Examples of Use of Mgfun

Frédéric Chyzak

To illustrate the course “Holonomic summation and integration” during ISSFQFT2012 (July 2012).

Set up

The package to be used is part of the Algolib library, which can be downloaded from http://algo.inria.fr/libraries/. After installing it, one needs to let Maple know about it:

```
> libname := "/home/chyzak", libname:
```

Here we use our package Mgfun (F. Chyzak + contributions by Shaoshi Chen, Cyril Germa, Lucien Pech, and Ziming Li). An homologue package for Mathematica was written by Christoph Koutschan.

```
> with(Mgfun);
```

[*MG_Internals*, *creative_telescoping*, *dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *rational_creative_telescoping*, *sum_of_sys*, *sys**sys*, *sys*+*sys*] **(1.1)**

```
> infolevel[:-CreativeTelescoping] := 5:
```

An example in P. Paule's course

```
> f := (-1)^k / 2^k * binomial(n, k) * binomial(2*k, k);
```

$$f := \frac{(-1)^k \binom{n}{k} \binom{2\,k}{k}}{2^k} \tag{2.1}$$

```
> seq(i = add(eval(f, {n = i, k = j}), j = 0 .. i), i = 0..10);
```

$$0 = 1,\ 1 = 0,\ 2 = \frac{1}{2},\ 3 = 0,\ 4 = \frac{3}{8},\ 5 = 0,\ 6 = \frac{5}{16},\ 7 = 0,\ 8 = \frac{35}{128},\ 9 = 0,\ 10 = \frac{63}{256} \tag{2.2}$$

Creative telescoping in view of summation over k so as to describe a sum parametrized by n is done like this:

```
> ct := creative_telescoping(f, n::shift, k::shift);
skew_poly_creative_telescoping: PROFILE - DIMENSION 1
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
```

```
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: PROFILE - LAST_D 2
creative_telescoping: Start to reconstruct rhs operators.
```

$$ct := \left[\left[(-n-1)\ \_F(n) + (n+2)\ \_F(n+2),\ -\frac{(n+1)\ k^2\ \_f(n,k)}{(-n-1+k)\ (-n-2+k)}\right]\right] \qquad \textbf{(2.3)}$$

This result can be interpreted in terms of the summand as:

```
> eval(ct[1][1], _F = unapply(_f(n, k), n)) = Delta[k] ( ct[1]
  [2] );
```

$$(-n-1)\ \_f(n,k) + (n+2)\ \_f(n+2,k) = \Delta_k\left(-\frac{(n+1)\ k^2\ \_f(n,k)}{(-n-1+k)\ (-n-2+k)}\right) \qquad \textbf{(2.4)}$$

This cannot be evaluated for $k = n$ or $k = n + 1$ or $k = n + 2$!

So we can only sum until $k = n - 1$ and have to adjust the relation:

```
> ct[1][1] = (-n-1) * _f(n, n) + (n+2) * add(_f(n+2, n+i), i =
  0..2) + eval(ct[1][2], k = n) - eval(ct[1][2], k = 0);
```

$$(-n-1)\ \_F(n) + (n+2)\ \_F(n+2) = (-n-1)\ \_f(n,n) + (n+2)\ (\_f(n+2,n) + \_f(n+2,n+1) + \_f(n+2,n+2)) - \frac{(n+1)\ n^2\ \_f(n,n)}{2} \qquad \textbf{(2.5)}$$

```
> eval(%, _f = unapply(f, n, k));
```

$$(-n-1)\ \_F(n) + (n+2)\ \_F(n+2) = \frac{(-n-1)\ (-1)^n \binom{2\,n}{n}}{2^n} + (n+2)\left(\frac{(-1)^n \binom{n+2}{n}\binom{2\,n}{n}}{2^n} + \frac{(-1)^{n+1}\ (n+2)\binom{2\,n+2}{n+1}}{2^{n+1}} + \frac{(-1)^{n+2}\binom{2\,n+4}{n+2}}{2^{n+2}}\right) - \frac{(n+1)\ n^2\ (-1)^n\binom{2\,n}{n}}{2\ 2^n} \qquad \textbf{(2.6)}$$

Now, the adjusted inhomogeneous part turns out to be zero.

```
> normal(rhs(%), expanded);
```

$$0 \qquad \textbf{(2.7)}$$

So, the left-hand component of the pair obtained by creative telescoping is a recurrence satisfied by the sum, and we can solve:

```
> sol := rsolve(ct[1][1] = 0, _F(n));
```

(2.8)

$$sol := \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\_F(0)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)} & n::even \\ \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\_F(1)}{2\,\Gamma\left(\frac{n}{2}+1\right)} & n::odd \end{cases} \tag{2.8}$$

Check a few values:

```
> seq(eval(sol/_F(0), n = 2 * p), p = 0..5);
```

$$1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256} \tag{2.9}$$

Home work in M. Schlosser's course

Our goal, (1.2) in Gasper and Schlosser's 2004 paper:

```
> f := (c - a * (a + t))^beta * (c - (a + 1) * (a + t))^(beta - 1) / (c - (a + t)^2)^(2 * beta) * t^beta * (1 - t)^(beta - 1);
```

$$f := \frac{(c-a(a+t))^{\beta}(c-(a+1)(a+t))^{\beta-1} t^{\beta}(1-t)^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \tag{3.1}$$

```
> GAMMA(beta)^2/2/GAMMA(2*beta) = (c - (a + 1)^2) * Int(f, t = 0 .. 1);
```

$$\frac{\Gamma(\beta)^2}{2\,\Gamma(2\beta)} = (c-(a+1)^2)\left(\int_0^1 \frac{(c-a(a+t))^{\beta}(c-(a+1)(a+t))^{\beta-1} t^{\beta}(1-t)^{\beta-1}}{(c-(a+t)^2)^{2\beta}}\,dt\right) \tag{3.2}$$

Encoding of the integrand by 1st-order linear functional equations (we just hide large coefficients under dots):

```
> dfinite_expr_to_sys(f, _f(beta::shift, t::diff)):  collect(%, {_f, diff}, p -> `...`);
```

$$\left\{\ldots\,\_f(\beta, t) + \ldots\,\_f(\beta+1, t), \ldots\,\_f(\beta, t) + \ldots\,\frac{\partial}{\partial t}\_f(\beta, t)\right\} \tag{3.3}$$

Mgfun does most of it by itself (20 seconds):

```
> ct := creative_telescoping(f, beta::shift, t::diff):
skew_poly_creative_telescoping: PROFILE - DIMENSION 1
skew_poly_creative_telescoping: Start uncoupling system.
```

```
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: PROFILE - LAST_D 2
creative_telescoping: Start to reconstruct rhs operators.
```

One typically doesn't want to see the result, as it can be large, but here it is for this example:

```
> ct;
```

$$\Big[\Big[\big(c^2 \beta^2 - 2\, a\, c\beta^2 + 2\, a^3\, \beta^2 - 2\, a^2\, \beta\, c - 2\, \beta\, c\, a + \beta\, c^2 + a^2\, \beta + a^4\, \beta + 2\, \beta\, a^3 + a^4\, \beta^2 + a^2\, \beta^2 - 2\, a^2\, c\beta^2\big)\, \_F(\beta) + \big(32\, \beta\, c + 16\, c\beta^2 + 12\, c\big)\, \_F(\beta+2) + \big(-2\, c - 2\, a^2 - 4\, a^3 + 4\, c\, a^2 - 2\, c^2 - 2\, a^4 - 4\, c\beta^2 - 4\, c^2\, \beta^2 - 8\, a^3\, \beta^2 + 4\, a\, c - 6\, \beta\, c^2 - 6\, \beta\, c - 6\, a^2\, \beta - 6\, a^4\, \beta - 12\, \beta\, a^3 - 4\, a^4\, \beta^2 - 4\, a^2\, \beta^2 + 8\, a^2\, c\beta^2 + 12\, \beta\, c\, a + 12\, a^2\, \beta\, c + 8\, a\, c\beta^2\big)\, \_F(\beta+1),\ \frac{1}{\left(-c + a^2 + 2\, a\, t + t^2\right)^3}\Big(\big(5\, t^3\, a^9 + 6\, t^3\, c^4 + 5\, t^3\, a^8 - 6\, t^4\, c^3 - t^4\, c^4 + 20\, t^4\, a^7 + 10\, t^4\, a^8 + t^5\, c^2 + t^5\, c^3 + 25\, t^5\, a^6 + 10\, t^5\, a^7 + 14\, t^6\, a^5 + 5\, t^6\, a^6 + t^7\, a^2 + 3\, t^7\, a^3 + 3\, t^7\, a^4 + t^7\, a^5 - 2\, t^2\, \beta\, c^3 + 5\, t^2\, \beta\, a^6 - 11\, t^2\, \beta\, c^4 - 4\, t^2\, a^7\, \beta - 21\, t^2\, a^8\, \beta + 5\, t^2\, c^4\, a^2 - 10\, t^2\, c^3\, a^4 - 2\, t^2\, a\, c^4 + 8\, t^2\, a^3\, c^3 - 12\, t^2\, a^5\, c^2 + 10\, t^2\, c^2\, a^6 + 8\, t^2\, c\, a^7 - 5\, t^2\, c\, a^8 - 10\, t^2\, \beta\, a^9 - 2\, t^2\, \beta\, c^5 + 2\, t^2\, \beta\, a^{10} - 33\, t^3\, c^2\, a^2 + 10\, t^3\, \beta\, c^3 - 25\, t^3\, \beta\, a^6 - 23\, t^3\, c^3\, a^2 + 33\, t^3\, c^2\, a^4 - 21\, t^3\, c\, a^6 + 10\, t^3\, \beta\, c^4 - 40\, t^3\, a^7\, \beta - 5\, t^3\, a^8\, \beta + 5\, t^3\, a\, c^4 - 20\, t^3\, a^3\, c^3 + 30\, t^3\, a^5\, c^2 - 20\, t^3\, c\, a^7 + 10\, t^3\, \beta\, a^9 - 2\, t^4\, \beta\, c^2 - 5\, t^4\, a^4\, \beta - 30\, t^4\, \beta\, a^5 - a^7\, t - 11\, t^4\, \beta\, c^3 - 25\, t^4\, \beta\, a^6 - 20\, t^4\, a\, c^3 + 60\, t^4\, a^3\, c^2 - 7\, t^4\, c^3\, a^2 + 27\, t^4\, c^2\, a^4 - 60\, t^4\, c\, a^5 - 29\, t^4\, c\, a^6 - 2\, t^4\, \beta\, c^4 + 20\, t^4\, a^7\, \beta + 20\, t^4\, a^8\, \beta + 15\, t^5\, a\, c^2 + 23\, t^5\, c^2\, a^2 - 49\, t^5\, c\, a^4 + 3\, t^5\, \beta\, c^2 - 13\, t^5\, a^4\, \beta - 4\, t^5\, \beta\, a^3 + 6\, t^5\, \beta\, a^5 + 3\, t^5\, \beta\, c^3 + 35\, t^5\, \beta\, a^6 + 3\, t^5\, a\, c^3 + 4\, t^5\, a^3\, c^2 - 17\, t^5\, c\, a^5 + 20\, t^5\, a^7\, \beta - 12\, t^6\, c\, a^2 - 2\, t^6\, a\, c - 2\, t^6\, a\, c^2 - 12\, t^6\, a^3\, c - 3\, t^6\, c^2\, a^2 - 2\, t^6\, c\, a^4 - t^6\, \beta\, c^2 - t^6\, a^2\, \beta + 13\, t^6\, a^4\, \beta + 22\, t^6\, \beta\, a^5 + 10\, t^6\, \beta\, a^6 + t^7\, a^3\, c + t^7\, a^2\, c + 4\, t^7\, a^3\, \beta + 2\, t^7\, a^5\, \beta + t^7\, a^2\, \beta + 5\, t^7\, a^4\, \beta + 10\, t\, \beta\, a^3\, c^2 - 3\, t\, \beta\, a\, c^3 - 12\, t\, \beta\, a^5\, c^2 + 8\, t\, \beta\, a^3\, c^3 - 2\, t\, \beta\, a\, c^4 + 15\, t\, \beta\, c\, a^8 - 30\, t\, \beta\, c^2\, a^6 + 30\, t\, \beta\, c^3\, a^4 - 15\, t\, \beta\, c^4\, a^2 - 18\, t\, c\, \beta\, a^6 - 11\, t\, c\, \beta\, a^5 + 24\, t\, \beta\, c^2\, a^4 - 14\, t\, \beta\, c^3\, a^2 + 8\, t\, \beta\, a^7\, c \quad \textbf{(3.4)}$$

