Bijections by Automata for Variants of Tandem Walks on the Square Lattice

Frédéric Chyzak

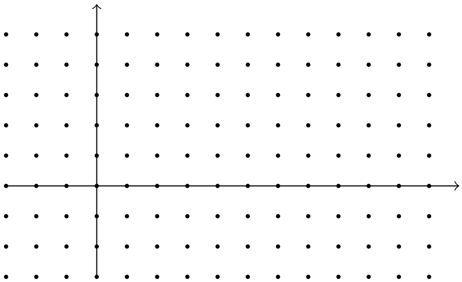
Inria — informatiques mathématiques

May 3, 2018 — *IRIF Combinatorics Seminar*

Based on ongoing works with A. Bostan, A. Mahboubi, and K. Yeats

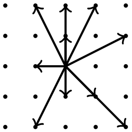
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

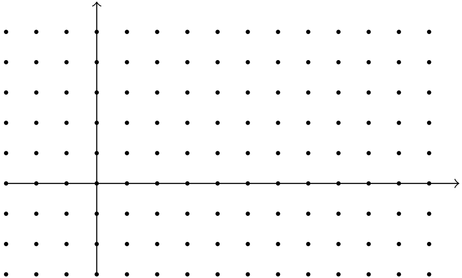
Bijections by Automata for Variants of Tandem Walks on the Square Lattice



Square lattice = $\mathbb{Z}^2$

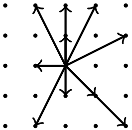
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

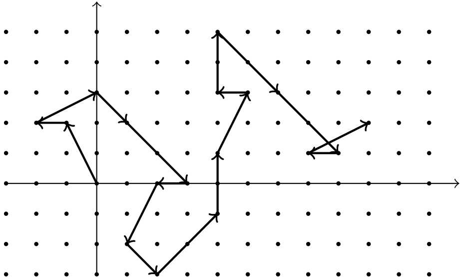




Step set $\subset \mathbb{Z}^2$

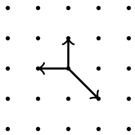
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

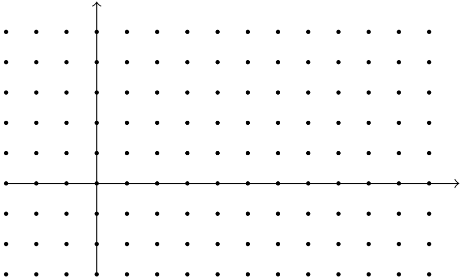




A walk = a series of steps (from the origin)

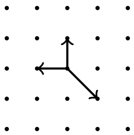
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

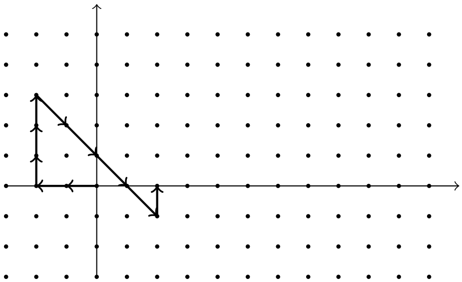




Step set for tandem walks = { ↑, ←, ↘ }

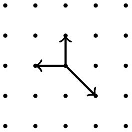
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

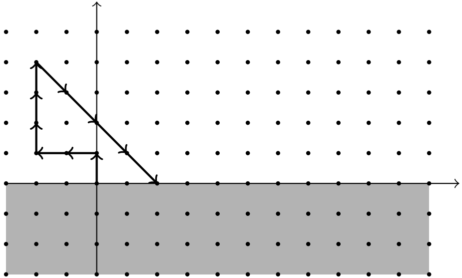




Step set for tandem walks = $\{ \boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow} \}$

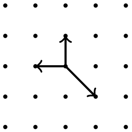
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

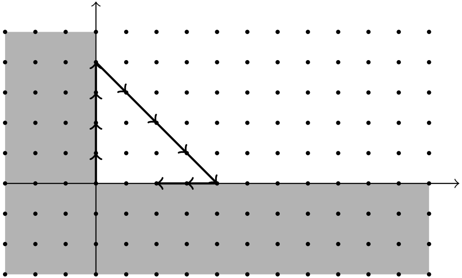




Variant: half-plane walks

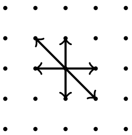
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

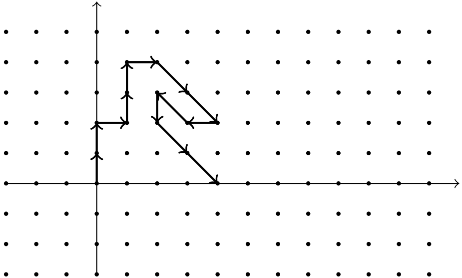




Variant: quarter-plane walks

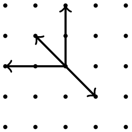
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

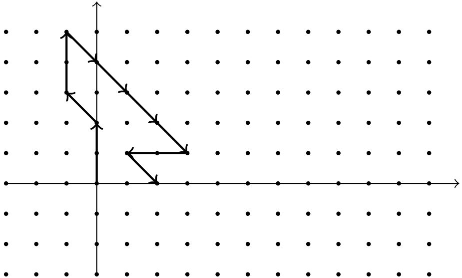




Variant: symmetrized step set

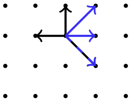
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

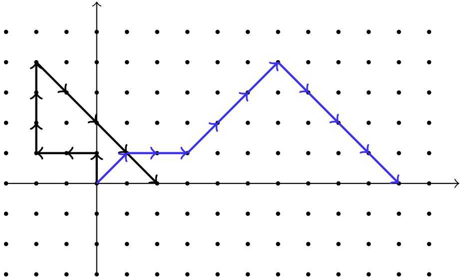




Variant: p-tandem walks

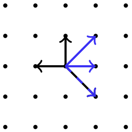
Bijections by Automata for Variants of Tandem Walks on the Square Lattice

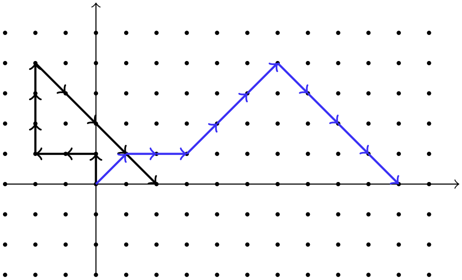




Bijection: a mapping?

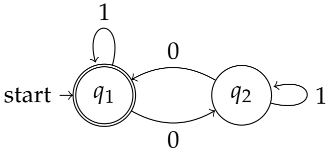
Bijections by Automata for Variants of Tandem Walks on the Square Lattice





Bijection: a mapping? Better, a calculation!

Bijections by Automata for Variants of Tandem Walks on the Square Lattice



Automata: . . . of automata theory

Origin of the work (I): Small-step walks and Motzkin numbers

nth Motzkin number = number of half-plane walks
of length n, using steps $\boxed{\nearrow}$, $\boxed{\rightarrow}$, $\boxed{\searrow}$, ending on x-axis

Two models of quarter-plane walks on the square lattice with counting related to Motzkin numbers (Bousquet-Mélou, Mishna, 2010)

- step set $S_1 = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}\}$, (already known)
- step set $S_{1,\mathrm{sym}} = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}, \boxed{\downarrow}, \boxed{\rightarrow}, \boxed{\nwarrow}\}$. (was new)

Origin of the work (I): Small-step walks and Motzkin numbers

nth Motzkin number = number of half-plane walks of length n, using steps $\boxed{\nearrow}$, $\boxed{\rightarrow}$, $\boxed{\searrow}$, ending on x-axis

Two models of quarter-plane walks on the square lattice with counting related to Motzkin numbers (Bousquet-Mélou, Mishna, 2010)

- step set $S_1 = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}\}$, (already known)
- step set $S_{1,\mathrm{sym}} = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}, \boxed{\downarrow}, \boxed{\rightarrow}, \boxed{\nwarrow}\}$. (was new)

Bousquet-Mélou and Mishna have an approach by generating series.

Open questions (Bousquet-Mélou, Mishna, 2010)

- Bijective proof for $S_{1,\mathrm{sym}}$?
- Counting sequences differ by a power of 2. Explain why bijectively?

Origin of the work (II): Half-plane walks vs quarter-plane walks

A problem posed by Alin Bostan (2015)

Give an explicit bijection between the two models with step set S_1:

- $\mathcal{H}(n,0) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning and ending at the origin $\}$ (half-plane excursions)
- $\mathcal{Q}(n,0) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending on the diagonal $\}$

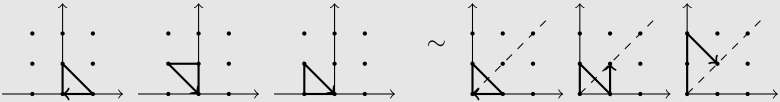
Origin of the work (II): Half-plane walks vs quarter-plane walks

A problem posed by Alin Bostan (2015)

Give an explicit bijection between the two models with step set S_1:

- $\mathcal{H}(n,0) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning and ending at the origin $\}$ (half-plane excursions)
- $\mathcal{Q}(n,0) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending on the diagonal $\}$

$n = 3$, three walks in each model



$$1, 0, 0, 3, 0, 0, 30, 0, 0, 420, 0, 0, 6930, 0, 0, 126126, 0, 0, \ldots$$

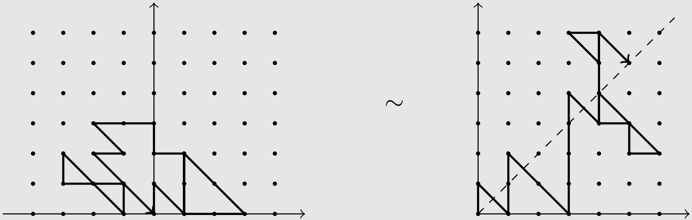
Origin of the work (II): Half-plane walks vs quarter-plane walks

A problem posed by Alin Bostan (2015)

Give an explicit bijection between the two models with step set S_1:

- $\mathcal{H}(n, 0) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning and ending at the origin $\}$ (half-plane excursions)
- $\mathcal{Q}(n, 0) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending on the diagonal $\}$

