Becker's Conjecture on Mahler Functions

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Joint work with Jason P. Bell, Michael Coons, and Philippe Dumas

Classes of Interest: the Generative Viewpoint

$$n = \overline{j_e \dots j_0}^k \quad \text{(expansion in base } k\text{)}$$

$$S(z) = \sum_{n \in \mathbb{N}} s_n z^n \in \mathbb{C}[[z]]$$

k-Automatic series
[Cobham (1972), Christol (1979), CKMR (1980), Allouche (1987)]

Fix a finite automaton and a map ϕ from states to $\mathbb{C}$, then set:

$$s_n = \phi\big(\text{final state after a run on the word } j_0 j_1 \dots j_e\big).$$

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Fix matrices $A_0, \dots, A_{k-1}$ and vectors L and C, then set: $s_n = L A_{j_e} \dots A_{j_0} C$.

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$$\{k\text{-automatic series}\} \subset \{k\text{-regular series}\}$$

Polynomial bound. Form an algebra. Fast to compute.

Classes of Interest: the Section Viewpoint

section operator $\Lambda_j : (s_n)_{n\in\mathbb{N}} \mapsto (s_{kn+j})_{n\in\mathbb{N}}$ for $j = 0, \ldots, k-1$

k-orbit of $(s_n)_{n\in\mathbb{N}}$ = set containing $(s_n)_{n\in\mathbb{N}}$ and closed under $\Lambda_0, \ldots, \Lambda_{k-1}$

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Cobham (1972)

$(s_n)_{n\in\mathbb{N}}$ is k-automatic iff its k-orbit is finite.

Allouche and Shallit (1992)

$(s_n)_{n\in\mathbb{N}}$ is k-regular iff the $\mathbb{C}$-span of its k-orbit is finite-dimensional.

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(Generalized) Cartier operators

For $j = 0, \dots, k-1$ and Laurent power series $S(z) \in \mathbb{C}((z))$,

$$\Lambda_j : S(z) = \sum_n s_n z^n \mapsto \sum_n s_{kn+j} z^n$$

An Equational Viewpoint: k-Mahler Series

k-Mahler equations

For polynomials $a_i(z) \in \mathbb{C}[z]$ with $a_0(z)a_d(z) \neq 0$, consider:

$$a_0(z)F(z) + \cdots + a_d(z)F(z^{k^d}) = 0.$$

A series solution in $\mathbb{C}((z))$ is called k-Mahler.

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Note: $S(z) = \sum_{j=0}^{k-1} z^j \Lambda_j(S)(z^k) = \sum_{j=0}^{k-1} \sum_{j'=0}^{k-1} z^{j+kj'} \Lambda_{j'} \Lambda_j(S)(z^{k^2}) = \cdots$

$\{k$-automatic series$\} \subset \{k$-regular series$\} \subset \{k$-Mahler series$\}$

Given V closed under the Λ_j and $m \geq 0$: $V \subset \sum_{H \in V} \mathbb{C}(z) H(z^{k^m})$.

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Special cases:

- V = a k-orbit of finite cardinality m,
- V = the $\mathbb{C}$-span of $\mathbb{C}$-dimension m of a k-orbit.

Computational Viewpoint: k-Regular Series Are Simpler

Divide-and-conquer recurrences

k-Mahler equation $\Longrightarrow$ a recurrence of the form

$$s_n = \sum_{j \in J,\ j>0} b_{0,j} s_{n-j} + \sum_{i=1}^{d} \sum_{j \in J} b_{i,j} s_{(n-j)/k^i} \qquad \text{(finite } J\text{)}.$$

Computing s_n requires:

- computing n terms in the general k-Mahler case;
- computing just $O(\log n)$ terms if $b_{0,j} = 0$ for $j \geq 1$ $(a_0(z) \in \mathbb{C})$.

Linear representation $(L, \{A_j\}, C)$

Computing s_n requires $O(\log n)$ operations in the k-regular case.

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How to determine if a k-Mahler is k-regular?

k-Regularity and Singularities

$$F(z) = \frac{1}{a_0(z)} \sum_{j=1}^{d} a_j(z)\, F(z^{k^j}) \quad \longrightarrow \begin{cases} |z| = 1 & \text{stay on unit circle,} \\ |z| < 1 & \text{approach it from } 0, \\ |z| > 1 & \text{approach it from } \infty. \end{cases}$$

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Classes with restricted singularities

A series satisfying a (potentially non-minimal) k-Mahler equation is called:

- k-Becker if $a_0 = 1$;

Becker's partial converse (1994)

A k-Becker series is k-regular.

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Classes with restricted singularities

$\alpha \in \mathbb{C}$ is k-calm if α is either 0 or a root of unity of order not coprime to k.

A series satisfying a (potentially non-minimal) k-Mahler equation is called:

- k-Becker if $a_0 = 1$;
- k-Dumas if any zero of a_0 is k-calm.

Becker's partial converse (1994)

A k-Becker series is k-regular.

Dumas's partial converse (1993)

A k-Dumas series is k-regular.