$$- 12\, t^{2} \beta\, c\, a^{4} + 9\, t^{2} \beta\, c^{2} a^{2} + 40\, t^{2} \beta\, a^{7} c - 60\, t^{2} \beta\, a^{5} c^{2} + 40\, t^{2} \beta\, a^{3} c^{3}$$
$$- 10\, t^{2} \beta\, a\, c^{4} - 10\, t^{2} \beta\, c\, a^{8} + 20\, t^{2} \beta\, c^{2} a^{6} - 20\, t^{2} \beta\, c^{3} a^{4} + 10\, t^{2} \beta\, c^{4} a^{2}$$
$$+ 74\, t^{2} c \beta\, a^{6} + 12\, t^{2} c \beta\, a^{5} - 96\, t^{2} \beta\, c^{2} a^{4} + 54\, t^{2} \beta\, c^{3} a^{2} - 12\, t^{2} \beta\, a^{3} c^{2}$$
$$+ 4\, t^{2} \beta\, a\, c^{3} - 40\, t^{3} \beta\, a^{7} c + 60\, t^{3} \beta\, a^{5} c^{2} - 40\, t^{3} \beta\, a^{3} c^{3} + 10\, t^{3} \beta\, a\, c^{4}$$
$$+ 5\, t^{3} c \beta\, a^{6} + 105\, t^{3} c \beta\, a^{5} + 15\, t^{3} \beta\, c^{2} a^{4} - 25\, t^{3} \beta\, c^{3} a^{2} - 90\, t^{3} \beta\, a^{3} c^{2}$$
$$+ 25\, t^{3} \beta\, a\, c^{3} + 60\, t^{3} \beta\, c\, a^{4} - 45\, t^{3} \beta\, c^{2} a^{2} + 84\, t^{4} \beta\, a^{3} c^{2} - 32\, t^{4} \beta\, a\, c^{3}$$
$$+ 7\, t^{4} a^{2} \beta\, c + 27\, t^{4} \beta\, c\, a^{4} + 9\, t^{4} \beta\, c^{2} a^{2} + 48\, t^{4} \beta\, a^{3} c - 18\, t^{4} \beta\, a\, c^{2} - 10\, t^{4} a^{4}$$
$$- 5\, t^{5} a^{3} - t^{6} a^{2} - 10\, t^{3} a^{5} - t^{4} c^{2} - t^{2} c^{3} - 5\, t^{2} a^{6} - c^{3} a\, t + c^{2} a^{3} t + c\, a^{5} t$$
$$- 58\, t^{4} c \beta\, a^{6} - 72\, t^{4} c \beta\, a^{5} + 54\, t^{4} \beta\, c^{2} a^{4} - 14\, t^{4} \beta\, c^{3} a^{2} + 7\, t^{4} c\, a^{2} + 2\, t^{5} c\, a$$
$$- t^{3} a\, c^{2} + 9\, t^{3} a^{3} c + t^{2} c^{2} a^{2} + 5\, t^{2} c\, a^{4} + 6\, t^{5} a^{2} \beta\, c - 71\, t^{5} \beta\, c\, a^{4}$$
$$+ 33\, t^{5} \beta\, c^{2} a^{2} - 34\, t^{5} \beta\, a^{3} c + 24\, t^{5} \beta\, a\, c^{2} - 34\, t^{5} c \beta\, a^{5} + 8\, t^{5} \beta\, a^{3} c^{2}$$
$$+ 6\, t^{5} \beta\, a\, c^{3} + 3\, t^{5} \beta\, c\, a - 4\, t^{6} \beta\, c\, a - 16\, t^{6} a^{2} \beta\, c - 4\, t^{6} \beta\, c\, a^{4} - 6\, t^{6} \beta\, c^{2} a^{2}$$
$$- 16\, t^{6} \beta\, a^{3} c - 6\, t^{6} \beta\, a\, c^{2} + t^{7} a \beta\, c + 3\, t^{7} a^{2} \beta\, c + 2\, t^{7} a^{3} \beta\, c - 2\, t^{2} a^{9} - t^{2} c^{5}$$
$$+ t^{2} a^{10} + 6\, t^{2} c^{3} a + 30\, t^{2} c^{3} a^{2} + 12\, t^{3} c^{3} a - 26\, t^{2} c^{2} a^{3} - 54\, t^{2} c^{2} a^{4}$$
$$- 39\, t^{3} c^{2} a^{3} + 12\, t^{4} c^{2} a^{2} - 12\, t^{4} c^{2} a - 6\, t^{2} c^{4} + 6\, t^{3} c^{3} - 14\, t^{2} a^{7} - 12\, t^{2} a^{8}$$
$$- 15\, t^{3} a^{7} - 25\, t^{3} a^{6} - 20\, t^{4} a^{5} + 15\, t^{5} a^{5} - 5\, t^{5} a^{4} + 12\, t^{6} a^{4} + 2\, t^{6} a^{3} + t\, c^{4}$$
$$- 3\, t\, a^{8} + a^{8} \beta + 34\, t^{2} c\, a^{5} + 42\, t^{2} c\, a^{6} + 42\, t^{3} c\, a^{5} - 6\, t^{4} c\, a^{4} + 52\, t^{3} c\, a^{4}$$
$$+ 32\, t^{4} c\, a^{3} - 30\, t^{5} c\, a^{3} + 4\, t^{5} c\, a^{2} - 6\, t\, c^{2} a^{4} + 8\, t\, c\, a^{6} - \beta\, c^{3} a^{2} + 3\, \beta\, c^{2} a^{4}$$
$$- 3\, c \beta\, a^{6} + 2\, \beta\, a^{9} - \beta\, c^{5} + \beta\, a^{10} + 5\, \beta\, c^{4} a^{2} - 10\, \beta\, c^{3} a^{4} + 10\, \beta\, c^{2} a^{6} - 5\, \beta\, c\, a^{8}$$
$$+ 2\, \beta\, a\, c^{4} - 8\, \beta\, a^{3} c^{3} + 12\, \beta\, a^{5} c^{2} - 8\, \beta\, a^{7} c + 3\, t \beta\, c^{4} + 4\, t\, a^{7} \beta + 5\, t\, a^{8} \beta$$
$$- 5\, t\, c^{4} a^{2} + 10\, t\, c^{3} a^{4} - 3\, t\, a\, c^{4} + 12\, t\, a^{3} c^{3} - 18\, t\, a^{5} c^{2} - 10\, t\, c^{2} a^{6} + 12\, t\, c\, a^{7}$$
$$+ 5\, t\, c\, a^{8} - 2\, t \beta\, a^{9} + 3\, t \beta\, c^{5} - 3\, t \beta\, a^{10} - 3\, t\, a^{9} + t\, c^{5} - t\, a^{10}\big)\, t\, \_f(\beta, t)\big)\big]\big]$$

The inhomogeneous part of the creative-telescoping equation behaves nicely:

```
> Qf := eval(ct[1][2], _f(beta, t) = f):
> eval(Qf, t = 1);
```

$$0 \qquad \textbf{(3.5)}$$

```
> eval(Qf, t = 0);
```

$$0 \qquad \textbf{(3.6)}$$

Thus, we directly get a homogeneous recurrence for the integral:

```
> rec := collect(ct[1][1], _F, factor);
```

$$rec := \beta\, (-c + a^{2} + a)^{2}\, (\beta + 1)\, \_F(\beta) - 2\, (\beta + 1)\, (2\, \beta + 1)\, \big(c + a^{2} + 2\, a^{3} - 2\, c\, a^{2} + c^{2} + a^{4} - 2\, a\, c\big)\, \_F(\beta + 1) + 4\, (3 + 2\, \beta)\, (2\, \beta + 1)\, c\, \_F(\beta + 2) \qquad \textbf{(3.7)}$$

At this point, we can check that the known closed-form for the integral satisfies this recurrence:

```
> eval(rec, _F = unapply(GAMMA(beta)^2 / 2 / GAMMA(2 * beta) /
  (c - (a + 1)^2), beta));
```

$$\frac{\beta\,(-c+a^2+a)^2\,(\beta+1)\,\Gamma(\beta)^2}{2\,\Gamma(2\,\beta)\,\left(c-(a+1)^2\right)} - \frac{(\beta+1)\,(2\,\beta+1)\,(c+a^2+2\,a^3-2\,c\,a^2+c^2+a^4-2\,a\,c)\,\Gamma(\beta+1)^2}{\Gamma(2+2\,\beta)\,\left(c-(a+1)^2\right)} + \frac{2\,(3+2\,\beta)\,(2\,\beta+1)\,c\,\Gamma(\beta+2)^2}{\Gamma(2\,\beta+4)\,\left(c-(a+1)^2\right)} \quad (3.8)$$

```
> simplify(%, GAMMA);
```

$$0 \quad (3.9)$$

Therefore, checking enough initial conditions would prove the integral identity.