$n = 24 = 3 \times 8$



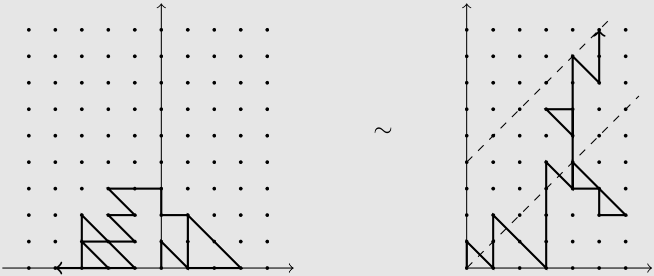
Origin of the work (II): Half-plane walks vs quarter-plane walks

Parametrized models

Give an explicit bijection between the parametrized models with step set S_1:

- $\mathcal{H}(n,k) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning at the origin and ending at $(-k, 0)$ $\}$
- $\mathcal{Q}(n,k) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending on $y - x = k$ $\}$

$n = 28 = 3 \times 8 + 4, \quad k = 4$



Origin of the work (II): Half-plane walks vs quarter-plane walks

Relaxed problem (already solved by Motzkin numbers)

Give an explicit bijection between the two models with step set S_1:

- $\mathcal{H}(n) = \bigcup_k \mathcal{H}(n,k) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning at the origin and ending on $y = 0$ $\}$
- $\mathcal{Q}(n) = \bigcup_k \mathcal{Q}(n,k) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending anywhere $\}$

Origin of the work (II): Half-plane walks vs quarter-plane walks

Relaxed problem (already solved by Motzkin numbers)

Give an explicit bijection between the two models with step set S_1:

- $\mathcal{H}(n) = \bigcup_k \mathcal{H}(n,k) = \{$ length n, confined to the half plane $\mathbb{Z} \times \mathbb{N}$, beginning at the origin and ending on $y = 0$ $\}$
- $\mathcal{Q}(n) = \bigcup_k \mathcal{Q}(n,k) = \{$ length n, confined to the quarter plane $\mathbb{N}^2$, beginning at the origin and ending anywhere $\}$

My initial goal: give a formal proof of a bijection at $k = 0$.

New results

Similarly described algorithmic bijections between

Half-plane walks returning to $y = 0$		Quarter-plane walks
using $S_1 \sim$ using Σ_1 ("Motzkin")	and	using S_1 ("tandem")

$S_1 = \{ \uparrow, \leftarrow, \searrow \}$ $\Sigma_1 = \{-1, 0, +1\} \sim \{ \searrow, \rightarrow, \nearrow \}$

Similarly described algorithmic bijections between

Half-plane walks returning to $y = 0$		Quarter-plane walks
using S_1 $\sim$ using Σ_1 ("Motzkin")	and	using S_1 ("tandem")
using $\Sigma_{1,\text{bicol}}$ ("bicoloured Motzkin")	and	using $S_{1,\text{sym}}$

$$S_1 = \{\uparrow, \leftarrow, \searrow\} \qquad \Sigma_1 = \{-1, 0, +1\} \sim \{\searrow, \rightarrow, \nearrow\}$$

$$S_{1,\text{sym}} = \{\uparrow, \leftarrow, \searrow, \downarrow, \rightarrow, \nwarrow\}$$

$$\Sigma_{1,\text{bicol}} = \{-1, 0, +1, -1, 0, +1\} \sim \{\nearrow, \rightarrow, \searrow, \nearrow, \rightarrow, \searrow\}$$

Similarly described algorithmic bijections between

Half-plane walks returning to $y = 0$		Quarter-plane walks
using $S_1 \sim$ using Σ_1 ("Motzkin")	and	using S_1 ("tandem")
using $\Sigma_{1,\text{bicol}}$ ("bicoloured Motzkin")	and	using $S_{1,\text{sym}}$
using Σ_p ("p-Łukasiewicz")	and	using S_p ("p-tandem")

$$S_1 = \{\uparrow, \leftarrow, \searrow\} \qquad \Sigma_1 = \{-1, 0, +1\} \sim \{\searrow, \rightarrow, \nearrow\}$$

$$S_{1,\text{sym}} = \{\uparrow, \leftarrow, \searrow, \downarrow, \rightarrow, \nwarrow\}$$

$$\Sigma_{1,\text{bicol}} = \{\color{blue}{-1, 0, +1}, \color{red}{-1, 0, +1}\} \sim \{\color{blue}{\nearrow, \rightarrow, \searrow}, \color{red}{\nearrow, \rightarrow, \searrow}\}$$

$$S_p = \{\uparrow, \dots, \nwarrow, \dots, \leftarrow, \searrow\} = \{\,(-i, p-i) \text{ for } 0 \le i \le p\,\} \cup \{\,(-1, 1)\,\}$$

$$\Sigma_p = \{-1, 0, 1, \dots, p\}$$

New results

Similarly described algorithmic bijections between

Half-plane walks returning to $y = 0$		Quarter-plane walks
using S_1 ∼ using Σ_1 ("Motzkin")	and	using S_1 ("tandem")
using $\Sigma_{1,\text{bicol}}$ ("bicoloured Motzkin")	and	using $S_{1,\text{sym}}$
using Σ_p ("p-Łukasiewicz")	and	using S_p ("p-tandem")

$$S_1 = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}\} \qquad \Sigma_1 = \{-1, 0, +1\} \sim \{\boxed{\searrow}, \boxed{\rightarrow}, \boxed{\nearrow}\}$$

$$S_{1,\text{sym}} = \{\boxed{\uparrow}, \boxed{\leftarrow}, \boxed{\searrow}, \boxed{\downarrow}, \boxed{\rightarrow}, \boxed{\nwarrow}\}$$

$$\Sigma_{1,\text{bicol}} = \{{\color{blue}-1}, {\color{blue}0}, {\color{blue}+1}, {\color{red}-1}, {\color{red}0}, {\color{red}+1}\} \sim \{{\color{blue}\boxed{\nearrow}}, {\color{blue}\boxed{\rightarrow}}, {\color{blue}\boxed{\searrow}}, {\color{red}\boxed{\nearrow}}, {\color{red}\boxed{\rightarrow}}, {\color{red}\boxed{\searrow}}\}$$

$$S_p = \{\boxed{\uparrow}, \dots, \boxed{\nwarrow}, \dots, \boxed{\leftarrow}, \boxed{\searrow}\} = \{\,(-i, p-i) \text{ for } 0 \le i \le p\,\} \cup \{\,(-1, 1)\,\}$$

$$\Sigma_p = \{-1, 0, 1, \dots, p\}$$

Parameters preserved/exchanged

- number of $(1, -1)$,
- relative endpoint position w.r.t. origin/diagonal, excess of level steps.

Related works

Earlier works: counting arguments + complex/composed bijections

- Schensted (1961); Regev (1981); Gouyou-Beauchamps (1989) S_1
- Bousquet-Mélou, Mishna (2010) $S_1 + S_{1,\text{sym}}$
- Eu (2010); Eu, Fu, Hou, Hsu (2013) S_1
- Yeats (2014) $S_{1,\text{sym}}$

Related works

Earlier works: counting arguments + complex/composed bijections

- Schensted (1961); Regev (1981); Gouyou-Beauchamps (1989) S_1
- Bousquet-Mélou, Mishna (2010) $S_1 + S_{1,\text{sym}}$
- Eu (2010); Eu, Fu, Hou, Hsu (2013) S_1
- Yeats (2014) $S_{1,\text{sym}}$

- $\mathcal{M}(n) = \{$ Motzkin walks of length $n \}$
- $\mathcal{I}_3(n) = \{$ involutions of $[n]$ with no length-4 decreasing subsequence $\}$
- $\mathcal{T}_3(n) = \{$ standard Young tableaux of size n and height $\leq 3 \}$
- $\mathcal{Y}_3(n) = \{$ Yamanouchi words using only 1, 2, 3 $\}$

Related works

Earlier works: counting arguments + complex/composed bijections

- Schensted (1961); Regev (1981); Gouyou-Beauchamps (1989) — S_1
- Bousquet-Mélou, Mishna (2010) — $S_1 + S_{1,\text{sym}}$
- Eu (2010); Eu, Fu, Hou, Hsu (2013) — S_1
- Yeats (2014) — $S_{1,\text{sym}}$

- $\mathcal{M}(n) = \{$ Motzkin walks of length $n \}$
- $\mathcal{I}_3(n) = \{$ involutions of $[n]$ with no length-4 decreasing subsequence $\}$
- $\mathcal{T}_3(n) = \{$ standard Young tableaux of size n and height $\leq 3 \}$
- $\mathcal{Y}_3(n) = \{$ Yamanouchi words using only 1, 2, 3 $\}$

Ongoing works regarding bijections

- Bostan, Chyzak, Mahboubi: bijective proof based on transducers theory for tandem walks + its formalization + three nonbijective proofs
- Chyzak, Yeats: symmetrized step set + p-tandem walks
- Bousquet-Mélou, Fusy, Raschel: generalized tandem walks via a planar-maps interpretation by Kenyon, Miller, Sheffield, and Wilson

Review of the classical bijection for (simple) tandem walks

$\boxed{\leftarrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\uparrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\leftarrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\searrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\leftarrow} \quad \mathcal{H}(n)$

Review of the classical bijection for (simple) tandem walks

$\boxed{\leftarrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\uparrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\leftarrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\searrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\uparrow}\boxed{\searrow}\boxed{\leftarrow}\boxed{\leftarrow} \quad \mathcal{H}(n)$

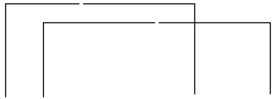
$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y \quad \mathcal{M}(n)$

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow \quad \mathcal{H}(n)$

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y \quad \mathcal{M}(n)$

For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

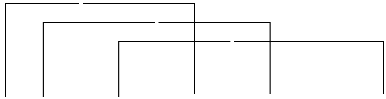
$\leftarrow \; \leftarrow \; \uparrow \; \uparrow \; \leftarrow \; \uparrow \; \leftarrow \; \searrow \; \leftarrow \; \searrow \; \leftarrow \; \uparrow \; \searrow \; \searrow \; \leftarrow \; \uparrow \; \searrow \; \leftarrow \; \leftarrow \qquad \mathcal{H}(n)$



$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y \qquad \mathcal{M}(n)$

For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

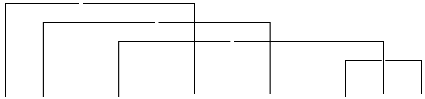
$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\mathcal{H}(n)$



$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$

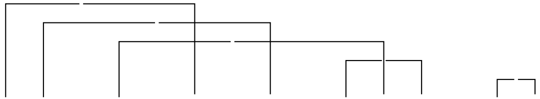


y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



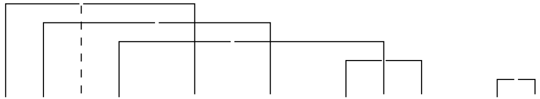
stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

Stack = nesting of archs without crossing.

