frac{(s-t)\cdot U}{2st\cdot q_1^2\cdot \mathrm{disc}_t(q_1)}, \quad T = \frac{t\cdot V}{2xs^2(3s-2)^2\cdot q_1^3\cdot \mathrm{disc}_t(q_1)^2}$$

where

$$\begin{aligned} P_2 = x(x-1)(64x-1)(3x-2)(6x+1)\partial_x^2 \\ + (4608x^4 - 6372x^3 + 813x^2 + 514x - 4)\partial_x \\ + 4(576x^3 - 801x^2 - 108x + 74), \end{aligned}$$

$$q_1 = q/(st), \quad \deg_{x,s,t} U = (5,8,3), \quad \deg_{x,s,t} V = (9,17,5).$$

Creative Telescoping (1): Almkvist & DZ's Rational/Hyperexponential Case

One variable less: simplified key equation

For $F \in \mathbb{Q}(u,v)$, find $P(u,\partial_u)$ and $S \in \mathbb{Q}(u,v)$ such that $P(F) = \frac{\partial S}{\partial v}$.

Algorithm (simplified)

For $r = 0, 1, 2, \dots$:

1. set $P = \eta_r(u)\partial_u^r + \dots + \eta_0(u)$ for undertermined $\eta_i \in \mathbb{Q}(u)$;
2. set $S = \phi(u,v)F$ for undertermined $\phi \in \mathbb{Q}(u,v)$;
3. derive first-order ODE on ϕ;
4. solve by a variant of Abramov's decision algorithm;
5. if solvable, output (P, S), else loop.

Original A & Z's algorithm

F and S are hyperexponential; solving uses this specific feature.

Creative Telescoping (2): Chyzak's D-Finite Case

Definition

$F(u, v)$ is *D-finite* if there exists a *finite* maximal set of derivatives $\partial_u^a \partial_v^b(F)$ that are linearly independent over $\mathbb{Q}(u, v)$.

Algorithm (Chyzak, 2000)

Adapt A & Z's algorithm (simplified) by:

1. Fix a set of (a, b) from the definition and change the ansatz into
$$S = \sum_{(a,b)} \phi_{a,b}(u, v) \partial_u^a \partial_v^b(F).$$
2. Equation on ϕ is replaced with a system of coupled ODE's on the $\phi_{a,b}$'s.
3. Solved by uncoupling or a direct approach.

Remark: The (a, b)'s can be obtained by a Gröbner-basis calculation.

A Failure and a Cure

Key equation: Solve for $P(x, \partial_x)$ and *rational functions* S and T

$$P(F) = \frac{\partial S}{\partial s} + \frac{\partial T}{\partial t}.$$

Obstruction

Even for rational F, no algorithm is known to solve the ansatz

$$P\big(F(s,t,x)\big) = \frac{\partial}{\partial s}\big(\phi_1(s,t,x)F\big) + \frac{\partial}{\partial t}\big(\phi_2(s,t,x)F\big).$$

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A Salvaging Observation

In A & Z, dependency of P in a single derivation ∂_u is inessential.
Extended ansatz (Chyzak, 2000):

$$P = \sum_{0 \le i+j \le r} \eta_{i,j}(u_1,u_2)\partial_{u_1}^i \partial_{u_2}^j \quad \text{for undetermined } \eta_{i,j} \in \mathbb{Q}(u_1,u_2).$$

Creative Telescoping (3): Iterated Chyzak Algorithm

with the “UFO” finish for non-natural boundaries

1. Extended A & Z algorithm for the rational function $F(s, x, t) \to$

$$P^{(\alpha)}(s, x, \partial_s, \partial_x)(F) = \frac{\partial}{\partial t}\left(\phi^{(\alpha)}(s, t, x)F\right).$$

2. Chyzak's algorithm for $\hat{F}(s, x)$ annihilated by all $P^{(\alpha)} \to$

$$P(x, \partial_x)(\hat{F}) = \frac{\partial}{\partial s}\big(Q(s, x, \partial_s, \partial_x)(\hat{F})\big).$$

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$$P(x, \partial_x)(\hat{F}) = \frac{\partial}{\partial s}\big(Q(s, x, \partial_s, \partial_x)(\hat{F})\big).$$

3. Obtain operators $L^{(\alpha)}(s, x, \partial_s, \partial_x)$ satisfying

$$P(x, \partial_x) - \partial_s Q(s, x, \partial_s, \partial_x) = \sum_\alpha L^{(\alpha)}(s, x, \partial_s, \partial_x) P^{(\alpha)}(s, x, \partial_s, \partial_x)$$

by non-commutative multivariate division.