On the other hand, we can fake that we don't know the closed-form evaluation of the integal and use a symbolic solver to get a closed form:

```
> sol := rsolve(rec, _F(beta));
```

$$sol := \begin{cases} \_F(0) & \beta = 0 \\ \dfrac{2\left(\left(\dfrac{a^4+2\,a^3-2\,c\,a^2+a^2-2\,a\,c+c^2}{c}\right)^{\beta}-1\right)\sqrt{\pi}\,c\,\_F(1)\,4^{-\beta}\,\Gamma(\beta)}{(a^4+2\,a^3-2\,c\,a^2+a^2-2\,a\,c+c^2-c)\,\Gamma\left(\beta+\frac{1}{2}\right)} & otherwise \end{cases} \quad (3.1$$

Last example in C. Raab's course

```
> f := GegenbauerC(m, mu, x) * GegenbauerC(n, nu, x) * (1 -
  x^2)^(nu - 1/2);
```

$$f := C_m^{(\mu)}(x)\,C_n^{(\nu)}(x)\,(1-x^2)^{\nu-\frac{1}{2}} \quad (4.1)$$

```
> ct := creative_telescoping(f, [m::shift, n::shift], x::diff):
skew_poly_creative_telescoping: PROFILE - DIMENSION 4
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
```

```
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: PROFILE - LAST_D 2
creative_telescoping: Start to reconstruct rhs operators.
```

```
> collect(map2(op, 1, ct), _F, factor);
```

$$[(n+1)(m-1-n+2\mu-2\nu)\,\_F(m, n+1) - (2\nu+n)(m+1-n)\,\_F(m+1, n), -(n+2\mu+m)(-n-2\nu+2\mu+m)\,\_F(m, n) + (m+2-n)(2\nu+2+n+m)\,\_F(m+2, n)] \tag{4.2}$$

These are the recurrences also found by Clemens Raab in his lecture.

The sequences (a_n) and (b_n) of Apéry's proof of irrationality of z(3)

Apéry's proof of irrationality of ζ(3) relies on showing that the sequences with general terms a_n and b_n below satisfy the same second-order recurrence. This session shows how to get this recurrence.

```
> a[n] = Sum(binomial(n, k)^2 * binomial(n + k, k)^2, k = 0..n)
  ;
```

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \tag{5.1}$$

```
> b[n] = Sum(binomial(n, k)^2 * binomial(n + k, k)^2 * (Sum
  (1/m^3, m = 1..n) + Sum((-1)^(m+1) / (2 * m^3 * binomial(n,
  m) * binomial(n + m, m)), m = 1..k)), k = 0..n);
```

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m+1}}{2\, m^3 \binom{n}{m} \binom{n+m}{m}} \right) \tag{5.2}$$

```
> with(Mgfun);
```

$$[MG\_Internals, creative\_telescoping, dfinite\_expr\_to\_diffeq, dfinite\_expr\_to\_rec, dfinite\_expr\_to\_sys, diag\_of\_sys, int\_of\_sys, pol\_to\_sys, rational\_creative\_telescoping, sum\_of\_sys, sys{*}sys, sys+sys] \tag{5.3}$$

Inner indefinite binomial sum:

$$DD(n, k) := \sum_{m=1}^{k} \frac{(-1)^{m+1}}{2\, m^3 \binom{n}{m} \binom{n+m}{m}}$$

Creative telescoping on the summand $d_{n,\,k,\,m}$:

```
> d_nkm := (-1)^(m+1) / (2 * m^3 * binomial(n, m) * binomial
  (n + m, m));
```

$$d\_nkm := \frac{(-1)^{m+1}}{2\, m^3 \binom{n}{m} \binom{n+m}{m}} \quad (5.1.1)$$

```
> ct_d := creative_telescoping(d_nkm, [n::shift, k::shift],
  m::shift);
```

skew_poly_creative_telescoping: PROFILE - DIMENSION 1
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j = 1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j = 1
skew_poly_creative_telescoping: PROFILE - LAST_D 1
creative_telescoping: Start to reconstruct rhs operators.

$$ct\_d := \left[\left[-\_F(n,k) + \_F(n+1,k),\ \frac{(2\,n\,m + 2\,m - 2\,m^2)\ \_f(n,k,m)}{n^2 + 2\,n + 1}\right], \left[-\_F(n,k) + \_F(n,1+k),\ 0\right]\right] \quad (5.1.2)$$

The meaning of this is given by the following two relations, where Δ_m means a forward finite difference operator w.r.t. m:

```
> eval(map(e -> e[1] = Delta[m](e[2]), eval(ct_d, _F =
  unapply(_f(n, k, m), n, k))), _f = d);
```

$$\left[-d(n,k,m) + d(n+1,k,m) = \Delta_m\left(\frac{(2\,n\,m + 2\,m - 2\,m^2)\ d(n,k,m)}{n^2 + 2\,n + 1}\right),\ -d(n,k,m) + d(n,1+k,m) = \Delta_m(0)\right] \quad (5.1.3)$$

(in other words:)

```
> eval(%, {d(n + 1, k, m) = Delta[n](d(n, k, m)) + d(n, k,
  m), d(n, k + 1, m) = Delta[k](d(n, k, m)) + d(n, k, m)});
```

$$\left[\Delta_n(d(n,k,m)) = \Delta_m\left(\frac{(2\,n\,m + 2\,m - 2\,m^2)\ d(n,k,m)}{n^2 + 2\,n + 1}\right),\ \Delta_k(d(n,k,m)) = \Delta_m(0)\right] \quad (5.1.4)$$

Deal with left-hand relation

Check it:

```
> eval(ct_d[1][1], _F = unapply(d_nkm, n, k));
```

$$-\frac{(-1)^{m+1}}{2\, m^3 \binom{n}{m} \binom{n+m}{m}} + \frac{(-1)^{m+1}}{2\, m^3 \binom{n+1}{m} \binom{n+m+1}{m}} \quad (5.1.1.1)$$

```
> eval(ct_d[1][2], _f(n, k, m) = d_nkm);
```

$$\frac{(2\,n\,m+2\,m-2\,m^2)\,(-1)^{m+1}}{2\,(n^2+2\,n+1)\,m^3\binom{n}{m}\binom{n+m}{m}} \qquad (5.1.1.2)$$

```
> normal(%% - (eval(%, m = m + 1) - %), expanded);
```

$$0 \qquad (5.1.1.3)$$

Exploit left-hand relation to get relation on sum:

```
> eval(ct_d[1][2], _f(n, k, m) = d_nkm);
```

$$\frac{(2\,n\,m+2\,m-2\,m^2)\,(-1)^{m+1}}{2\,(n^2+2\,n+1)\,m^3\binom{n}{m}\binom{n+m}{m}} \qquad (5.1.1.4)$$

```
> eval(%, m = k + 1) - eval(%, m = 1);
```

$$\frac{(2\,n\,(1+k)+2+2\,k-2\,(1+k)^2)\,(-1)^{2+k}}{2\,(n^2+2\,n+1)\,(1+k)^3\binom{n}{1+k}\binom{n+1+k}{1+k}} - \frac{1}{(n^2+2\,n+1)\,(n+1)} \qquad (5.1.1.5)$$

```
> inhom1 := map(normal@expand, %, expanded);
```

$$inhom1 := \frac{(-1)^k}{\binom{n+k}{k}\,(n+1+k)\binom{n}{k}\,(n^2+2\,n+1)} - \frac{1}{(n^2+2\,n+1)\,(n+1)} \qquad (5.1.1.6)$$

```
> dfinite_expr_to_sys(inhom1, u(n::shift, k::shift));
```

$$\{(-3\,k^2-2\,k^2\,n+2\,n^2\,k+3\,n\,k-n^3-1-3\,n^2-3\,n)\,u(n,k)+(3\,k^2-2\,n^2\,k-3\,n\,k+2\,k^2\,n)\,u(n,1+k)+(n^3+6\,n^2+12\,n+8)\,u(n+1,k),\ (-2\,n^4\,k-17\,n\,k-5\,k-21\,n^2\,k-11\,n^3\,k+5+38\,n^2+2\,n^5+22\,n+32\,n^3+13\,n^4)\,u(n,k)+(2\,n^4\,k+81\,k+135\,n\,k+81\,n^2\,k+21\,n^3\,k+378\,n^2+144\,n^3+486\,n+243+2\,n^5+27\,n^4)\,u(n+2,k)+(2\,n^3\,k+12\,n^2\,k+24\,n\,k+16\,k-112-40\,n^4-308\,n^2-296\,n-158\,n^3-4\,n^5)\,u(n+1,k)\} \qquad (5.1.1.7)$$

```
> sys1 := eval(%, u = unapply(DD(n + 1, k) - DD(n, k), n,
  k));
```

$$sys1 := \{(-3\,k^2-2\,k^2\,n+2\,n^2\,k+3\,n\,k-n^3-1-3\,n^2-3\,n)\,(DD(n+1,k)-DD(n,k))+(3\,k^2-2\,n^2\,k-3\,n\,k+2\,k^2\,n)\,(DD(n+1,1+k)-DD(n,1+k))+(n^3+6\,n^2+12\,n+8)\,(DD(n+2,k)-DD(n+1,k)),\ (-2\,n^4\,k-17\,n\,k-5\,k-21\,n^2\,k-11\,n^3\,k+5+38\,n^2+2\,n^5+22\,n+32\,n^3+13\,n^4)\,(DD(n+1,k)-DD(n,k)) \qquad (5.1.1.8)$$