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$

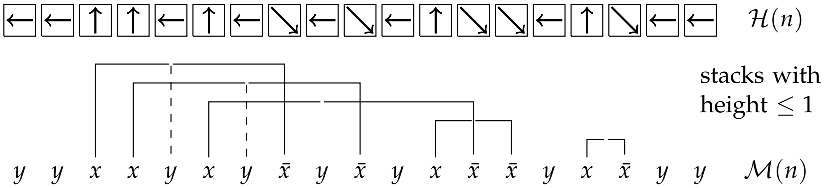


stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

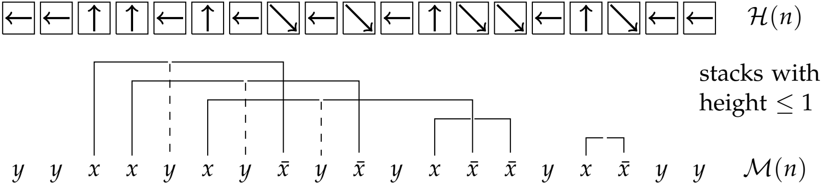
For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

For $\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: distinguish a y if it is the leftmost under an x–$\bar{x}$.



For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

For $\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: distinguish a y if it is the leftmost under an x–$\bar{x}$.

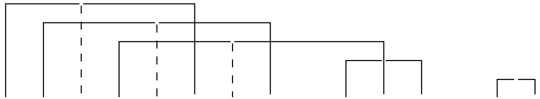


For $\mathcal{M}(n) \sim \mathcal{I}_3(n)$: from left to right, associate x with the first following $\bar{x}$.

For $\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: distinguish a y if it is the leftmost under an x–$\bar{x}$.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$

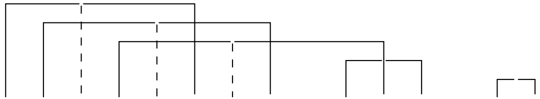


stacks with height ≤ 1

y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 $\mathcal{I}_3(n)$

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



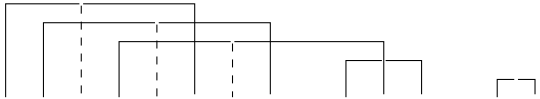
stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
		8					3												

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



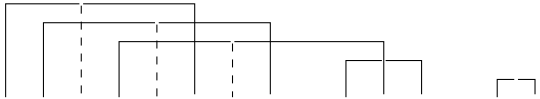
stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
		8	10				3		4										

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

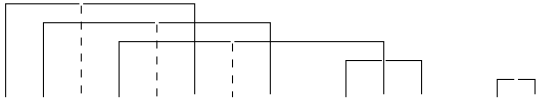
$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\quad \mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
		8	10		13		3		4			6							

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$

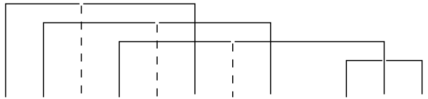
stacks with height ≤ 1



y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
		8	10		13		3		4		14	6	12						

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x}$ → transposition

$\leftarrow \leftarrow \uparrow \uparrow \leftarrow \uparrow \leftarrow \searrow \leftarrow \searrow \leftarrow \uparrow \searrow \searrow \leftarrow \uparrow \searrow \leftarrow \leftarrow$ $\mathcal{H}(n)$



stacks with height ≤ 1

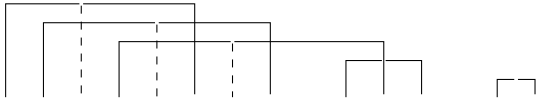
$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
		8	10		13		3		4		14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



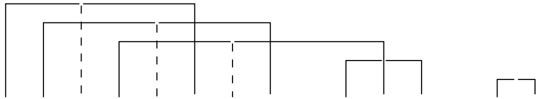
stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1		8	10		13		3		4		14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



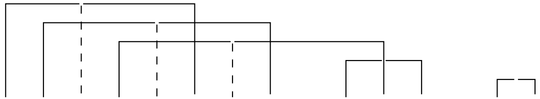
stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10		13		3		4		14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\mathcal{M}(n)$

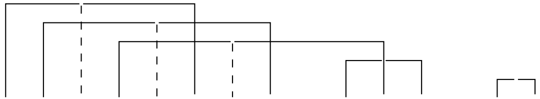
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13		3		4		14	6	12		17	16		

$\mathcal{I}_3(n)$

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \rightarrow$ transposition and $y \rightarrow$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

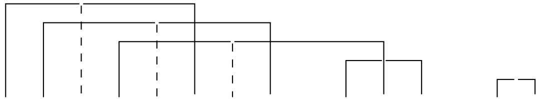
$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3		4		14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

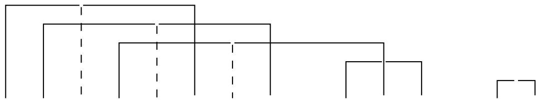
y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4		14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$

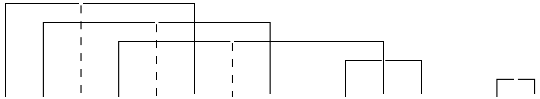


stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12		17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \rightarrow$ transposition and $y \rightarrow$ fixed point.

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\mathcal{H}(n)$



stacks with height ≤ 1

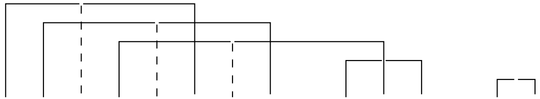
$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16			

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\mathcal{M}(n)$

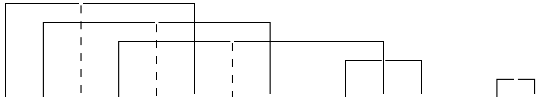
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	

$\mathcal{I}_3(n)$

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\mathcal{M}(n)$

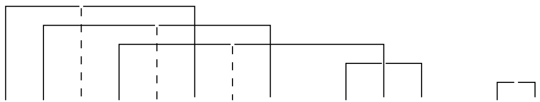
$\mathcal{I}_3(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

$\mathcal{M}(n) \sim \mathcal{I}_3(n)$: x–$\bar{x} \to$ transposition and $y \to$ fixed point.

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

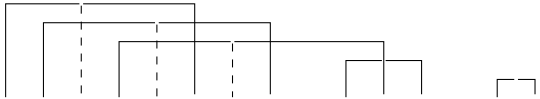
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

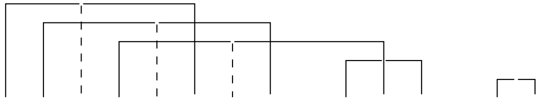
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

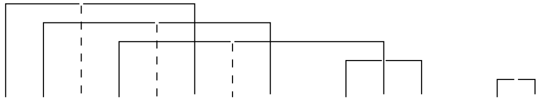
$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2 8

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

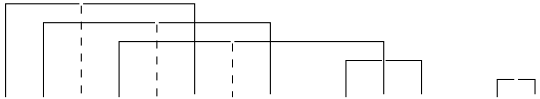
$\mathcal{I}_3(n)$

1 2 8 10

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\qquad \mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\qquad \mathcal{M}(n)$

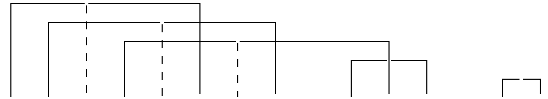
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	5	10
		8	

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

$\leftarrow$	$\leftarrow$	$\uparrow$	$\uparrow$	$\leftarrow$	$\uparrow$	$\leftarrow$	$\searrow$	$\leftarrow$	$\searrow$	$\leftarrow$	$\uparrow$	$\searrow$	$\searrow$	$\leftarrow$	$\uparrow$	$\searrow$	$\leftarrow$	$\leftarrow$	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

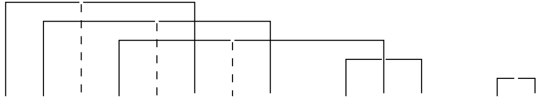
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	5	10	13
		8		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence

(using the P-symbol only.)

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

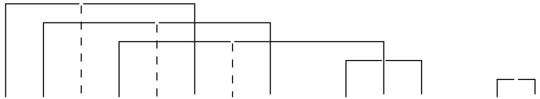
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	5	7	13
		8	10	

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$

stacks with height ≤ 1



y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

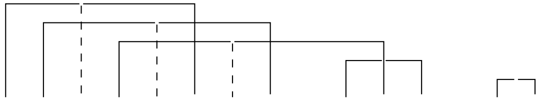
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	7	13
		5	10	
		8		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence

(using the P-symbol only.)