Raising roots of unity to kth power

- order not coprime to k $\iff$ on the tails of the ρ's
- order coprime to k $\iff$ on the cycles of the ρ's

From Becker's Conjecture to Our Proof

Becker's conjecture (1994)

If F is k-regular, then $\exists$ a k-regular $R \in \mathbb{C}(z)$ s.t. $F(z)/R(z)$ is k-Becker.

Factorization of singularities vs Desingularization of operator

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Factorization of singularities vs Desingularization of operator

No singularities at roots of unity: Kisielewski (2017)

Let F be k-Mahler with a minimal-order k-Mahler equation whose a_0 has no roots at roots of unity. Then:

- F is k-regular iff it is k-Dumas;
- F is k-regular iff $\exists$ a k-regular $R \in \mathbb{C}(z)$ s.t. $F(z)/R(z)$ is k-Becker.

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General case: our result (2018)

- If F is k-regular, then $\exists \gamma \in \mathbb{N}$, $\exists Q \in \mathbb{C}[z]$ s.t. $1/Q(z)$ is k-regular and $F(z)/(z^\gamma Q(z))$ is k-Becker.
- Q can be obtained from an initial equation for F.
- F is k-regular iff it is k-Dumas.

A Matrix Viewpoint on k-Mahler Equations

$$a_0(z)F(z) + \cdots + a_d(z)F(z^{k^d}) = 0 \implies F(z) = \sum_{i=0}^{d-1} c_{i,n}(z)F(z^{k^{n+i}})$$

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$$\Phi(z) := \left(F(z), \ldots, F(z^{k^{d-1}})\right)^T$$

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$$A(z) := \begin{pmatrix} -\frac{a_1}{a_0}(z) & \dots & & -\frac{a_d}{a_0}(z) \\ 1 & & 0 & 0 \\ & \ddots & & 0 \\ 0 & & 1 & 0 \end{pmatrix} =: B_1(z) \qquad B_n(z) := A(z)B_{n-1}(z^k)$$

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$$B_n(z) = \begin{pmatrix} c_{1,n}(z) & \dots & c_{d,n}(z) \\ c_{1,n-1}(z^k) & \dots & c_{d,n-1}(z^k) \\ \vdots & & \vdots \\ c_{1,n-d+1}(z^{k^{d-1}}) & \dots & c_{d,n-d+1}(z^{k^{d-1}}) \end{pmatrix} = \left(c_{j,n+1-i}(z^{k^{i-1}})\right)_{i,j}$$

More Properties of k-Orbits

Note: $\Lambda_j\big(S(z)\,T(z^k)\big) = \Lambda_j\big(S(z)\big)\,T(z)$.

k-Orbit of a k-Mahler series

For $n \geq 0$, since $F(z) = \sum_{i=0}^{d-1} c_{i,n}(z) F\big(z^{k^{n+i}}\big)$,

$$\forall (j), \quad \Lambda_{j_n} \cdots \Lambda_{j_1}(F)(z) = \sum_{i=0}^{d-1} \Lambda_{j_n} \cdots \Lambda_{j_1}(c_{i,n})(z)\, F\big(z^{k^i}\big).$$

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k-Orbit of a k-regular series

Since $V := \sum_{n,(j)} \mathbb{C}\,\Lambda_{j_n} \cdots \Lambda_{j_1}(F)(z)$ has finite $\mathbb{C}$-dimension,

$$\exists h \in \mathbb{C}[z], \quad V \subset \frac{1}{h(z)} \sum_{i=0}^{d-1} \mathbb{C}[z]\, F\big(z^{k^i}\big).$$

Bounding Denominators in the $B_n(z)$

$$\omega_\alpha(S) := \text{order of the pole of } S(z) \text{ at } \alpha$$

Cartiers operators and pole orders (Kisielewski, 2017)

Given $c \in \mathbb{C}(z)$ and a non-zero $\alpha \in \mathbb{C}$,

$$\exists j, \quad \omega_\alpha\Big(\Lambda_j(c)(z^k)\Big) \geq \omega_\alpha\big(c(z)\big).$$

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Uniform order bound for k-regular series at nonzero "fixed point" (2018)

Given a k-regular $F(z)$ and $\xi \neq 0$ s.t. $\xi^k = \xi$,

$$\omega_\xi\big(B_n(z)\big)$$

is bounded uniformly for $n \geq 1$.

Proof: $\quad \Lambda_{j_n} \cdots \Lambda_{j_1}(c_{i,n})(z) \in h(z)^{-1}\mathbb{C}[z]; \quad \omega_\xi\big(c(z)\big) = \omega_\xi\big(c(z^k)\big)$;
Kisielewski's lemma implies $\omega_\xi\big(c_{i,n}(z)\big) \leq \omega_\xi\big(h(z)^{-1}\big)$; structure of $B_n(z)$.

Fixed Roots of Unity Cannot Make a k-Regular Series

Fixed roots of unity and k-regular series (2018)

Given a k-regular $F(z)$, consider its minimal-order Mahler equation. If $a_0(\xi) = 0$ and $\xi^k = \xi$, then $\xi = 0$.