$+ (2 n^4 k + 81 k + 135 n k + 81 n^2 k + 21 n^3 k + 378 n^2 + 144 n^3 + 486 n + 243 + 2 n^5 + 27 n^4) (DD(n+3, k) - DD(n+2, k)) + (2 n^3 k + 12 n^2 k + 24 n k + 16 k - 112 - 40 n^4 - 308 n^2 - 296 n - 158 n^3 - 4 n^5) (DD(n+2, k) - DD(n+1, k))\}$

```
> sys1;
```

$$\{(-3 k^2 - 2 k^2 n + 2 n^2 k + 3 n k - n^3 - 1 - 3 n^2 - 3 n) (DD(n+1, k) - DD(n, k)) + (3 k^2 - 2 n^2 k - 3 n k + 2 k^2 n) (DD(n+1, 1+k) - DD(n, 1+k)) + (n^3 + 6 n^2 + 12 n + 8) (DD(n+2, k) - DD(n+1, k)), (-2 n^4 k - 17 n k - 5 k - 21 n^2 k - 11 n^3 k + 5 + 38 n^2 + 2 n^5 + 22 n + 32 n^3 + 13 n^4) (DD(n+1, k) - DD(n, k)) + (2 n^4 k + 81 k + 135 n k + 81 n^2 k + 21 n^3 k + 378 n^2 + 144 n^3 + 486 n + 243 + 2 n^5 + 27 n^4) (DD(n+3, k) - DD(n+2, k)) + (2 n^3 k + 12 n^2 k + 24 n k + 16 k - 112 - 40 n^4 - 308 n^2 - 296 n - 158 n^3 - 4 n^5) (DD(n+2, k) - DD(n+1, k))\} \quad (5.1.5)$$

Deal with right-hand relation

```
> inhom2 := factor(normal(expand(eval(d_nkm, m = k + 1)), expanded));
```

$$inhom2 := -\frac{(-1)^k}{2 (1+k) (-n+k) \binom{n}{k} (n+1+k) \binom{n+k}{k}} \quad (5.1.2.1)$$

```
> dfinite_expr_to_sys(inhom2, u(n::shift, k::shift));
```

$$\{(-n+k) u(n, k) + (n+2+k) u(n+1, k), (-k^3 - 3 k^2 - 3 k - 1) u(n, k) + (k^3 + 5 k^2 + 8 k - n k - n^2 k - 2 n + 4 - 2 n^2) u(n, 1+k)\} \quad (5.1.2.2)$$

```
> sys2 := eval(%, u = unapply(DD(n, k + 1) - DD(n, k), n, k));
```

$$sys2 := \{(-n+k) (DD(n, 1+k) - DD(n, k)) + (n+2+k) (DD(n+1, 1+k) - DD(n+1, k)), (-k^3 - 3 k^2 - 3 k - 1) (DD(n, 1+k) - DD(n, k)) + (k^3 + 5 k^2 + 8 k - n k - n^2 k - 2 n + 4 - 2 n^2) (DD(n, 2+k) - DD(n, 1+k))\} \quad (5.1.2.3)$$

```
> sys2;
```

$$\{(-n+k) (DD(n, 1+k) - DD(n, k)) + (n+2+k) (DD(n+1, 1+k) - DD(n+1, k)), (-k^3 - 3 k^2 - 3 k - 1) (DD(n, 1+k) - DD(n, k)) + (k^3 + 5 k^2 + 8 k - n k - n^2 k - 2 n + 4 - 2 n^2) (DD(n, 2+k) - DD(n, 1 \quad (5.1.6)$$

$+ k))\}$

Finally, gather both systems:

```
> sys_for_DD := collect(sys1 union sys2, DD, expand);
```

$$\begin{aligned}
&\mathit{sys_for_DD} := \{(k^3 + 3k^2 + 3k + 1)\,DD(n, k) + (-2k^3 - 8k^2 - 11k - 5 \\
&\quad + nk + n^2k + 2n + 2n^2)\,DD(n, 1 + k) + (k^3 + 5k^2 + 8k - nk - n^2k \\
&\quad - 2n + 4 - 2n^2)\,DD(n, 2 + k),\ (n - k)\,DD(n, k) + (-n + k)\,DD(n, 1 \\
&\quad + k) + (-n - 2 - k)\,DD(n + 1, k) + (n + 2 + k)\,DD(n + 1, 1 + k),\ (2n^4k \\
&\quad + 17nk + 5k + 21n^2k + 11n^3k - 5 - 38n^2 - 2n^5 - 22n - 32n^3 \\
&\quad - 13n^4)\,DD(n, k) + (-2n^4k - 41nk - 21k - 33n^2k - 13n^3k + 117 \\
&\quad + 346n^2 + 6n^5 + 318n + 190n^3 + 53n^4)\,DD(n + 1, k) + (-2n^4k \\
&\quad - 65k - 111nk - 69n^2k - 19n^3k - 686n^2 - 302n^3 - 782n - 355 \\
&\quad - 6n^5 - 67n^4)\,DD(n + 2, k) + (2n^4k + 81k + 135nk + 81n^2k \\
&\quad + 21n^3k + 378n^2 + 144n^3 + 486n + 243 + 2n^5 + 27n^4)\,DD(n + 3, k), \\
&\quad (3k^2 + 2k^2n - 2n^2k - 3nk + n^3 + 1 + 3n^2 + 3n)\,DD(n, k) + (3nk \\
&\quad - 3k^2 + 2n^2k - 2k^2n)\,DD(n, 1 + k) + (-3k^2 - 2k^2n + 2n^2k + 3nk \\
&\quad - 2n^3 - 9 - 9n^2 - 15n)\,DD(n + 1, k) + (3k^2 - 2n^2k - 3nk \\
&\quad + 2k^2n)\,DD(n + 1, 1 + k) + (n^3 + 6n^2 + 12n + 8)\,DD(n + 2, k)\}
\end{aligned} \qquad \textbf{(5.1.7)}$$

```
> sys_for_DD;
```

$$\begin{aligned}
&\{(k^3 + 3k^2 + 3k + 1)\,DD(n, k) + (-2k^3 - 8k^2 - 11k - 5 + nk + n^2k + 2n \\
&\quad + 2n^2)\,DD(n, 1 + k) + (k^3 + 5k^2 + 8k - nk - n^2k - 2n + 4 \\
&\quad - 2n^2)\,DD(n, 2 + k),\ (n - k)\,DD(n, k) + (-n + k)\,DD(n, 1 + k) + (-n - 2 \\
&\quad - k)\,DD(n + 1, k) + (n + 2 + k)\,DD(n + 1, 1 + k),\ (2n^4k + 17nk + 5k \\
&\quad + 21n^2k + 11n^3k - 5 - 38n^2 - 2n^5 - 22n - 32n^3 - 13n^4)\,DD(n, k) + (\\
&\quad -2n^4k - 41nk - 21k - 33n^2k - 13n^3k + 117 + 346n^2 + 6n^5 + 318n \\
&\quad + 190n^3 + 53n^4)\,DD(n + 1, k) + (-2n^4k - 65k - 111nk - 69n^2k \\
&\quad - 19n^3k - 686n^2 - 302n^3 - 782n - 355 - 6n^5 - 67n^4)\,DD(n + 2, k) \\
&\quad + (2n^4k + 81k + 135nk + 81n^2k + 21n^3k + 378n^2 + 144n^3 + 486n \\
&\quad + 243 + 2n^5 + 27n^4)\,DD(n + 3, k),\ (3k^2 + 2k^2n - 2n^2k - 3nk + n^3 + 1 \\
&\quad + 3n^2 + 3n)\,DD(n, k) + (3nk - 3k^2 + 2n^2k - 2k^2n)\,DD(n, 1 + k) + (\\
&\quad -3k^2 - 2k^2n + 2n^2k + 3nk - 2n^3 - 9 - 9n^2 - 15n)\,DD(n + 1, k) + (3k^2 \\
&\quad - 2n^2k - 3nk + 2k^2n)\,DD(n + 1, 1 + k) + (n^3 + 6n^2 + 12n + 8)\,DD(n \\
&\quad + 2, k)\}
\end{aligned} \qquad \textbf{(5.4)}$$

Second inner indefinite binomial sum: $iz3(n,k) := \sum_{m=1}^{n} \frac{1}{m^3}$

We use a similar process, but simplified because of the simpler summand: some of the operations can be done "by hand."

```
> dfinite_expr_to_sys(1/(n + 1)^3, u(n::shift, k::shift));
```

$$\{(-n^3-1-3n^2-3n)\,u(n,k)+(n^3+6n^2+12n+8)\,u(n+1,k),\ -u(n,k)+u(n,1+k)\} \quad (5.2.1)$$

```
> sys_for_iz3 := eval(%, u = unapply(iz3(n + 1, k) - iz3(n, k), n, k)) union {iz3(n, k + 1) - iz3(n, k)};
```

$$sys\_for\_iz3 := \{(-n^3-1-3n^2-3n)\,(iz3(n+1,k)-iz3(n,k))+(n^3+6n^2+12n+8)\,(iz3(n+2,k)-iz3(n+1,k)),\ iz3(n,1+k)-iz3(n,k),\ -iz3(n+1,k)+iz3(n,k)+iz3(n+1,1+k)-iz3(n,1+k)\} \quad (5.2.2)$$

```
> sys_for_iz3;
```

$$\{(-n^3-1-3n^2-3n)\,(iz3(n+1,k)-iz3(n,k))+(n^3+6n^2+12n+8)\,(iz3(n+2,k)-iz3(n+1,k)),\ iz3(n,1+k)-iz3(n,k),\ -iz3(n+1,k)+iz3(n,k)+iz3(n+1,1+k)-iz3(n,1+k)\} \quad (5.5)$$

```
> cofactor := binomial(n, k)^2 * binomial(n + k, k)^2;
```

$$cofactor := \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (5.6)$$

System for the summand of the outer sum: $cofactor \cdot (DD(n,k)+iz3(n,k))$

$$= \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m+1}}{2\,m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