4. A solution (P, S, T) of the key equation is given upon setting

$$S := Q(s, x, \partial_s, \partial_x)(F) \quad \text{and} \quad T := \sum_\alpha L^{(\alpha)}(s, x, \partial_s, \partial_x)\big(\phi^{(\alpha)}(s, t, x)F\big).$$

Proving the Conjectured Recurrence, and a Shorter One

Fourth-order conjectured recurrence

Proved classically from the differential operator P.

Third-order recurrence

1. Observe that $-1/6$ is an apparent singularity of P.
2. The differential operator of order 4 provides a second fourth-order recurrence.
3. Elimination of a_{n+4} yields the announced third-order recurrence.

Variant: Guess it, then show it has the same solution as the fourth-order one.

Solution in Explicit Form: Van Hoeij's Heuristic

General speculation of Dwork's (about nilpotent p-curvature) applies to P_2: $\partial_x\big(\mathrm{Diag}(f)\big)$ is likely to be of the form

$$(r_0 y + r_1 y') \cdot \exp\left(\int r\right) \quad \text{for} \quad y = {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \,\middle|\, f\right)$$

where $a, b, c \in \mathbb{Q}$ and $r, r_0, r_1, f \in \mathbb{Q}(x)$.

Principles for determining y

1. f must transfer the location and behaviour at the singularities of the Gauss ODE to those of P_2.
2. "Exponent difference" (attached to each singular point) is a natural guide to f.
3. For the Gauss hypergeometric function: $(e_0, e_1, e_\infty) = \big(\pm(1-c), \pm(c-a-b), \pm(a-b)\big)$.

Solution in Explicit Form: Calculations

1. First candidate: $(e_0, e_1, e_\infty) = (0, 1/3, 0)$ and
$$f = 1 - \frac{(4x-1)^3}{(x-1)^2(64x-1)} = -\frac{81x(x-2/3)}{64(x-1)^2(x-1/64)}.$$

2. Optimized candidate: $(e_0, e_1, e_\infty) = (1, 1, 1/3)$ and
$$\bar{f} = \frac{f}{1-f} = \frac{27x(2-3x)}{(1-4x)^3}.$$

3. $(a, b, c) = (1/3, 2/3, 2)$.

4. Identification: $r = r_1 = 0$ and $r_0 = \dfrac{6}{(1-4x)(1-64x)}$.

5. Taking initial conditions into account:
$$G(x) = 1 + 6 \cdot \int_0^x \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix} \,\middle|\, \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\, dw.$$

Asymptotic Expression for a_n

Recall: $\partial_x\big(\mathrm{Diag}(f)\big) = \dfrac{6 \cdot {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}.$

Dominant singularity is $1/64$, with residue $r =$

$$\frac{6}{(1-\frac{4}{64})} \cdot {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, \frac{\frac{27}{64}(2-\frac{3}{64})}{(1-\frac{4}{64})^3}\right) = \frac{32}{5} \cdot {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, 1\right) = \frac{72\sqrt{3}}{5\pi}.$$

Thus: $a_n \sim \dfrac{1}{64n}(r\,64^n) = \dfrac{9\sqrt{3}}{40\pi} \cdot \dfrac{64^n}{n} \simeq 0.124 \cdot 64^n.$

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Thus: $$a_n \sim \frac{1}{64n}(r\,64^n) = \frac{9\sqrt{3}}{40\pi} \cdot \frac{64^n}{n} \simeq 0.124 \cdot 64^n.$$

This avoids general multivariate asymptotics *à la* Raichev and Wilson.

Concluding remarks

1. Series is D-finite but not algebraic.
2. Beukers communicated a derivation by differential forms, leading to a similar but different representation.
3. 3D queens paths is a challenge: order 6, degree 71, only guessed.
4. Apply same technique to all nearest-neighbour types of walks (in progress, same authors + Kauers).