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

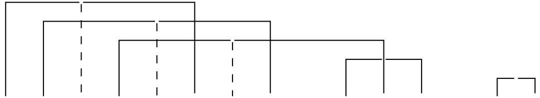
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	7	9
		5	10	13
		8		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

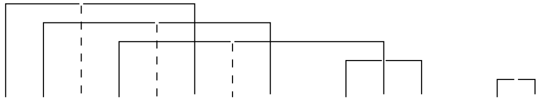
$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\quad \mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	9
		5	7	13
		8	10	

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\quad \mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\quad \mathcal{M}(n)$

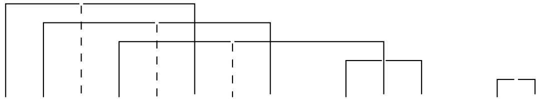
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	9	11
		5	7	13	
		8	10		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

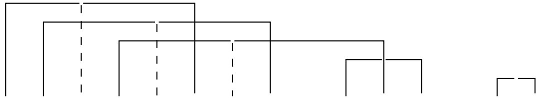
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	9	11	14
		5	7	13		
		8	10			

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

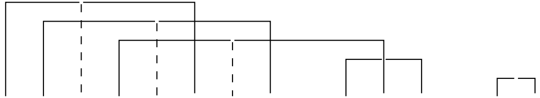
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	14
		5	7	9		
		8	10	13		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ \ y\ \ x\ \ x\ \ y\ \ x\ \ y\ \ \bar{x}\ \ y\ \ \bar{x}\ \ y\ \ x\ \ \bar{x}\ \ \bar{x}\ \ y\ \ x\ \ \bar{x}\ \ y\ \ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

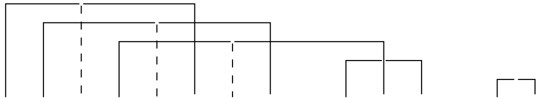
$\mathcal{I}_3(n)$

1	2	3	4	6	11	12
		5	7	9		14
		8	10	13		

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\quad\mathcal{H}(n)$



stacks with height ≤ 1

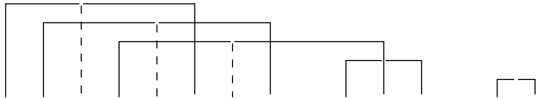
$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\quad\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15
		5	7	9		14	
		8	10	13			

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

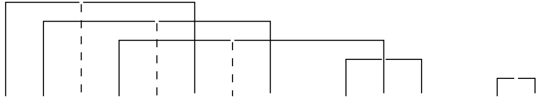
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	17
		5	7	9		14		
		8	10	13				

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

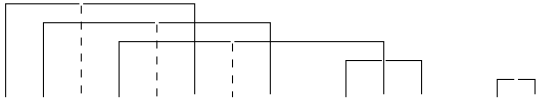
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16
		5	7	9		14		17
		8	10	13				

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y \qquad \mathcal{M}(n)$

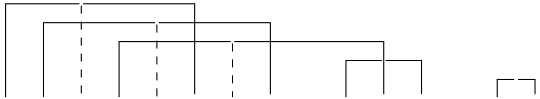
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18
		5	7	9		14		17	
		8	10	13					

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

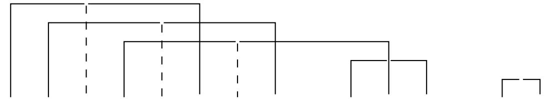
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19
		5	7	9		14		17		
		8	10	13						

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence
(using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\quad \mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\quad \mathcal{M}(n)$

$\mathcal{I}_3(n)$:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

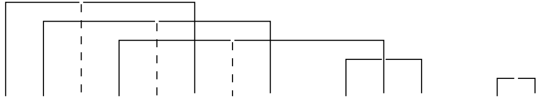
$\mathcal{T}_3(n)$:

1	2	3	4	6	11	12	15	16	18	19
5	7	9	14	17						
8	10	13								

$\mathcal{I}_3(n) \sim \mathcal{T}_3(n)$: by the Robinson–Schensted correspondence (using the P-symbol only.)

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2 3 4 6 11 12 15 16 18 19

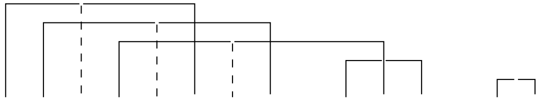
5 7 9 14 17 $\mathcal{T}_3(n)$

8 10 13

1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

$\leftarrow\ \leftarrow\ \uparrow\ \uparrow\ \leftarrow\ \uparrow\ \leftarrow\ \searrow\ \leftarrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \searrow\ \leftarrow\ \uparrow\ \searrow\ \leftarrow\ \leftarrow$ $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

$\mathcal{I}_3(n)$

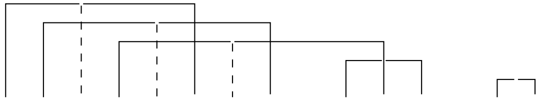
1	2	3	4	6	11	12	15	16	18	19
5	7	9	14	17						
8	10	13								

$\mathcal{T}_3(n)$

1 1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

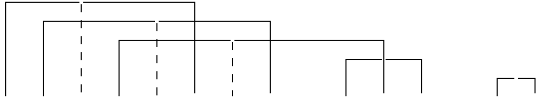
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

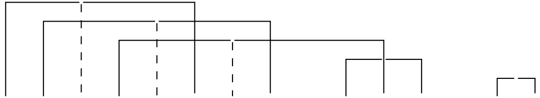
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1 1 1 1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

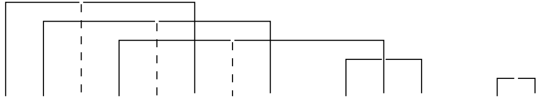
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$ $\mathcal{M}(n)$

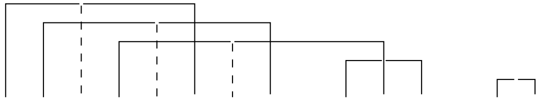
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1 1 1 1 2 1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

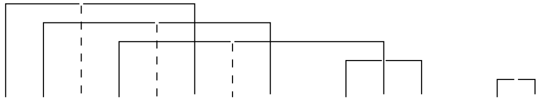
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$ $\mathcal{M}(n)$

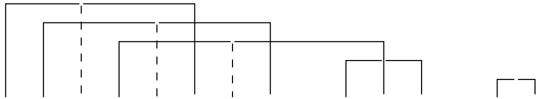
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1 1 1 1 2 1 2 3 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

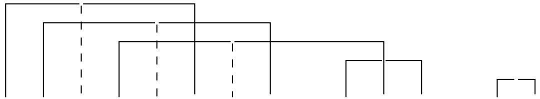
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	3	2	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

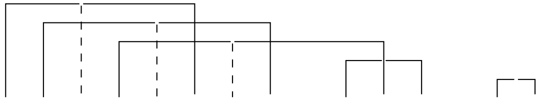
$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1 1 1 1 2 1 2 3 2 3 $\mathcal{Y}_3(n)$

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

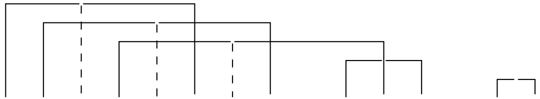
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	3	2	3	1	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

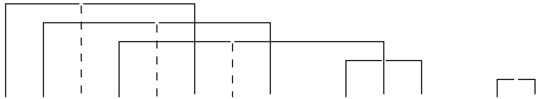
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	3	2	3	1	1	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

$\mathcal{I}_3(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19

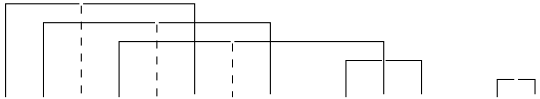
$\mathcal{T}_3(n)$

1	2	3	4	6	11	12	15	16	18	19
5	7	9	14	17						
8	10	13								

1 1 1 1 2 1 2 3 2 3 1 1 3 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

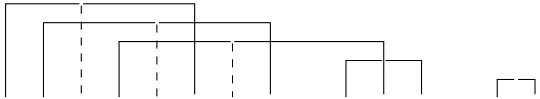
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	3	2	3	1	1	3	2	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

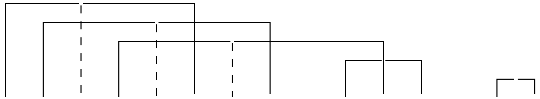
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1	2	3	4	6	11	12	15	16	18	19	
5	7	9	14	17							$\mathcal{T}_3(n)$
8	10	13									

1	1	1	1	2	1	2	3	2	3	1	1	3	2	1	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2 3 4 6 11 12 15 16 18 19

5 7 9 14 17

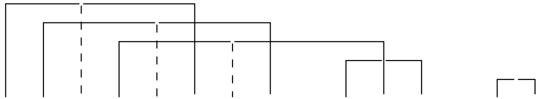
8 10 13

$\mathcal{T}_3(n)$

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

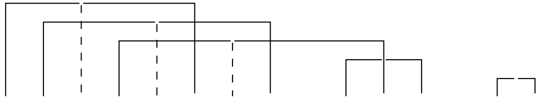
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
1 2 8 10 5 13 7 3 9 4 11 14 6 12 15 17 16 18 19 $\mathcal{I}_3(n)$

1 2 3 4 6 11 12 15 16 18 19
5 7 9 14 17 $\mathcal{T}_3(n)$
8 10 13

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$ $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2 3 4 6 11 12 15 16 18 19

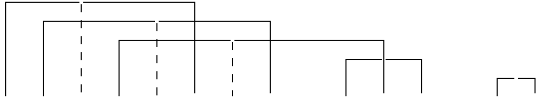
5 7 9 14 17 $\mathcal{T}_3(n)$

8 10 13

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 $\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

←	←	↑	↑	←	↑	←	↘	←	↘	←	↑	↘	↘	←	↑	↘	←	←	$\mathcal{H}(n)$



stacks with height ≤ 1

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y	$\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

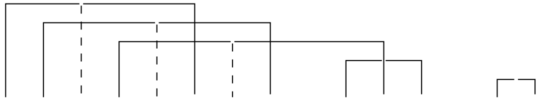
1	2	3	4	6	11	12	15	16	18	19
5	7	9	14	17						
8	10	13								

$\mathcal{T}_3(n)$

1	1	1	1	2	1	2	3	2	3	1	1	3	2	1	1	2	1	1	$\mathcal{Y}_3(n)$

Review of the classical bijection for (simple) tandem walks

← ← ↑ ↑ ← ↑ ← ↘ ← ↘ ← ↑ ↘ ↘ ← ↑ ↘ ← ← $\mathcal{H}(n)$



stacks with height ≤ 1

y y x x y x y $\bar{x}$ y $\bar{x}$ y x $\bar{x}$ $\bar{x}$ y x $\bar{x}$ y y $\mathcal{M}(n)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\mathcal{I}_3(n)$
1	2	8	10	5	13	7	3	9	4	11	14	6	12	15	17	16	18	19	