Now, it would be great just to put the two systems side by side in a call, but the routine requires the same name for both systems; so the following makes an error:

```
> `sys+sys`(sys_for_DD, sys_for_iz3);
Error, (in recognize_operator_algebra) multiple functions not dealt with yet
```

Therefore we help Mgfun to add the two sums:

```
> sys_for_2_indef_sums := `sys+sys`(eval(sys_for_DD, DD = u), eval(sys_for_iz3, iz3 = u));
```

$$sys\_for\_2\_indef\_sums := \{(k^3+3k^2+3k+1)\,u(n,k)+(-2k^3-8k^2-11k-5+nk+n^2k+2n+2n^2)\,u(n,1+k)+(k^3+5k^2+8k-nk \quad (5.3.1)$$

$- n^2 k - 2 n + 4 - 2 n^2) u(n, 2 + k), (n - k) u(n, k) + (-n + k) u(n, 1 + k) + (-n - 2 - k) u(n + 1, k) + (n + 2 + k) u(n + 1, 1 + k), (2 + 7 n + k + 6 k^2 + 9 n^2 - 3 n k - 2 k^2 n - 4 k^2 n^2 + 4 k^3 n + 6 k^3 + n^3 k + n^4 - n^2 k + 5 n^3) u(n, k) + (-6 k^3 - 6 k^2 + 4 k^2 n^2 - 4 k^3 n + 6 n k + 4 n^2 k + 2 k^2 n) u(n, 1 + k) + (-18 - 39 n - 9 k - 13 n^3 - 33 n^2 - 2 n^4 - 2 n^3 k - 9 n^2 k - 15 n k) u(n + 1, k) + (n^4 + 8 n^3 + n^3 k + 24 n^2 + 6 n^2 k + 32 n + 12 n k + 16 + 8 k) u(n + 2, k)\}$

Next, we prepare a system for the left-hand factor of the product:

```
> sys_for_cofactor := dfinite_expr_to_sys(cofactor, u(n::shift, k::shift));
```

$$sys\_for\_cofactor := \{(-k^2 - 2 k - 2 n k - 1 - 2 n - n^2) u(n, k) + (n^2 + 2 n - 2 n k + 1 - 2 k + k^2) u(n + 1, k), (-n^4 - 2 n^3 - n^2 + 2 n^2 k + 2 k^2 n^2 + 2 n k + 2 k^2 n - k^2 - 2 k^3 - k^4) u(n, k) + (4 k^3 + 6 k^2 + 4 k + 1 + k^4) u(n, 1 + k)\} \quad \textbf{(5.3.2)}$$

And we do the multiplication:

```
> sys_for_summand := `sys*sys`(sys_for_cofactor, sys_for_2_indef_sums);
```

$$sys\_for\_summand := \{(k^6 + 5 k^5 + 9 k^4 - 3 k^4 n^2 - 3 k^4 n + 7 k^3 - 10 k^3 n - 10 k^3 n^2 + 3 k^2 n^4 - 11 k^2 n + 2 k^2 - 8 k^2 n^2 + 6 k^2 n^3 - 4 n k + 10 n^3 k + 5 n^4 k + n^2 k + 2 n^2 - 3 n^5 + 3 n^3 - n^6 - n^4) u(n, k) + (3 k^4 n^2 - 2 n^4 - 4 n^3 - 2 k^6 - 6 n^3 k - 16 k^5 - 53 k^4 - 3 n^4 k + 3 k^4 n - 2 k^2 n^3 + 16 k^3 n^2 - 10 + 9 n - 47 k - 93 k^3 - 91 k^2 + 7 n^2 + 31 k^2 n^2 + 16 k^3 n - k^2 n^4 + 28 n k + 25 n^2 k + 32 k^2 n) u(n, 1 + k) + (k^6 + 11 k^5 + 50 k^4 + 120 k^3 + 160 k^2 + 112 k + 32) u(n, 2 + k), (8 k^2 n^2 - 5 k^3 + 5 n^2 k - 5 n^3 + 2 k^3 n^2 - 2 k^2 - n^5 + 4 n k + 5 k^2 n - k^5 - 2 n^2 - 4 k^4 + 2 k^2 n^3 - n^4 k - k^4 n - 4 n^4) u(n, k) + (-k^4 n - 4 k^3 n - 4 n k - 6 k^2 n + k^5 + 4 k^4 + 6 k^3 + 4 k^2 - n + k) u(n + 1, 1 + k) + (k^5 + 9 k + k^4 n + 16 k^2 + 4 n k + 2 + n + 6 k^4 + 14 k^3 + 6 k^2 n + 4 k^3 n) u(n, 1 + k) + (8 k^3 n - n^4 k - 4 k + 4 n + 12 n^2 - 4 k^2 n^2 + 13 n^3 - 8 n^3 k + 6 n^4 - 2 k^2 n^3 - k^5 + 2 k^3 n^2 + k^4 n - 2 k^4 - 19 n^2 k - 16 n k + 3 k^3 + 4 k^2 + n^5 + 3 k^2 n) u(n + 1, k), (-n^8 - 4 n + 4 k - 9 n^7 + 12 k^2 - 24 n^2 + 12 n k + 25 k^2 n + 41 k^2 n^2 + 5 k^3 n + 25 k^3 + 48 k^4 - 9 n^3 k + 30 k^6 + 6 k^7 + 4 k^6 n^2 + 26 k^6 n + 4 k^7 n + 20 k^2 n^5 + 16 k^3 n^4 - 17 n^6 k + 2 k^2 n^6 + 6 k^3 n^5 - 3 n^7 k - 53 k^4 n^2 - 37 k^4 n^3 + 45 k^5 n - 5 k^5 n^2 - 5 k^4 n^4 - 7 k^5 n^3 + 55 k^5 - 85 n^4 + 67 k^2 n^3 - 37 k^3 n^2 + 19 k^4 n + 11 n^2 k + 57 k^2 n^4 - 9 k^3 n^3 - 36 n^5 k - 70 n^5 \quad \textbf{(5.3.3)}$$

$$
\begin{aligned}
&-34\,n^6-34\,n^4\,k-61\,n^3)\;u(n,k)+(-110\,k^5\,n-40\,k^4\,n^2-6\,k^7-42\,k^6\\
&-120\,k^5-180\,k^4-180\,k^4\,n-40\,k^3\,n^2-12\,k-150\,k^3-66\,k^2-4\,k^7\,n\\
&-4\,k^6\,n^2-34\,k^6\,n-20\,k^5\,n^2-20\,k^2\,n^2-160\,k^3\,n-14\,n\,k-4\,n^2\,k\\
&-74\,k^2\,n)\;u(n,1+k)+(2\,n^8+36\,n-36\,k+21\,n^7+36\,k^2+168\,n^2\\
&-204\,n\,k+87\,k^2\,n+45\,k^2\,n^2+117\,k^3\,n+27\,k^3-18\,k^4-509\,n^3\,k\\
&-26\,k^2\,n^5+34\,k^3\,n^4-25\,n^6\,k-4\,k^2\,n^6+4\,k^3\,n^5-2\,n^7\,k-3\,k^4\,n^2\\
&+5\,k^4\,n^3-15\,k^5\,n-9\,k^5\,n^2+2\,k^4\,n^4-2\,k^5\,n^3-9\,k^5+365\,n^4-43\,k^2\,n^3\\
&+165\,k^3\,n^2-21\,k^4\,n-447\,n^2\,k-60\,k^2\,n^4+108\,k^3\,n^3-125\,n^5\,k\\
&+240\,n^5+95\,n^6-332\,n^4\,k+333\,n^3)\;u(n+1,k)+(-n^8-32\,n+32\,k\\
&-12\,n^7-96\,k^2-144\,n^2+240\,n\,k-456\,k^2\,n-828\,k^2\,n^2+348\,k^3\,n\\
&+104\,k^3-48\,k^4+808\,n^3\,k-96\,k^2\,n^5+84\,k^3\,n^4+54\,n^6\,k-10\,k^2\,n^6\\
&+10\,k^3\,n^5+5\,n^7\,k-96\,k^4\,n^2-36\,k^4\,n^3+12\,k^5\,n+6\,k^5\,n^2-5\,k^4\,n^4\\
&+k^5\,n^3+8\,k^5-280\,n^4-758\,k^2\,n^3+446\,k^3\,n^2-112\,k^4\,n+624\,n^2\,k\\
&-375\,k^2\,n^4+277\,k^3\,n^3+243\,n^5\,k-170\,n^5-61\,n^6+586\,n^4\,k-272\,n^3)\\
&u(n+2,k)\}
\end{aligned}
$$