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Proof: Assume $\xi^k = \xi \neq 0$. Pattern of pole orders in first row of $A = B_1$:

$$\begin{matrix} (\omega < X & \dots & \omega < X & \omega = X & \omega \leq X & \omega \leq X) \\ 1 & \dots & & N & \dots & d \end{matrix}$$

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As F is k-regular, because of the structure of $B_n(z)$, the maximal pole order Y among the B_n occurs as some $(B_m)_{1,J}$ for minimal m, and satisfies $Y \ge X > 0$.

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As F is k-regular, because of the structure of $B_n(z)$, the maximal pole order Y among the B_n occurs as some $(B_m)_{1,J}$ for minimal m, and satisfies $Y \ge X > 0$. Pattern of pole orders in Jth column of B_{m+N-1}:

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Now, the $(1, N)$-entry of $B_{m+N}(z) = A(z)\, B_{m+N-1}(z^k)$ is the sum of $d - 1$ elements of order $< X + Y$ and one of order $X + Y$. Contradiction.

Periodic Roots of Unity Cannot Make a k-Regular Series

Periodic roots of unity and k-regular series (2018)

Given a k-regular $F(z)$, consider $q \in \mathbb{C}[z]$ of minimal degree s.t.

$$q(z)F(z) \in \sum_{j\geq 1} \mathbb{C}[z]F(z^{k^j}).$$

If $q(\xi) = 0$ and $\xi^{k^M} = \xi$ for some $M \geq 1$, then $\xi = 0$.

Proof: If F is k-regular, it is k^M-regular. By previous lemma (for k^M) and because $q \mid a_0$, ξ must be zero.

Dumas's Structure Theorem and a Consequence

Structure theorem for k-Mahler functions (Dumas, 1993)

$$\begin{cases} a_0(z)F(z) + \cdots + a_d(z)F(z^{k^d}) = 0 \\ a_0(z) = \rho z^\delta P(z),\ P(0) = 1 \end{cases} \implies \begin{cases} \exists J(z)\ k\text{-regular}, \\ K(z) := \prod_{j\geq 1} P(z^{k^j})\ k\text{-regular}, \\ \quad F(z) = J(z)/K(z). \end{cases}$$

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Our (preliminary) structure theorem for k-regular functions (2018)

$$\begin{cases} F(z)\text{ is }k\text{-regular} \\ a_0(z)F(z)+\cdots+a_d(z)F(z^{k^d})=0 \end{cases} \implies \begin{cases} \exists Q\in\mathbb{C}[z],\ \text{s.t.} \\ 1/Q(z)\text{ is }k\text{-regular}, \\ \quad F(z)=z^\gamma Q(z)G(z), \end{cases}$$

with $G(z)$ given by an equation $q_0(z)G(z)+\cdots+q_d(z)G(z^{k^d})=0$ satisfying:

- $q_0(0)\neq 0$
- if $q_0(\xi)=0$ for a root of unity ξ, then $\xi^{k^M}=\xi$ for some non-zero $M\in\mathbb{N}$.

Proof: Gather zeroes of a_0 that are ultimately periodic but not periodic roots of unity into Q. Apply Dumas's theorem to F. Simplify infinite

Final Result: Reducing k-Regular to k-Becker

$$\frac{k\text{-regular } F(z)}{z^\gamma Q(z)} = k\text{-Becker}$$

Our main theorem (2018)

If F is k-regular, then $\exists \gamma \in \mathbb{N}$, $\exists Q \in \mathbb{C}[z]$ s.t. $1/Q(z)$ is k-regular and $F(z)/(z^\gamma Q(z))$ is k-Becker.

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Proof: For $Q(z)$ and $G(z)$ as in the previous theorem, consider $q \in \mathbb{C}[z]$ of minimal degree s.t.

$$q(z)G(z) \in \sum_{j \geq 1} \mathbb{C}[z]G(z^{k^j}),$$

$q(0) \neq 0$, and if $q(\xi) = 0$ for a root of unity ξ, then $\xi^{k^M} = \xi$ for some $M \geq 1$.

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$q(0) \neq 0$, and if $q(\xi) = 0$ for a root of unity ξ, then $\xi^{k^M} = \xi$ for some $M \geq 1$. Applying the lemma about periodic roots to $H(z) = z^\gamma G(z)$ (k-regular) forbids those roots of unity. So no zero of $q(z)$ is 0 or a root of unity.

Final Result: Reducing k-Regular to k-Becker

$$\frac{k\text{-regular } F(z)}{z^\gamma Q(z)} = k\text{-Becker}$$

Our main theorem (2018)

If F is k-regular, then $\exists\gamma \in \mathbb{N}$, $\exists Q \in \mathbb{C}[z]$ s.t. $1/Q(z)$ is k-regular and $F(z)/(z^\gamma Q(z))$ is k-Becker.

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If $\exists\lambda,\ q(\lambda) = 0$, consider denominator bound $h(z)$ for k-orbit V of $H(z)$.
For $N \gg 1$, $h(\lambda^{k^N}) \neq 0$. Write $H(z) \in \sum_{j=1}^d \mathbb{C}(z)H(z^{k^{N+j}})$ and use Kisielewski's lemma for a contradiction.

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