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HAPPY BIRTHDAY DORON!

```
> F := (s-t)*(-1+s)*(t-x)/s/t/(-s*t+2*s^2*t+2*t^2+2*x*s-3*s*t^2-3*x*t-3*x*s^2+4*x*s*t);
```

$$F := \frac{(s-t)\,(-1+s)\,(t-x)}{s\,t\,\left(-s\,t+2\,s^2\,t+2\,t^2+2\,x\,s-3\,s\,t^2-3\,x\,t-3\,x\,s^2+4\,x\,s\,t\right)} \quad \textbf{(1)}$$

```
> P_, S_, T_ := op(op(Mgfun:-creative_telescoping(F, x::diff, [s::diff, t::diff]))):
> P := collect(DEtools[de2diffop](P_, _F(x), [dx,x]), dx, factor);
```

$$P := x\,(x-1)\,(64\,x-1)\,(3\,x-2)\,(6\,x+1)\,dx^3+\left(4608\,x^4-6372\,x^3+813\,x^2+514\,x-4\right)dx^2+\left(2304\,x^3-3204\,x^2-432\,x+296\right)dx \quad \textbf{(2)}$$

```
> S := eval(S_, _f = unapply(F, x, s, t)):
> length(S);
```

$$1163 \quad \textbf{(3)}$$

```
> S;
```

$$\frac{(324\,x\,s-216\,x+159\,s-106)\,(s-t)\,(t-x)}{t\,\left(-s\,t+2\,s^2\,t+2\,t^2+2\,x\,s-3\,s\,t^2-3\,x\,t-3\,x\,s^2+4\,x\,s\,t\right)}-\frac{1}{2}\Bigg(\Big(-56\,s^2+42420\,x^4\,s^3-83632\,x^4\,s^4+72\,x\,s-1008\,x\,s^2-1620\,x^2\,s+19730\,x^2\,s^2+616\,s^3+4228\,x\,s^3-82024\,x^2\,s^3-19512\,x^3\,s+93084\,x^3\,s^2+5724\,x^4\,s+972\,x^3-6924\,x\,s^4-2555\,s^4+2624\,s^5\,x+5112\,s^5+165704\,x^2\,s^4-178716\,x^2\,s^5+99852\,x^2\,s^6-5008\,s^6-196372\,x^3\,s^3+202768\,x^3\,s^4-81872\,x^3\,s^5-17064\,x^3\,s^6+18432\,s^7\,x^3-23040\,x^2\,s^7+1944\,s^7+84096\,x^4\,s^5-15516\,x^4\,s^2+3952\,x\,s^6-3048\,x\,s^7-82944\,x^5\,s^3+93312\,x^5\,s^2-46656\,x^5\,s+27648\,x^5\,s^4-32256\,x^4\,s^6-1215\,x^4+8748\,x^5\Big)\left(-\frac{(s-t)\,(-1+s)}{s\,t\,\left(-s\,t+2\,s^2\,t+2\,t^2+2\,x\,s-3\,s\,t^2-3\,x\,t-3\,x\,s^2+4\,x\,s\,t\right)}-\frac{(s-t)\,(-1+s)\,(t-x)\,\left(2\,s-3\,t-3\,s^2+4\,s\,t\right)}{s\,t\,\left(-s\,t+2\,s^2\,t+2\,t^2+2\,x\,s-3\,s\,t^2-3\,x\,t-3\,x\,s^2+4\,x\,s\,t\right)^2}\right)\Bigg)\Bigg/\left((-1+s)^2\,(s-x)\,\left(-16\,x\,s^2+24\,x\,s-9\,x-4\,s^2+4\,s^3+s\right)\right) \quad \textbf{(4)}$$

```
> T := eval(T_, _f = unapply(F, x, s, t)):
> length(T);
```

$$50860 \quad \textbf{(5)}$$

```
> normal(eval(P_, _F(x) = F) - diff(S, s) - diff(T, t));
```