1 2 3 4 6 11 12 15 16 18 19

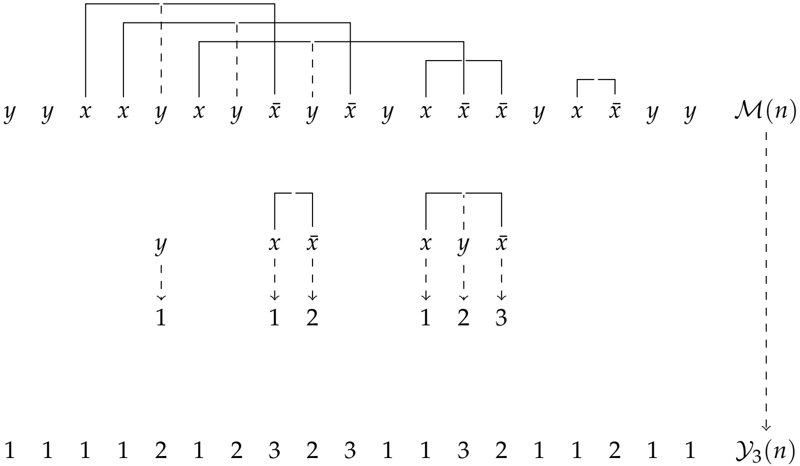
5 7 9 14 17 $\mathcal{T}_3(n)$

8 10 13

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1 $\mathcal{Y}_3(n)$

↑ ↑ ↑ ↑ ↘ ↑ ↘ ← ↘ ← ↑ ↑ ← ↘ ↑ ↑ ↘ ↑ ↑ $\mathcal{Q}(n)$

Review of the classical bijection for (simple) tandem walks

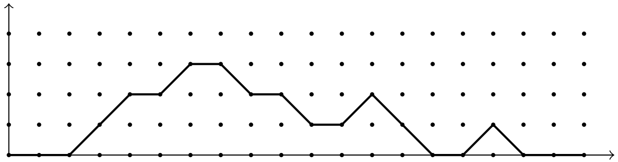


This bijection is implicit in Gouyou-Beauchamps (1989).

The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

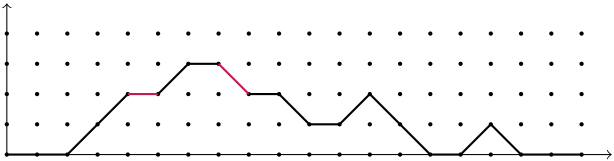
$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

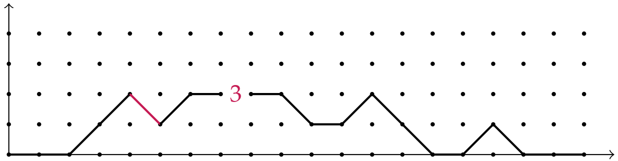
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

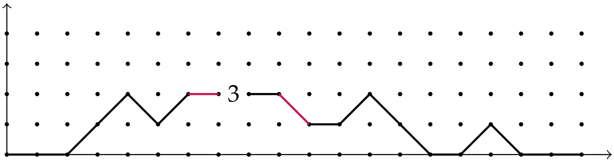
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

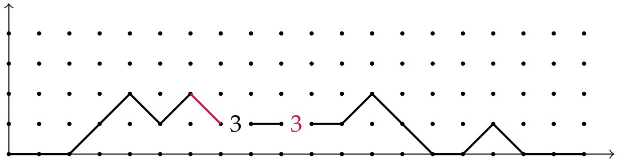
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$



Transformation Motzkin → Yamanouchi: searches for successive patterns

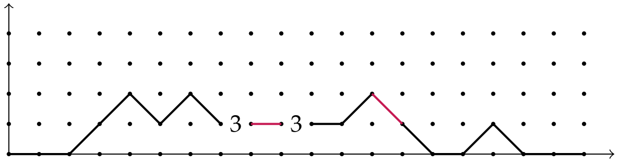
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \rightarrow (\bar{x}, 3)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

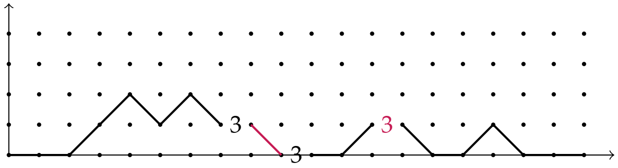
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$



Transformation Motzkin → Yamanouchi: searches for successive patterns

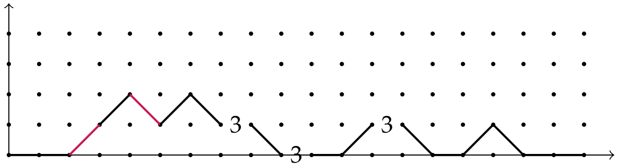
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

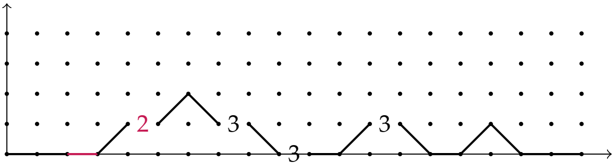
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



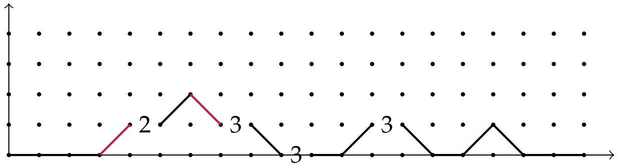
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



Transformation Motzkin → Yamanouchi: searches for successive patterns

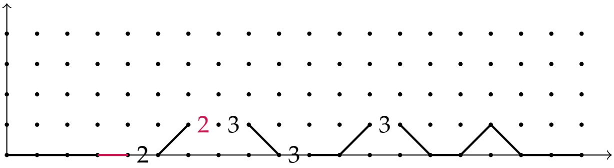
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

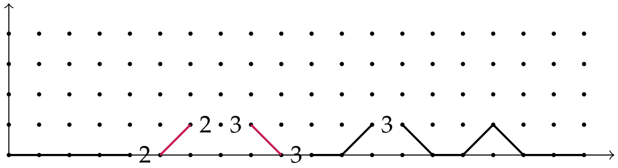
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

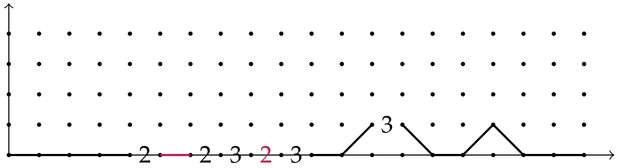
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

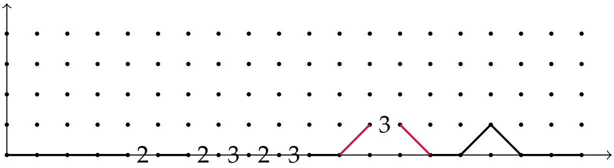
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

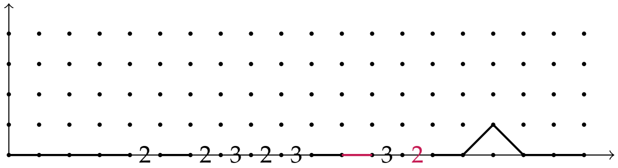
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



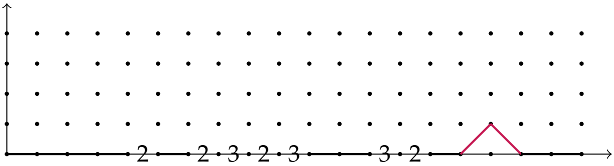
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



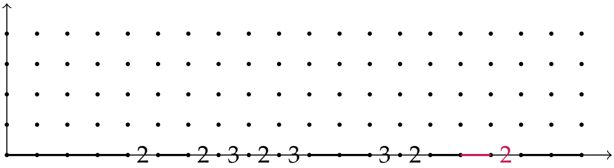
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



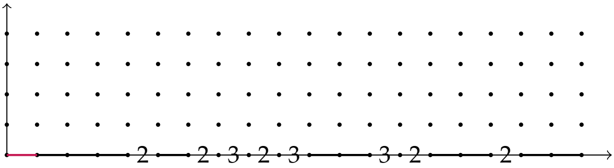
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$



Transformation Motzkin → Yamanouchi: searches for successive patterns

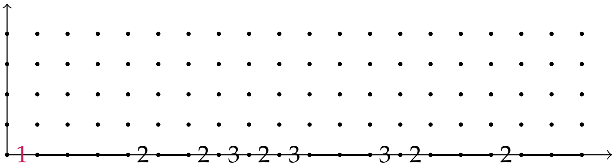
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

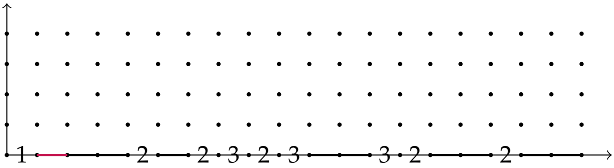
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin → Yamanouchi: searches for successive patterns

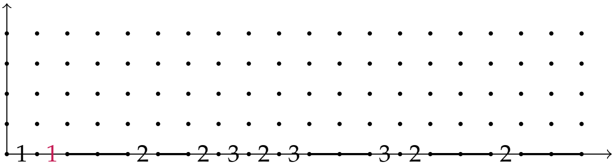
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

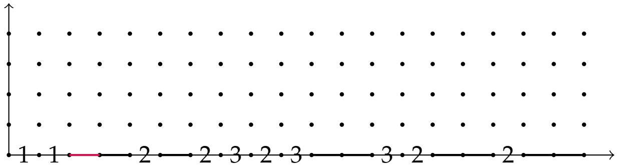
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



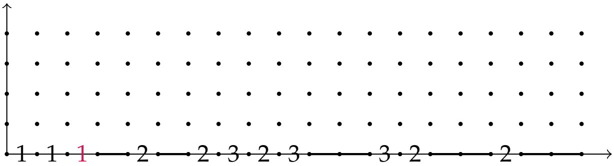
Transformation Motzkin $\to$ Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