> sys_for_summand;

$$
\begin{aligned}
&\{((k^6+5\,k^5+9\,k^4-3\,k^4\,n^2-3\,k^4\,n+7\,k^3-10\,k^3\,n-10\,k^3\,n^2+3\,k^2\,n^4-11\,k^2\,n\\
&+2\,k^2-8\,k^2\,n^2+6\,k^2\,n^3-4\,n\,k+10\,n^3\,k+5\,n^4\,k+n^2\,k+2\,n^2-3\,n^5+3\,n^3\\
&-n^6-n^4)\;u(n,k)+(3\,k^4\,n^2-2\,n^4-4\,n^3-2\,k^6-6\,n^3\,k-16\,k^5-53\,k^4\\
&-3\,n^4\,k+3\,k^4\,n-2\,k^2\,n^3+16\,k^3\,n^2-10+9\,n-47\,k-93\,k^3-91\,k^2+7\,n^2\\
&+31\,k^2\,n^2+16\,k^3\,n-k^2\,n^4+28\,n\,k+25\,n^2\,k+32\,k^2\,n)\;u(n,1+k)+(k^6\\
&+11\,k^5+50\,k^4+120\,k^3+160\,k^2+112\,k+32)\;u(n,2+k),\;(8\,k^2\,n^2-5\,k^3\\
&+5\,n^2\,k-5\,n^3+2\,k^3\,n^2-2\,k^2-n^5+4\,n\,k+5\,k^2\,n-k^5-2\,n^2-4\,k^4\\
&+2\,k^2\,n^3-n^4\,k-k^4\,n-4\,n^4)\;u(n,k)+(-k^4\,n-4\,k^3\,n-4\,n\,k-6\,k^2\,n+k^5\\
&+4\,k^4+6\,k^3+4\,k^2-n+k)\;u(n+1,1+k)+(k^5+9\,k+k^4\,n+16\,k^2\\
&+4\,n\,k+2+n+6\,k^4+14\,k^3+6\,k^2\,n+4\,k^3\,n)\;u(n,1+k)+(8\,k^3\,n-n^4\,k\\
&-4\,k+4\,n+12\,n^2-4\,k^2\,n^2+13\,n^3-8\,n^3\,k+6\,n^4-2\,k^2\,n^3-k^5+2\,k^3\,n^2\\
&+k^4\,n-2\,k^4-19\,n^2\,k-16\,n\,k+3\,k^3+4\,k^2+n^5+3\,k^2\,n)\;u(n+1,k),\;(-n^8\\
&-4\,n+4\,k-9\,n^7+12\,k^2-24\,n^2+12\,n\,k+25\,k^2\,n+41\,k^2\,n^2+5\,k^3\,n\\
&+25\,k^3+48\,k^4-9\,n^3\,k+30\,k^6+6\,k^7+4\,k^6\,n^2+26\,k^6\,n+4\,k^7\,n+20\,k^2\,n^5\\
&+16\,k^3\,n^4-17\,n^6\,k+2\,k^2\,n^6+6\,k^3\,n^5-3\,n^7\,k-53\,k^4\,n^2-37\,k^4\,n^3\\
&+45\,k^5\,n-5\,k^5\,n^2-5\,k^4\,n^4-7\,k^5\,n^3+55\,k^5-85\,n^4+67\,k^2\,n^3-37\,k^3\,n^2\\
&+19\,k^4\,n+11\,n^2\,k+57\,k^2\,n^4-9\,k^3\,n^3-36\,n^5\,k-70\,n^5-34\,n^6-34\,n^4\,k
\end{aligned}
\qquad \textbf{(5.7)}
$$

$-61\,n^3)\; u(n, k) + (-110\,k^5\,n - 40\,k^4\,n^2 - 6\,k^7 - 42\,k^6 - 120\,k^5 - 180\,k^4$
$- 180\,k^4\,n - 40\,k^3\,n^2 - 12\,k - 150\,k^3 - 66\,k^2 - 4\,k^7\,n - 4\,k^6\,n^2 - 34\,k^6\,n$
$- 20\,k^5\,n^2 - 20\,k^2\,n^2 - 160\,k^3\,n - 14\,n\,k - 4\,n^2\,k - 74\,k^2\,n)\; u(n, 1+k)$
$+ (2\,n^8 + 36\,n - 36\,k + 21\,n^7 + 36\,k^2 + 168\,n^2 - 204\,n\,k + 87\,k^2\,n + 45\,k^2\,n^2$
$+ 117\,k^3\,n + 27\,k^3 - 18\,k^4 - 509\,n^3\,k - 26\,k^2\,n^5 + 34\,k^3\,n^4 - 25\,n^6\,k$
$- 4\,k^2\,n^6 + 4\,k^3\,n^5 - 2\,n^7\,k - 3\,k^4\,n^2 + 5\,k^4\,n^3 - 15\,k^5\,n - 9\,k^5\,n^2 + 2\,k^4\,n^4$
$- 2\,k^5\,n^3 - 9\,k^5 + 365\,n^4 - 43\,k^2\,n^3 + 165\,k^3\,n^2 - 21\,k^4\,n - 447\,n^2\,k$
$- 60\,k^2\,n^4 + 108\,k^3\,n^3 - 125\,n^5\,k + 240\,n^5 + 95\,n^6 - 332\,n^4\,k + 333\,n^3)\; u(n$
$+ 1, k) + (-n^8 - 32\,n + 32\,k - 12\,n^7 - 96\,k^2 - 144\,n^2 + 240\,n\,k - 456\,k^2\,n$
$- 828\,k^2\,n^2 + 348\,k^3\,n + 104\,k^3 - 48\,k^4 + 808\,n^3\,k - 96\,k^2\,n^5 + 84\,k^3\,n^4$
$+ 54\,n^6\,k - 10\,k^2\,n^6 + 10\,k^3\,n^5 + 5\,n^7\,k - 96\,k^4\,n^2 - 36\,k^4\,n^3 + 12\,k^5\,n$
$+ 6\,k^5\,n^2 - 5\,k^4\,n^4 + k^5\,n^3 + 8\,k^5 - 280\,n^4 - 758\,k^2\,n^3 + 446\,k^3\,n^2 - 112\,k^4\,n$
$+ 624\,n^2\,k - 375\,k^2\,n^4 + 277\,k^3\,n^3 + 243\,n^5\,k - 170\,n^5 - 61\,n^6 + 586\,n^4\,k$
$- 272\,n^3)\; u(n+2, k)\}$

Fourth-order recurrence for b_n

Here, we use the command creative_telescoping by providing a system (40 seconds):

```
> ct_for_double_sum := creative_telescoping(LFSol
  (sys_for_summand), n::shift, k::shift):
skew_poly_creative_telescoping: PROFILE - DIMENSION 3
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: Test operator P of order 3
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: Test operator P of order 4
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
```

skew_poly_creative_telescoping: j = 3
skew_poly_creative_telescoping: PROFILE - LAST_D 4
creative_telescoping: Start to reconstruct rhs operators.

Read off the recurrence for the sum:

```
> rec_for_double_sum := collect(ct_for_double_sum[1][1], _F, factor);
```

$$rec\_for\_double\_sum := (2n+7)(12n^4+144n^3+643n^2+1266n+928)(n+1)^6\,\_F(n) - (3+2n)(2n+7)(408n^9+7956n^8+68086n^7+336284n^6+1058890n^5+2209767n^4+3063206n^3+2724789n^2+1413006n+325664)\,\_F(n+1) + (2n+5)(13896n^{10}+347400n^9+3868998n^8+25269960n^7+107159724n^6+308199360n^5+608681313n^4+814935630n^3+707785777n^2+360083510n+81495208)\,\_F(n+2) - (3+2n)(2n+7)(408n^9+10404n^8+117046n^7+761526n^6+3153520n^5+8607233n^4+15461616n^3+17602001n^2+11509566n+3291016)\,\_F(n+3) + (3+2n)(12n^4+96n^3+283n^2+364n+173)(n+4)^6\,\_F(n+4) \quad \textbf{(5.4.1)}$$

```
> rec_for_double_sum;
```

$$(2n+7)(12n^4+144n^3+643n^2+1266n+928)(n+1)^6\,\_F(n) - (3+2n)(2n+7)(408n^9+7956n^8+68086n^7+336284n^6+1058890n^5+2209767n^4+3063206n^3+2724789n^2+1413006n+325664)\,\_F(n+1) + (2n+5)(13896n^{10}+347400n^9+3868998n^8+25269960n^7+107159724n^6+308199360n^5+608681313n^4+814935630n^3+707785777n^2+360083510n+81495208)\,\_F(n+2) - (3+2n)(2n+7)(408n^9+10404n^8+117046n^7+761526n^6+3153520n^5+8607233n^4+15461616n^3+17602001n^2+11509566n+3291016)\,\_F(n+3) + (3+2n)(12n^4+96n^3+283n^2+364n+173)(n+4)^6\,\_F(n+4) \quad \textbf{(5.8)}$$

Second-order recurrence for a_n

```
> ct_for_single_sum := creative_telescoping(cofactor, n::shift, k::shift);
```

skew_poly_creative_telescoping: PROFILE - DIMENSION 1
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j = 1

```
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: PROFILE - LAST_D 2
creative_telescoping: Start to reconstruct rhs operators.
```

$$ct\_for\_single\_sum := \left[\left[(n^3+1+3n^2+3n)\ \_F(n) + (-153n^2-34n^3-231n-117)\ \_F(n+1) + (n^3+6n^2+12n+8)\ \_F(n+2), \frac{4k^4(-36n^2-52n-24-9k+6k^2-8n^3-6nk+4k^2n)\ \_f(n,k)}{(n^2+2n-2nk+1-2k+k^2)(k^2-2nk-4k+4+n^2+4n)}\right]\right] \quad (5.5.1)$$

```
> rec_for_single_sum := collect(ct_for_single_sum[1][1], _F, factor);
```

$$rec\_for\_single\_sum := (n+1)^3\ \_F(n) - (17n^2+51n+39)(3+2n)\ \_F(n+1) + (n+2)^3\ \_F(n+2) \quad (5.5.2)$$

```
> rec_for_single_sum;
```

$$(n+1)^3\ \_F(n) - (17n^2+51n+39)(3+2n)\ \_F(n+1) + (n+2)^3\ \_F(n+2) \quad (5.9)$$