$$0 \quad \textbf{(6)}$$

```
> S := factor(S):
> T := factor(T):
> q := factor(denom(F));
```
$$q := s\,t\left(-s\,t+2\,s^2\,t+2\,t^2+2\,x\,s-3\,s\,t^2-3\,x\,t-3\,x\,s^2+4\,x\,s\,t\right) \quad (7)$$
```
> q1 := q/s/t:
> disc := factor(discrim(q1, t));
```
$$disc := \left(-16\,x\,s^2+24\,x\,s-9\,x-4\,s^2+4\,s^3+s\right)(s-x) \quad (8)$$
```
> U := 2 * s*t * q1^2 * disc * S / (s-t):
> V := 2 * x * s^2 * q1^3 * disc^2 * (3*s-2)^2 * T / t:
> map2(degree, q1, [x,s,t]);
```
$$[1,2,2] \quad (9)$$
```
> map2(degree, U, [x,s,t]);
```
$$[5,8,3] \quad (10)$$
```
> map2(degree, V, [x,s,t]);
```
$$[9,17,5] \quad (11)$$
```
> map2(degree, P, [x,dx]);
```
$$[5,3] \quad (12)$$

```
> collect(P_, {_F, diff}, factor);
```
$$\left(2304\,x^3-3204\,x^2-432\,x+296\right)\left(\frac{d}{dx}\_F(x)\right)+\left(4608\,x^4-6372\,x^3+813\,x^2+514\,x-4\right)\left(\frac{d^2}{dx^2}\_F(x)\right)+x\,(x-1)\,(64\,x-1)\,(3\,x-2)\,(6\,x+1)\left(\frac{d^3}{dx^3}\_F(x)\right) \quad (13)$$
```
> rec4 := gfun[diffeqtorec](%, _F(x), u(n));
```
$$rec4 := \left(1152\,n^2+1152\,n^3\right)u(n)+\left(-7830\,n-3204-6372\,n^2-1746\,n^3\right)u(n+1)+\left(2957\,n+762+2238\,n^2+475\,n^3\right)u(n+2)+\left(4197\,n+4698+1240\,n^2+121\,n^3\right)u(n+3)+\left(-22\,n^2-80\,n-96-2\,n^3\right)u(n+4) \quad (14)$$
```
> collect(eval(rec4, n=n-4), u, factor);
```
$$-2\,n^2\,(n-1)\,u(n)+1152\,(n-3)\,(n-4)^2\,u(n-4)-18\,(n-3)\left(97\,n^2-519\,n+702\right)u(n-3)+(n-2)\left(475\,n^2-2512\,n+2829\right)u(n-2)+(n-1)\left(121\,n^2-91\,n-6\right)u(n-1) \quad (15)$$

```
> Alg := Ore_algebra[diff_algebra]([dx, x]):
> dxP := Ore_algebra[skew_product](dx, P, Alg);
```
$$dxP := \left(-432+6912\,x^2-6408\,x\right)dx+\left(10368\,x^4-13356\,x^3-6+756\,x+2238\,x^2\right)dx^3+\left(810-22320\,x^2+20736\,x^3+1194\,x\right)dx^2+\left(1152\,x^5-1746\,x^4+475\,x^3+121\,x^2-2\,x\right)dx^4 \quad (16)$$
```
> eval(dxP, x=-1/6);
```
(17)

$$828\,dx - 105\,dx^2 \quad \textbf{(17)}$$

```
> eval(P, x=-1/6);
```

$$-\frac{1225}{36}dx^2 + \frac{805}{3}dx \quad \textbf{(18)}$$

```
> normal(%%/%);
```

$$\frac{108}{35} \quad \textbf{(19)}$$

```
> collect((35*dxP - 108*P) / (6*x+1), dx, factor);
```

$$35\,(x-1)\,(64\,x-1)\,(3\,x-2)\,x\,dx^4 - 6\,(x-1)\,(3456\,x^3 - 12438\,x^2 + 4621\,x - 35)\,dx^3 + (-82944\,x^3 + 249480\,x^2 - 186414\,x + 28782)\,dx^2 + (-41472\,x^2 + 104904\,x - 47088)\,dx \quad \textbf{(20)}$$

```
> Ore_algebra[applyopr](%, _F(x), Alg);
```