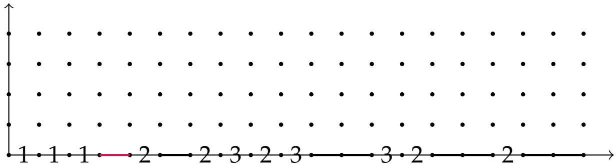
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



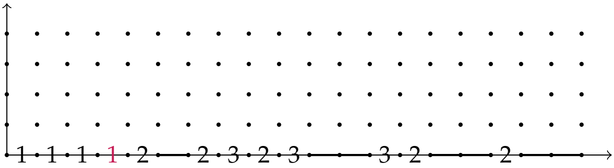
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



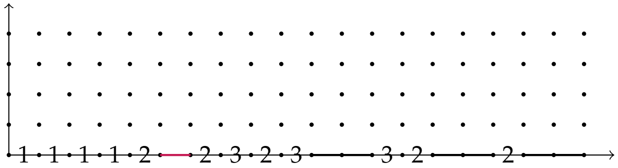
Transformation Motzkin $\to$ Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



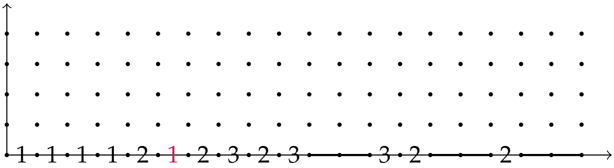
Transformation Motzkin $\rightarrow$ Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \rightarrow (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \rightarrow (y, 2)$
3. from left to right, deal with each step y and doing: $y \rightarrow 1$



Transformation Motzkin → Yamanouchi: searches for successive patterns

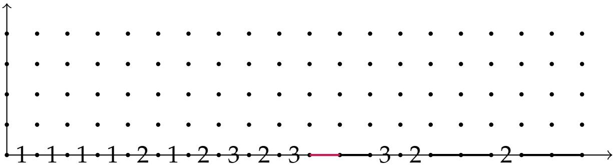
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

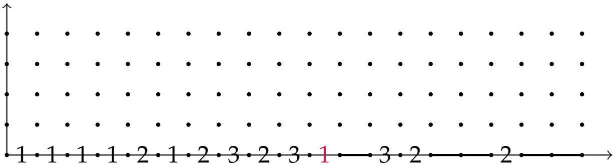
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

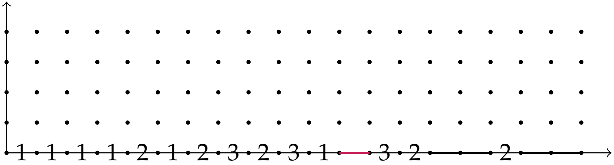
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

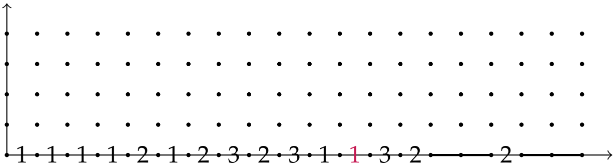
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



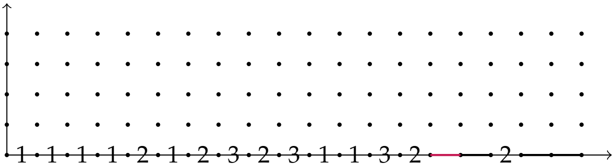
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin $\to$ Yamanouchi: searches for successive patterns

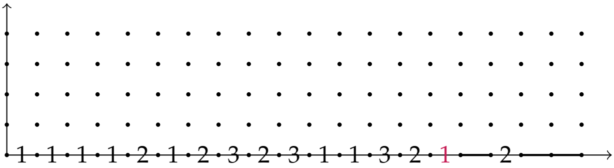
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

Transformation Motzkin → Yamanouchi: searches for successive patterns

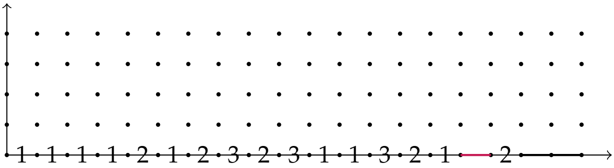
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

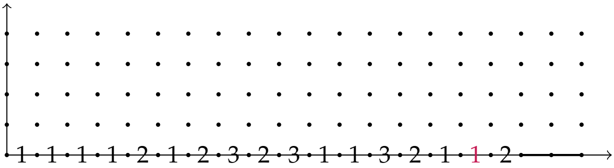
Transformation Motzkin $\to$ Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin → Yamanouchi: searches for successive patterns

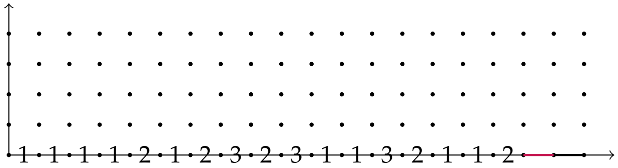
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



The computation as seen by Eu, Fu, Hou, and Hsu (2013)

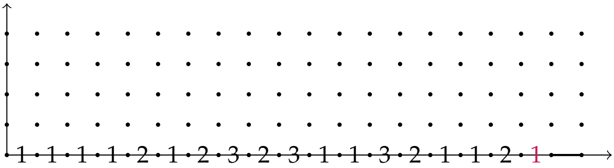
Transformation Motzkin $\to$ Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



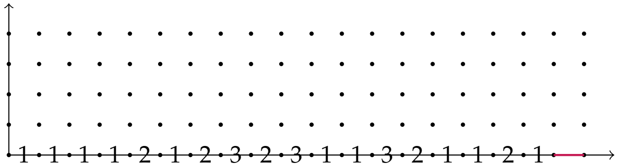
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



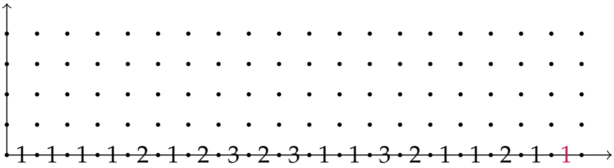
Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin → Yamanouchi: searches for successive patterns

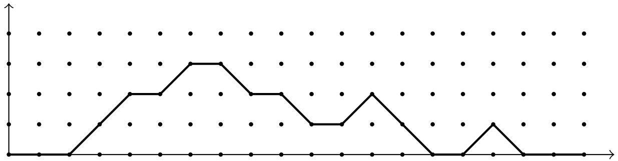
1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$

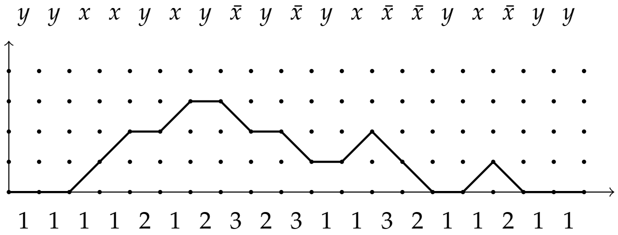
$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$



1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

Transformation Motzkin → Yamanouchi: searches for successive patterns

1. from left to right, deal with each step y at altitude > 0 by pairing it with the first next step $\bar{x}$ and doing: $(y, \bar{x}) \to (\bar{x}, 3)$
2. from left to right, deal with each step x by pairing it with the first next step $\bar{x}$ and doing: $(x, \bar{x}) \to (y, 2)$
3. from left to right, deal with each step y and doing: $y \to 1$



Transformation Motzkin ← Yamanouchi

By strict reversal of the steps.

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{array}{c} s \\ \begin{array}{cc} h & h' \\ v & v' \end{array} \\ s' \end{array}} = \boxed{\begin{array}{c} x \\ \begin{array}{cc} \alpha & \alpha \\ \beta & \beta+1 \end{array} \\ 1 \end{array}}, \boxed{\begin{array}{c} y \\ \begin{array}{cc} \alpha & \alpha \\ 0 & 0 \end{array} \\ 1 \end{array}}, \boxed{\begin{array}{c} y \\ \begin{array}{cc} \alpha & \alpha+1 \\ \beta+1 & \beta \end{array} \\ 2 \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{array}{cc} 0 & - \\ 0 & - \end{array} \\ \text{err.} \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{array}{cc} 0 & 0 \\ \beta+1 & \beta \end{array} \\ 2 \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{array}{cc} \alpha+1 & \alpha \\ \beta & \beta \end{array} \\ 3 \end{array}}$$

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{array}{cc} \multicolumn{2}{c}{s} \\ h & h' \\ v & v' \\ \multicolumn{2}{c}{s'} \end{array}} = \boxed{\begin{array}{cc} \multicolumn{2}{c}{x} \\ \alpha & \alpha \\ \beta & \beta+1 \\ \multicolumn{2}{c}{1} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{y} \\ \alpha & \alpha \\ 0 & 0 \\ \multicolumn{2}{c}{1} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{y} \\ \alpha & \alpha+1 \\ \beta+1 & \beta \\ \multicolumn{2}{c}{2} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ 0 & - \\ 0 & - \\ \multicolumn{2}{c}{\text{err.}} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ 0 & 0 \\ \beta+1 & \beta \\ \multicolumn{2}{c}{2} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ \alpha+1 & \alpha \\ \beta & \beta \\ \multicolumn{2}{c}{3} \end{array}}$$

$$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$$

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

0
0

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s (input)	h	v	h'	v'	s' (output)
x	α	β	α	$\beta+1$	1
y	α	0	α	0	1
y	α	$\beta+1$	$\alpha+1$	β	2
$\bar{x}$	0	0	-	-	err.
$\bar{x}$	0	$\beta+1$	0	β	2
$\bar{x}$	$\alpha+1$	β	α	β	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

h	v	h'	v'	output
0	0	0	0	1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1,2,3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0		0	0																
0		0	0																
	1	1																	