Prove that b_n also satisfies the second-order recurrence

Observe that a solution of the 4th-order recurrence is fully determined by its first four values, as its leading coefficient never vanishes on natural integers:

```
> factor(coeff(rec_for_double_sum, _F(n + 4)));
```

$$(3+2n)(12n^4+96n^3+283n^2+364n+173)(n+4)^6 \quad (5.6.1)$$

Therefore, we can introduce the sequence starting by b_0, b_1, b_2, b_3 and prolonged by unrolling the second-order recurrence. If we show that it satisfies the fourth-order recurrence, it will be exactly the sequence (b_n). This will show that the latter also satisfies the second-order recurrence.

```
> _F(n + 2) = solve(rec_for_single_sum, _F(n + 2));
```

$$\_F(n+2) = -\frac{1}{(n+2)^3}\left(\_F(n)\,n^3 + \_F(n) + 3\,\_F(n)\,n^2 + 3\,\_F(n)\,n - 153\,\_F(n+1)\,n^2 - 34\,\_F(n+1)\,n^3 - 231\,\_F(n+1)\,n - 117\,\_F(n+1)\right) \quad (5.6.2)$$

```
> collect(eval(%, n = n - 2), _F, factor);
```

$$\_F(n) = -\frac{(n-1)^3\ \_F(n-2)}{n^3} + \frac{(-1+2n)(17n^2-17n+5)\ \_F(n-1)}{n^3} \quad (5.6.3)$$

```
> simpl := subs(a = %, n -> a);
```

$$simpl := n \mapsto \_F(n) = -\frac{(n-1)^3\ \_F(n-2)}{n^3} + \frac{(-1+2\,n)\,(17\,n^2-17\,n+5)\ \_F(n-1)}{n^3} \qquad \textbf{(5.6.4)}$$

```
> eval(rec_for_double_sum, simpl(n + 4));
```

$$(2\,n+7)\,(12\,n^4+144\,n^3+643\,n^2+1266\,n+928)\,(n+1)^6\ \_F(n) - (3+2\,n)\,(2\,n+7)\,(408\,n^9+7956\,n^8+68086\,n^7+336284\,n^6+1058890\,n^5+2209767\,n^4+3063206\,n^3+2724789\,n^2+1413006\,n+325664)\ \_F(n+1) + (2\,n+5)\,(13896\,n^{10}+347400\,n^9+3868998\,n^8+25269960\,n^7+107159724\,n^6+308199360\,n^5+608681313\,n^4+814935630\,n^3+707785777\,n^2+360083510\,n+81495208)\ \_F(n+2) - (3+2\,n)\,(2\,n+7)\,(408\,n^9+10404\,n^8+117046\,n^7+761526\,n^6+3153520\,n^5+8607233\,n^4+15461616\,n^3+17602001\,n^2+11509566\,n+3291016)\ \_F(n+3) + (3+2\,n)\,(12\,n^4+96\,n^3+283\,n^2+364\,n+173)\,(n+4)^6\left(-\frac{(n+3)^3\ \_F(n+2)}{(n+4)^3} + \frac{(2\,n+7)\,(17\,(n+4)^2-17\,n-63)\ \_F(n+3)}{(n+4)^3}\right) \qquad \textbf{(5.6.5)}$$

```
> eval(%, simpl(n + 3));
```

$$(2\,n+7)\,(12\,n^4+144\,n^3+643\,n^2+1266\,n+928)\,(n+1)^6\ \_F(n) - (3+2\,n)\,(2\,n+7)\,(408\,n^9+7956\,n^8+68086\,n^7+336284\,n^6+1058890\,n^5+2209767\,n^4+3063206\,n^3+2724789\,n^2+1413006\,n+325664)\ \_F(n+1) + (2\,n+5)\,(13896\,n^{10}+347400\,n^9+3868998\,n^8+25269960\,n^7+107159724\,n^6+308199360\,n^5+608681313\,n^4+814935630\,n^3+707785777\,n^2+360083510\,n+81495208)\ \_F(n+2) - (3+2\,n)\,(2\,n+7)\,(408\,n^9+10404\,n^8+117046\,n^7+761526\,n^6+3153520\,n^5+8607233\,n^4+15461616\,n^3+17602001\,n^2+11509566\,n+3291016)\left(-\frac{(n+2)^3\ \_F(n+1)}{(n+3)^3} + \frac{(2\,n+5)\,(17\,(n+3)^2-17\,n-46)\ \_F(n+2)}{(n+3)^3}\right) + (3+2\,n)\,(12\,n^4+96\,n^3+283\,n^2+364\,n+173)\,(n+4)^6\left(-\frac{(n+3)^3\ \_F(n+2)}{(n+4)^3}\right. \qquad \textbf{(5.6.6)}$$

$$+\frac{1}{(n+4)^3}\left((2n+7)\,(17\,(n+4)^2-17\,n-63)\,\left(-\frac{(n+2)^3\,\_F(n+1)}{(n+3)^3}+\frac{(2\,n+5)\,(17\,(n+3)^2-17\,n-46)\,\_F(n+2)}{(n+3)^3}\right)\right)\Bigg)$$

```
> normal(eval(%, simpl(n + 2)));
```

$$0 \tag{5.6.7}$$

Computing $\int_0^{\infty} x^a\, e^{-bx}\, K_m(cx)\, K_n(dx)\, dx$

```
> _F(a) = Int(_f(a, x), x = 0..infinity);
```

$$\_F(a)=\int_0^{\infty}\_f(a,x)\,dx \tag{6.1}$$

for:

```
> f := x^a*exp(-b*x)*BesselK(m,c*x)*BesselK(n,d*x);
```

$$f := x^a\,e^{-bx}\,K_m(c\,x)\,K_n(d\,x) \tag{6.2}$$

under the knowledge:

(close to 0)

```
> MultiSeries:-series(BesselK(nu, x), x = 0, 3) assuming nu >
  5;
```

$$\Gamma(\nu)\,e^{\ln(2)\,(\nu-1)}\,x^{-\nu}-\frac{\Gamma(\nu)\,e^{\ln(2)\,(\nu-1)}\,x^{2-\nu}}{4\,(\nu-1)}+O\left(x^{4-\nu}\right) \tag{6.3}$$

(close to infinity)

```
> MultiSeries:-asympt(BesselK(nu, x), x, 1);
```

$$\frac{\frac{\sqrt{2}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}}{2}+O\left(\left(\frac{1}{x}\right)^{3/2}\right)}{e^x} \tag{6.4}$$

The following will be used to display results more nicely.

```
> pnice := proc(expr, $) local l := [a, b, c, d, m, n]; lcoeff
  (expr, l) * convert(map((v, e) -> v^degree(e, v), l, expr),
  `*`) + `...` end proc:
> rnice := proc(expr, $) local de := denom(expr); pnice(numer
  (expr)) / `if`(de = 1, 1, pnice(de)) end proc:
> nice := proc(expr, $) collect(expr, {_F, _f, diff}, rnice)
```

```
end proc:
```

"Creative telescoping" (15 seconds):

```
> res := creative_telescoping(f, [a::shift], x::diff):
skew_poly_creative_telescoping: PROFILE - DIMENSION 4
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: Test operator P of order 3
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: Test operator P of order 4
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: PROFILE - LAST_D 4
creative_telescoping: Start to reconstruct rhs operators.
> P_applied_to_F := res[1][1]:
> cPF := collect(P_applied_to_F, _F, nice);
```

$$cPF := \left(a^6 b^2 c^2 d^2 m^6 n^6 + \ldots\right) \_F(a) + \left(-4\, a^5 b^3 c^2 d^2 m^4 n^4 + \ldots\right) \_F(a+1) + \left(6\, a^4 b^4 c^4 d^4 m^4 n^4 + \ldots\right) \_F(a+2) + \left(-4\, a^3 b^5 c^4 d^4 m^2 n^2 + \ldots\right) \_F(a+3) + \left(a^2 b^6 c^6 d^6 m^2 n^2 + \ldots\right) \_F(a+4) \tag{6.5}$$

```
> Q_applied_to_f := res[1][2]:
> cQf := collect(Q_applied_to_f, _f, nice);
```

$$cQf := \frac{\left(4\, x^3 a^5 b^5 c^6 d^6 m^6 n^6 + \ldots\right) \_f(a, x)}{c^2 n^2 m^2 d^2 x^2 + \ldots} + \frac{\left(-6\, x^4 a^4 b^4 c^6 d^6 m^6 n^6 + \ldots\right) \frac{\partial}{\partial x} \_f(a, x)}{c^2 n^2 m^2 d^2 x^2 + \ldots} \tag{6.6}$$

$$+\frac{\left(4\,x^{5}\,a^{3}\,b^{3}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\frac{\partial^{2}}{\partial x^{2}}\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}$$

$$+\frac{\left(-x^{6}\,a^{2}\,b^{2}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\frac{\partial^{3}}{\partial x^{3}}\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}$$

Meaning:

```
> eval(cPF, _F = unapply(_f(a, x), a)) = Diff( cQf, x );
```

$$\left(a^{6}\,b^{2}\,c^{2}\,d^{2}\,m^{6}\,n^{6}+\ldots\right)\,\_f(a,x)+\left(-4\,a^{5}\,b^{3}\,c^{2}\,d^{2}\,m^{4}\,n^{4}+\ldots\right)\,\_f(a+1,x)+\left(6\,a^{4}\,b^{4}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\_f(a+2,x)+\left(-4\,a^{3}\,b^{5}\,c^{4}\,d^{4}\,m^{2}\,n^{2}+\ldots\right)\,\_f(a+3,x)+\left(a^{2}\,b^{6}\,c^{6}\,d^{6}\,m^{2}\,n^{2}+\ldots\right)\,\_f(a+4,x)=\frac{\partial}{\partial x}\left(\frac{\left(4\,x^{3}\,a^{5}\,b^{5}\,c^{6}\,d^{6}\,m^{6}\,n^{6}+\ldots\right)\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}+\frac{\left(-6\,x^{4}\,a^{4}\,b^{4}\,c^{6}\,d^{6}\,m^{6}\,n^{6}+\ldots\right)\,\frac{\partial}{\partial x}\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}+\frac{\left(4\,x^{5}\,a^{3}\,b^{3}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\frac{\partial^{2}}{\partial x^{2}}\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}+\frac{\left(-x^{6}\,a^{2}\,b^{2}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\frac{\partial^{3}}{\partial x^{3}}\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots}\right) \tag{6.7}$$