$$(-41472\,x^2 + 104904\,x - 47088)\left(\frac{d}{dx}\_F(x)\right) - 6\,(x-1)\,(3456\,x^3 - 12438\,x^2 + 4621\,x - 35)\left(\frac{d^3}{dx^3}\_F(x)\right) + (-82944\,x^3 + 249480\,x^2 - 186414\,x + 28782)\left(\frac{d^2}{dx^2}\_F(x)\right) + 35\,(x-1)\,(64\,x-1)\,(3\,x-2)\,x\left(\frac{d^4}{dx^4}\_F(x)\right) \quad \textbf{(21)}$$

```
> gfun[diffeqtorec](%, _F(x), u(n));
```

$$(-20736\,n^2 - 20736\,n^3)\,u(n) + (272460\,n + 104904 + 242760\,n^2 + 81924\,n^3 + 6720\,n^4)\,u(n+1) + (-788428\,n - 467004 - 482171\,n^2 - 124964\,n^3 - 11305\,n^4)\,u(n+2) + (247603\,n^2 + 479136\,n + 340308 + 55866\,n^3 + 4655\,n^4)\,u(n+3) + (-910\,n^3 - 4340\,n^2 - 8960\,n - 6720 - 70\,n^4)\,u(n+4) \quad \textbf{(22)}$$

```
> rec4bis := collect(%, u, factor);
```

$$rec4bis := -20736\,n^2\,(n+1)\,u(n) + 12\,(n+1)\,(560\,n^3 + 6267\,n^2 + 13963\,n + 8742)\,u(n+1) - (n+2)\,(11305\,n^3 + 102354\,n^2 + 277463\,n + 233502)\,u(n+2) + (n+3)\,(n+2)\,(4655\,n^2 + 32591\,n + 56718)\,u(n+3) - 70\,(n+3)\,(n+2)\,(n+4)^2\,u(n+4) \quad \textbf{(23)}$$

```
> solve(rec4, u(n+4));
```

$$-\frac{1}{2}\,\frac{1}{11\,n^2 + 40\,n + 48 + n^3}\,(-1152\,u(n)\,n^2 - 1152\,u(n)\,n^3 + 7830\,u(n+1)\,n + 3204\,u(n+1) + 6372\,u(n+1)\,n^2 + 1746\,u(n+1)\,n^3 - 2957\,u(n+2)\,n - 762\,u(n+2) - 2238\,u(n+2)\,n^2 - 475\,u(n+2)\,n^3 - 4197\,u(n+3)\,n - 4698\,u(n+3) - 1240\,u(n+3)\,n^2 - 121\,u(n+3)\,n^3) \quad \textbf{(24)}$$

```
> solve(rec4bis, u(n+4));
```

$$\frac{1}{70}\,\frac{1}{(n+3)\,(n+2)\,(n+4)^2}\,(-20736\,u(n)\,n^3 - 20736\,u(n)\,n^2 + 272460\,u(n+1)\,n \quad \textbf{(25)}$$

$$+ 104904\,u(n+1) + 242760\,u(n+1)\,n^2 + 81924\,u(n+1)\,n^3 + 6720\,u(n+1)\,n^4$$
$$- 788428\,u(n+2)\,n - 467004\,u(n+2) - 482171\,u(n+2)\,n^2 - 124964\,u(n+2)\,n^3$$
$$- 11305\,u(n+2)\,n^4 + 247603\,u(n+3)\,n^2 + 479136\,u(n+3)\,n + 340308\,u(n+3)$$
$$+ 55866\,u(n+3)\,n^3 + 4655\,u(n+3)\,n^4\Big)$$

```
> collect(numer(normal(%%-%)/3), u, factor);
```

$$192\,n^2\,(35\,n+88)\,(n+1)\,u(n) - (n+1)\,\big(11305\,n^3 + 59889\,n^2 + 100586\,n + 54864\big)\,u(n+1) + (n+2)\,\big(4655\,n^3 + 30114\,n^2 + 63493\,n + 43362\big)\,u(n+2) - 2\,(n+2)\,(35\,n+53)\,(n+3)^2\,u(n+3) \quad \textbf{(26)}$$