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$
\begin{array}{|c|}\hline s \\ \begin{matrix} h & h' \\ v & v' \end{matrix} \\ s' \\ \hline\end{array}
=
\begin{array}{|c|}\hline x \\ \begin{matrix} \alpha & \alpha \\ \beta & \beta+1 \end{matrix} \\ 1 \\ \hline\end{array},
\begin{array}{|c|}\hline y \\ \begin{matrix} \alpha & \alpha \\ 0 & 0 \end{matrix} \\ 1 \\ \hline\end{array},
\begin{array}{|c|}\hline y \\ \begin{matrix} \alpha & \alpha+1 \\ \beta+1 & \beta \end{matrix} \\ 2 \\ \hline\end{array},
\begin{array}{|c|}\hline \bar{x} \\ \begin{matrix} 0 & - \\ 0 & - \end{matrix} \\ \text{err.} \\ \hline\end{array},
\begin{array}{|c|}\hline \bar{x} \\ \begin{matrix} 0 & 0 \\ \beta+1 & \beta \end{matrix} \\ 2 \\ \hline\end{array},
\begin{array}{|c|}\hline \bar{x} \\ \begin{matrix} \alpha+1 & \alpha \\ \beta & \beta \end{matrix} \\ 3 \\ \hline\end{array}
$$

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0															
0	0	0	1															
1	1	1																

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0														
0	0	0	1	2														
1	1	1	1															

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1													
0	0	0	1	2	1													
1	1	1	1	2														

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$	$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'	1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1												
0	0	0	1	2	1	2												
1	1	1	1	2	1													

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \to h'$	$\alpha \to \alpha$	$\alpha \to \alpha$	$\alpha \to \alpha+1$	$0 \to$ -	$0 \to 0$	$\alpha+1 \to \alpha$
$v \to v'$	$\beta \to \beta+1$	$0 \to 0$	$\beta+1 \to \beta$	$0 \to$ -	$\beta+1 \to \beta$	$\beta \to \beta$
s'	1	1	2	err.	2	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

h: 0 0 0 0 0 1 1 2

v: 0 0 0 1 2 1 2 1

1 1 1 1 2 1 2

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\begin{array}{|c|}\hline s \\ h \quad h' \\ v \quad v' \\ s' \\ \hline\end{array} = \begin{array}{|c|}\hline x \\ \alpha \quad \alpha \\ \beta \quad \beta+1 \\ 1 \\ \hline\end{array}, \begin{array}{|c|}\hline y \\ \alpha \quad \alpha \\ 0 \quad 0 \\ 1 \\ \hline\end{array}, \begin{array}{|c|}\hline y \\ \alpha \quad \alpha+1 \\ \beta+1 \quad \beta \\ 2 \\ \hline\end{array}, \begin{array}{|c|}\hline \bar{x} \\ 0 \quad - \\ 0 \quad - \\ \text{err.} \\ \hline\end{array}, \begin{array}{|c|}\hline \bar{x} \\ 0 \quad 0 \\ \beta+1 \quad \beta \\ 2 \\ \hline\end{array}, \begin{array}{|c|}\hline \bar{x} \\ \alpha+1 \quad \alpha \\ \beta \quad \beta \\ 3 \\ \hline\end{array}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1											
0	0	0	1	2	1	2	1	1											
	1	1	1	1	2	1	2	3											

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2										
0	0	0	1	2	1	2	1	1	0										
	1	1	1	1	2	1	2	3	2										

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	h	v	h'	v'	s'
x	α	β	α	$\beta+1$	1
y	α	0	α	0	1
y	α	$\beta+1$	$\alpha+1$	β	2
$\bar{x}$	0	0	-	-	err.
$\bar{x}$	0	$\beta+1$	0	β	2
$\bar{x}$	$\alpha+1$	β	α	β	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

h: 0 0 0 0 0 1 1 2 1 2 1

v: 0 0 0 1 2 1 2 1 1 0 0

1 1 1 1 2 1 2 3 2 3

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\begin{array}{|cc|}\hline \multicolumn{2}{|c|}{s} \\ h & h' \\ v & v' \\ \multicolumn{2}{|c|}{s'} \\ \hline\end{array} = \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{x} \\ \alpha & \alpha \\ \beta & \beta+1 \\ \multicolumn{2}{|c|}{1} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{y} \\ \alpha & \alpha \\ 0 & 0 \\ \multicolumn{2}{|c|}{1} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{y} \\ \alpha & \alpha+1 \\ \beta+1 & \beta \\ \multicolumn{2}{|c|}{2} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ 0 & - \\ 0 & - \\ \multicolumn{2}{|c|}{\text{err.}} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ 0 & 0 \\ \beta+1 & \beta \\ \multicolumn{2}{|c|}{2} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ \alpha+1 & \alpha \\ \beta & \beta \\ \multicolumn{2}{|c|}{3} \\ \hline\end{array}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1								
0	0	0	1	2	1	2	1	1	0	0	0								
	1	1	1	1	2	1	2	3	2	3	1								

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1							
0	0	0	1	2	1	2	1	1	0	0	0	1							
	1	1	1	1	2	1	2	3	2	3	1	1							

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0						
0	0	0	1	2	1	2	1	1	0	0	0	1	1						
	1	1	1	1	2	1	2	3	2	3	1	1	3						

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	=	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h h'		α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'		β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'		1	1	2	err.	2	3

	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0					
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0					
	1	1	1	1	2	1	2	3	2	3	1	1	3	2					

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1,2,3\}$ and change counters from (h,v) to (h',v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h → h'	=	α → α	α → α	α → $\alpha+1$	0 → -	0 → 0	$\alpha+1$ → α
v → v'		β → $\beta+1$	0 → 0	$\beta+1$ → β	0 → -	$\beta+1$ → β	β → β
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0	0			
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0	0			
1	1	1	1	2	1	2	3	2	3	1	1	3	2	1				

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	=	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \to h'$		$\alpha \to \alpha$	$\alpha \to \alpha$	$\alpha \to \alpha+1$	$0 \to$ -	$0 \to 0$	$\alpha+1 \to \alpha$
$v \to v'$		$\beta \to \beta+1$	$0 \to 0$	$\beta+1 \to \beta$	$0 \to$ -	$\beta+1 \to \beta$	$\beta \to \beta$
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0	0	0		
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0	0	1		
1	1	1	1	2	1	2	3	2	3	1	1	3	2	1	1			

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h h'	=	α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'		β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0	0	0	0	
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0	0	1	0	
1	1	1	1	2	1	2	3	2	3	1	1	3	2	1	1	2		

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0	0	0	0	0
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0	0	1	0	0
1	1	1	1	2	1	2	3	2	3	1	1	3	2	1	1	2	1	

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	h	v	h'	v'	s'
x	α	β	α	$\beta+1$	1
y	α	0	α	0	1
y	α	$\beta+1$	$\alpha+1$	β	2
$\bar{x}$	0	0	-	-	err.
$\bar{x}$	0	$\beta+1$	0	β	2
$\bar{x}$	$\alpha+1$	β	α	β	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

h: 0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0 0

v: 0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0 0

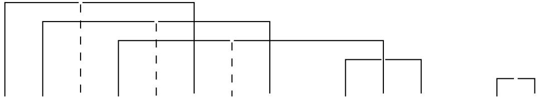
1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \quad \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \quad \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \quad \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \quad \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \quad \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$



$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0 0

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

0

0

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

$\begin{matrix} 0 & & 0 \\ 0 & & 0 \\ & 1 & \end{matrix}$

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$

$$\begin{matrix} & y & & y & & x & & x & & y & & x & & y & & \bar{x} & & y & & \bar{x} & & y & & x & & \bar{x} & & \bar{x} & & y & & x & & \bar{x} & & y & & y \\ 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \\ & 1 & & 1 \end{matrix}$$

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h → h'	α → α	α → α	α → $\alpha+1$	0 → -	0 → 0	$\alpha+1$ → α
v → v'	β → $\beta+1$	0 → 0	$\beta+1$ → β	0 → -	$\beta+1$ → β	β → β
s'	1	1	2	err.	2	3

$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

0 0 0 0

0 0 0 1

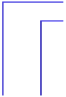
1 1 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h h'	α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'	β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'	1	1	2	err.	2	3



$$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$$

0	0	0	0	0
0	0	0	1	2
1	1	1	1	

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

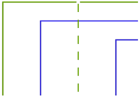
s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \to h'$	$\alpha \to \alpha$	$\alpha \to \alpha$	$\alpha \to \alpha+1$	$0 \to$ -	$0 \to 0$	$\alpha+1 \to \alpha$
$v \to v'$	$\beta \to \beta+1$	$0 \to 0$	$\beta+1 \to \beta$	$0 \to$ -	$\beta+1 \to \beta$	$\beta \to \beta$
s'	1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1													
0	0	0	1	2	1													
1	1	1	1	2														

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\begin{array}{|cc|}\hline \multicolumn{2}{|c|}{s} \\ h & h' \\ v & v' \\ \multicolumn{2}{|c|}{s'} \\ \hline\end{array} = \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{x} \\ \alpha & \alpha \\ \beta & \beta+1 \\ \multicolumn{2}{|c|}{1} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{y} \\ \alpha & \alpha \\ 0 & 0 \\ \multicolumn{2}{|c|}{1} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{y} \\ \alpha & \alpha+1 \\ \beta+1 & \beta \\ \multicolumn{2}{|c|}{2} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ 0 & - \\ 0 & - \\ \multicolumn{2}{|c|}{\text{err.}} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ 0 & 0 \\ \beta+1 & \beta \\ \multicolumn{2}{|c|}{2} \\ \hline\end{array}, \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{\bar{x}} \\ \alpha+1 & \alpha \\ \beta & \beta \\ \multicolumn{2}{|c|}{3} \\ \hline\end{array}$$



$y \quad y \quad x \quad x \quad y \quad x \quad y \quad \bar{x} \quad y \quad \bar{x} \quad y \quad x \quad \bar{x} \quad \bar{x} \quad y \quad x \quad \bar{x} \quad y \quad y$

0 0 0 0 0 1 1

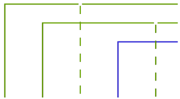
0 0 0 1 2 1 2

1 1 1 1 2 1

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1,2,3\}$ and change counters from (h,v) to (h',v'), following the unique applicable rule

s	h	h'	v	v'	s'
x	α	α	β	$\beta+1$	1
y	α	α	0	0	1
y	α	$\alpha+1$	$\beta+1$	β	2
$\bar{x}$	0	-	0	-	err.
$\bar{x}$	0	0	$\beta+1$	β	2
$\bar{x}$	$\alpha+1$	α	β	β	3



y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2											
0	0	0	1	2	1	2	1											
1	1	1	1	2	1	2												