(now in terms of example)

```
> eval(eval(P_applied_to_F, _F = unapply(_f(a, x), a)) = diff(
  Q_applied_to_f, x ), _f = unapply(f, a, x)):
> simplify( lhs(%) / rhs(%) );
```

$$1 \tag{6.8}$$

After integration:

```
> cPF = Limit(cQf, x = infinity) - Limit(cQf, x = 0);
```

$$\left(a^{6}\,b^{2}\,c^{2}\,d^{2}\,m^{6}\,n^{6}+\ldots\right)\,\_F(a)+\left(-4\,a^{5}\,b^{3}\,c^{2}\,d^{2}\,m^{4}\,n^{4}+\ldots\right)\,\_F(a+1)+\left(6\,a^{4}\,b^{4}\,c^{4}\,d^{4}\,m^{4}\,n^{4}+\ldots\right)\,\_F(a+2)+\left(-4\,a^{3}\,b^{5}\,c^{4}\,d^{4}\,m^{2}\,n^{2}+\ldots\right)\,\_F(a+3)+\left(a^{2}\,b^{6}\,c^{6}\,d^{6}\,m^{2}\,n^{2}+\ldots\right)\,\_F(a+4)=\lim_{x\to\infty}\left(\frac{\left(4\,x^{3}\,a^{5}\,b^{5}\,c^{6}\,d^{6}\,m^{6}\,n^{6}+\ldots\right)\,\_f(a,x)}{c^{2}\,n^{2}\,m^{2}\,d^{2}\,x^{2}+\ldots} \tag{6.9}$$

$$+\frac{\left(-6\,x^4\,a^4\,b^4\,c^6\,d^6\,m^6\,n^6+\ldots\right)\frac{\partial}{\partial x}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}$$

$$+\frac{\left(4\,x^5\,a^3\,b^3\,c^4\,d^4\,m^4\,n^4+\ldots\right)\frac{\partial^2}{\partial x^2}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}$$

$$+\left.\frac{\left(-x^6\,a^2\,b^2\,c^4\,d^4\,m^4\,n^4+\ldots\right)\frac{\partial^3}{\partial x^3}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}\right) -$$

$$\lim_{x\to 0}\left(\frac{\left(4\,x^3\,a^5\,b^5\,c^6\,d^6\,m^6\,n^6+\ldots\right)\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}\right.$$

$$+\frac{\left(-6\,x^4\,a^4\,b^4\,c^6\,d^6\,m^6\,n^6+\ldots\right)\frac{\partial}{\partial x}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}$$

$$+\frac{\left(4\,x^5\,a^3\,b^3\,c^4\,d^4\,m^4\,n^4+\ldots\right)\frac{\partial^2}{\partial x^2}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}$$

$$+\left.\frac{\left(-x^6\,a^2\,b^2\,c^4\,d^4\,m^4\,n^4+\ldots\right)\frac{\partial^3}{\partial x^3}\,\_f(a,x)}{c^2\,n^2\,m^2\,d^2\,x^2+\ldots}\right)$$

Inhomogeneous term is exponentiallyl small at infinity:

```
> MultiSeries:-asympt(f, x, 2) assuming b > 0 and c > 0 and d >
  0 ;
```

$$\left(\frac{\pi\left(\frac{1}{x}\right)^{1-a}}{2\sqrt{c}\sqrt{d}}+\frac{\pi\left(4\,d\,m^2-d+4\,c\,n^2-c\right)\left(\frac{1}{x}\right)^{2-a}}{16\,c^{3/2}\,d^{3/2}}+\mathrm{O}\left(\left(\frac{1}{x}\right)^{3-a}\right)\right)\left(\frac{1}{e^{b\,x}}\right)^{\frac{d}{b}+\frac{c}{b}+1} \qquad \textbf{(6.10)}$$

Inhomogeneous term is 0 at 0 for large enough a (a few seconds):

```
> MultiSeries:-series(eval(Q_applied_to_f, _f = unapply(f, a,
  x)) / x^(a-m-n), x = 0, 5) assuming m > 5 and n > 5:
> x^(a-m-n) * collect(convert(%, polynom), x, e -> `...`);
```

$$x^{a-m-n}\left(\ldots x^5+\ldots x^4+\ldots x^3+\ldots x^2+\ldots x\right) \qquad \textbf{(6.11)}$$

Internally, what has been obtained first is a system:

```
> map(nice, dfinite_expr_to_sys(f, _f(a::shift, x::diff)));
```

$$\left\{(-x+\ldots)\,\_f(a,x)+(1+\ldots)\,\_f(a+1,x),\ (x^2a^4b^4c^6d^6m^6n^6+\ldots)\,\_f(a,x)+(-4x^3a^3b^3c^4d^4m^4n^4+\ldots)\,\frac{\partial}{\partial x}\_f(a,x)+(6x^4a^2b^2c^4d^4m^4n^4+\ldots)\,\frac{\partial^2}{\partial x^2}\_f(a,x)+(-4x^5abc^2d^2m^2n^2+\ldots)\,\frac{\partial^3}{\partial x^3}\_f(a,x)+(x^6c^2d^2m^2n^2+\ldots)\,\frac{\partial^4}{\partial x^4}\_f(a,x)\right\} \quad (6.12)$$

ODE for $\int_0^\infty\int_0^\infty J_1(x)J_1(y)J_2(c\sqrt{xy})\,\mathrm{d}y\,\mathrm{d}x$

Interested in:

```
> _F2(c) = Int(Int(_f2(c, x, y), y = 0..infinity), x = 0..
  infinity);
```

$$\_F2(c)=\int_0^\infty\int_0^\infty \_f2(c,x,y)\,\mathrm{d}y\,\mathrm{d}x \quad (7.1)$$

```
> f2 := BesselJ(1, x) * BesselJ(1, y) * BesselJ(2, c * sqrt(x*
  y)) / exp(x + y);
```

$$f2 := \frac{J_1(x)\,J_1(y)\,J_2(c\sqrt{x\,y})}{e^{x+y}} \quad (7.2)$$

```
> res2 := creative_telescoping(f2, c::diff, [x::diff, y::diff])
  :
```

```
skew_poly_creative_telescoping: PROFILE - DIMENSION 8
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  8
skew_poly_creative_telescoping: j =  7
skew_poly_creative_telescoping: j =  6
skew_poly_creative_telescoping: j =  5
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  5
skew_poly_creative_telescoping: j =  6
```

```
skew_poly_creative_telescoping: j =  7
skew_poly_creative_telescoping: j =  8
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  5
skew_poly_creative_telescoping: j =  6
skew_poly_creative_telescoping: j =  7
skew_poly_creative_telescoping: j =  8
skew_poly_creative_telescoping: PROFILE - LAST_D 2
skew_poly_creative_telescoping: PROFILE - DIMENSION 4
skew_poly_creative_telescoping: Start uncoupling system.
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: Test operator P of order 1
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: Test operator P of order 2
skew_poly_creative_telescoping: j =  1
skew_poly_creative_telescoping: j =  2
skew_poly_creative_telescoping: j =  3
skew_poly_creative_telescoping: j =  4
skew_poly_creative_telescoping: PROFILE - LAST_D 2
creative_telescoping: Start to reconstruct rhs operators.
```

The local identity obtained by creative telescoping is the following:

```
> subs(_F(c) = _f(c, x, y), res2[1][1]) = Diff(res2[1][2], x) +
  Diff(res2[1][3], y);
```

$$(-5120\,c^2+16384+16\,c^6-256\,c^4)\,\_f(c,x,y)+(-4096\,c^2+c^{10})\,\frac{\partial^2}{\partial c^2}\,\_f(c,x,y)+(-2048\,c^3-256\,c^5+5\,c^9-4096\,c-32\,c^7)\,\frac{\partial}{\partial c}\,\_f(c,x,y)=\frac{\partial}{\partial x}\Big(4\,(-128\,c^2+1024+c^6\,x-128\,c^2\,x-512\,x+2\,x^2\,c^6-128\,c^2\,x^2)\,c^2\,\_f(c,x,y)-4\,(32\,c^2+256+c^6\,x-16\,c^4\,x+64\,c^2\,x)\,c^2\,x\,\frac{\partial}{\partial x}\,\_f(c,x,y)+4\,(256+c^6\,x-512\,x+32\,c^2)\,c^3\,\frac{\partial}{\partial c}\,\_f(c,x,y)+(-1024\,c^3\,x+128\,x\,c^5)\,\frac{\partial^2}{\partial c\,\partial x}\,\_f(c,x,y)\Big)+\frac{\partial}{\partial y}\Big((256\,c^4\,y+1024\,c^4\,x\,y-64\,c^6\,x\,y-4096\,c^2\,x\,y \tag{7.3}$$

$$- 96\, c^6\, y)\ \_f(c, x, y) + (1024\, c^4\, x\, y - 128\, c^6\, x\, y)\ \frac{\partial}{\partial x}\ \_f(c, x, y) + (128\, c^6 - 16\, c^8)\ \frac{\partial^2}{\partial c^2}\ \_f(c, x, y) + (-32\, c^6\, x\, y - 2048\, c^2\, x\, y - 48\, c^6\, y + 512\, c^4\, x\, y + 128\, c^4\, y)\ \frac{\partial}{\partial y}\ \_f(c, x, y) + (32\, c^7\, x - 256\, c^5\, y - 512\, x\, c^5 + 2048\, c^3\, x + 32\, c^7 + 32\, c^7\, y)\ \frac{\partial}{\partial c}\ \_f(c, x, y) + (64\, c^7\, x - 512\, x\, c^5)\ \frac{\partial^2}{\partial c\, \partial x}\ \_f(c, x, y) + (-64\, c^6\, x\, y + 512\, c^4\, x\, y)\ \frac{\partial^2}{\partial x\, \partial y}\ \_f(c, x, y) + (16\, c^7\, y - 128\, c^5\, y)\ \frac{\partial^2}{\partial c\, \partial y}\ \_f(c, x, y)\Big)$$

It can be integrated over x aan y, which leads to boundary terms that are 0. Therefore, the first component of the triple is a differential equation satisfied by the parametrized double integral:

```
> res2[1][1] = 0;
```

$$(-5120\, c^2 + 16384 + 16\, c^6 - 256\, c^4)\ \_F(c) + (-4096\, c^2 + c^{10})\ \frac{d^2}{dc^2}\ \_F(c) + (-2048\, c^3 - 256\, c^5 + 5\, c^9 - 4096\, c - 32\, c^7)\ \frac{d}{dc}\ \_F(c) = 0 \tag{7.4}$$