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

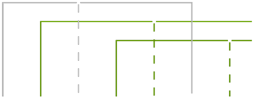
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1										
0	0	0	1	2	1	2	1	1										
1	1	1	1	2	1	2	3											

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	h	v	h'	v'	s'
x	α	β	α	$\beta+1$	1
y	α	0	α	0	1
y	α	$\beta+1$	$\alpha+1$	β	2
$\bar{x}$	0	0	-	-	err.
$\bar{x}$	0	$\beta+1$	0	β	2
$\bar{x}$	$\alpha+1$	β	α	β	3

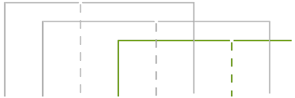


y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2									
0	0	0	1	2	1	2	1	1	0									
1	1	1	1	2	1	2	3	2										

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	h	v	h'	v'	s'
x	α	β	α	$\beta+1$	1
y	α	0	α	0	1
y	α	$\beta+1$	$\alpha+1$	β	2
$\bar{x}$	0	0	-	-	err.
$\bar{x}$	0	$\beta+1$	0	β	2
$\bar{x}$	$\alpha+1$	β	α	β	3



$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$

0 0 0 0 0 1 1 2 1 2 1

0 0 0 1 2 1 2 1 1 0 0

1 1 1 1 2 1 2 3 2 3

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

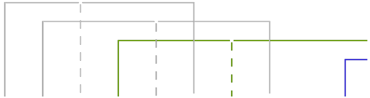
s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1							
0	0	0	1	2	1	2	1	1	0	0	0							
1	1	1	1	2	1	2	3	2	3	1								

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h , h'	α , α	α , α	α , $\alpha+1$	0 , -	0 , 0	$\alpha+1$, α
v , v'	β , $\beta+1$	0 , 0	$\beta+1$, β	0 , -	$\beta+1$, β	β , β
s'	1	1	2	err.	2	3



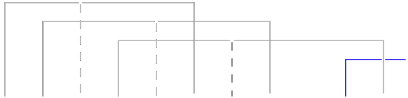
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1						
0	0	0	1	2	1	2	1	1	0	0	0	1						
1	1	1	1	2	1	2	3	2	3	1	1							

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad -$	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad -$	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3

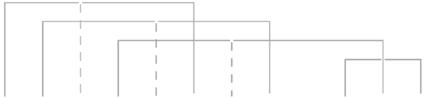


	y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0						
0	0	0	1	2	1	2	1	1	0	0	0	1	1						
	1	1	1	1	2	1	2	3	2	3	1	1	3						

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	=	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h h'		α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'		β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'		1	1	2	err.	2	3



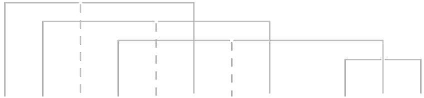
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0				
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0				
1	1	1	1	2	1	2	3	2	3	1	1	3	2					

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin $\to$ Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$



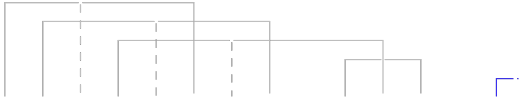
y	y	x	x	y	x	y	$\bar{x}$	y	$\bar{x}$	y	x	$\bar{x}$	$\bar{x}$	y	x	$\bar{x}$	y	y
0	0	0	0	0	1	1	2	1	2	1	1	1	0	0	0			
0	0	0	1	2	1	2	1	1	0	0	0	1	1	0	0			
1	1	1	1	2	1	2	3	2	3	1	1	3	2	1				

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad$ -	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$	$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad$ -	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'	1	1	2	err.	2	3



$y \ y \ x \ x \ y \ x \ y \ \bar{x} \ y \ \bar{x} \ y \ x \ \bar{x} \ \bar{x} \ y \ x \ \bar{x} \ y \ y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1

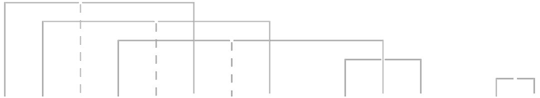
1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{matrix} & s & \\ h & & h' \\ v & & v' \\ & s' & \end{matrix}} = \boxed{\begin{matrix} & x & \\ \alpha & & \alpha \\ \beta & & \beta+1 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha \\ 0 & & 0 \\ & 1 & \end{matrix}}, \boxed{\begin{matrix} & y & \\ \alpha & & \alpha+1 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & - \\ 0 & & - \\ & \text{err.} & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ 0 & & 0 \\ \beta+1 & & \beta \\ & 2 & \end{matrix}}, \boxed{\begin{matrix} & \bar{x} & \\ \alpha+1 & & \alpha \\ \beta & & \beta \\ & 3 & \end{matrix}}$$



$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0

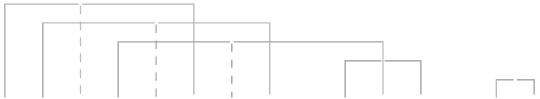
1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{array}{cc} \multicolumn{2}{c}{s} \\ h & h' \\ v & v' \\ \multicolumn{2}{c}{s'} \end{array}} = \boxed{\begin{array}{cc} \multicolumn{2}{c}{x} \\ \alpha & \alpha \\ \beta & \beta+1 \\ \multicolumn{2}{c}{1} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{y} \\ \alpha & \alpha \\ 0 & 0 \\ \multicolumn{2}{c}{1} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{y} \\ \alpha & \alpha+1 \\ \beta+1 & \beta \\ \multicolumn{2}{c}{2} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ 0 & - \\ 0 & - \\ \multicolumn{2}{c}{\text{err.}} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ 0 & 0 \\ \beta+1 & \beta \\ \multicolumn{2}{c}{2} \end{array}}, \boxed{\begin{array}{cc} \multicolumn{2}{c}{\bar{x}} \\ \alpha+1 & \alpha \\ \beta & \beta \\ \multicolumn{2}{c}{3} \end{array}}$$



$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0

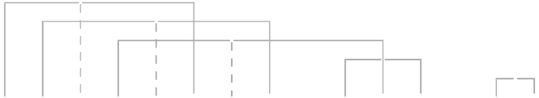
1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s		x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
$h \quad h'$	=	$\alpha \quad \alpha$	$\alpha \quad \alpha$	$\alpha \quad \alpha+1$	$0 \quad -$	$0 \quad 0$	$\alpha+1 \quad \alpha$
$v \quad v'$		$\beta \quad \beta+1$	$0 \quad 0$	$\beta+1 \quad \beta$	$0 \quad -$	$\beta+1 \quad \beta$	$\beta \quad \beta$
s'		1	1	2	err.	2	3



$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0 0

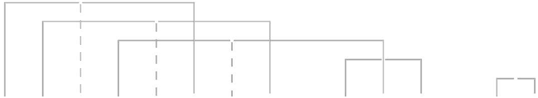
1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

New viewpoint: Reinterpretation by an explicit push-down transducer

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

s	=	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h → h'		α → α	α → α	α → $\alpha+1$	0 → -	0 → 0	$\alpha+1$ → α
v → v'		β → $\beta+1$	0 → 0	$\beta+1$ → β	0 → -	$\beta+1$ → β	β → β
s'		1	1	2	err.	2	3



$y\ y\ x\ x\ y\ x\ y\ \bar{x}\ y\ \bar{x}\ y\ x\ \bar{x}\ \bar{x}\ y\ x\ \bar{x}\ y\ y$

h: 0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0 0

v: 0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0 0

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

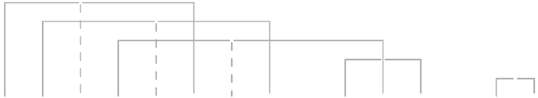
Transformation Motzkin ← Yamanouchi

By nonambiguous reversal of the rules (specific to the present automaton).

Transformation Motzkin → Yamanouchi

1. initialize two counters $h \in \mathbb{N}$ et $v \in \mathbb{N}$ to 0
2. on input letter $s \in \{x, y, \bar{x}\}$, output the letter $s' \in \{1, 2, 3\}$ and change counters from (h, v) to (h', v'), following the unique applicable rule

$$\boxed{\begin{array}{c} s \\ \begin{smallmatrix} h & h' \\ v & v' \end{smallmatrix} \\ s' \end{array}} = \boxed{\begin{array}{c} x \\ \begin{smallmatrix} \alpha & \alpha \\ \beta & \beta+1 \end{smallmatrix} \\ 1 \end{array}}, \boxed{\begin{array}{c} y \\ \begin{smallmatrix} \alpha & \alpha \\ 0 & 0 \end{smallmatrix} \\ 1 \end{array}}, \boxed{\begin{array}{c} y \\ \begin{smallmatrix} \alpha & \alpha+1 \\ \beta+1 & \beta \end{smallmatrix} \\ 2 \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{smallmatrix} 0 & - \\ 0 & - \end{smallmatrix} \\ \text{err.} \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{smallmatrix} 0 & 0 \\ \beta+1 & \beta \end{smallmatrix} \\ 2 \end{array}}, \boxed{\begin{array}{c} \bar{x} \\ \begin{smallmatrix} \alpha+1 & \alpha \\ \beta & \beta \end{smallmatrix} \\ 3 \end{array}}$$



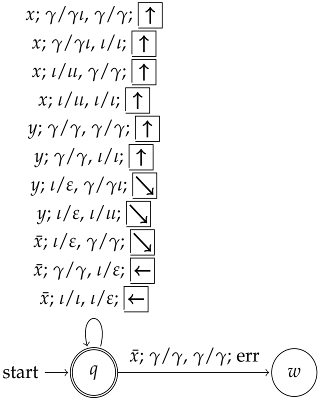
$y \; y \; x \; x \; y \; x \; y \; \bar{x} \; y \; \bar{x} \; y \; x \; \bar{x} \; \bar{x} \; y \; x \; \bar{x} \; y \; y$

0 0 0 0 0 1 1 2 1 2 1 1 1 0 0 0 0 0 0 0

0 0 0 1 2 1 2 1 1 0 0 0 1 1 0 0 1 0 0 0

1 1 1 1 2 1 2 3 2 3 1 1 3 2 1 1 2 1 1

Proofs without introducing tableaux/involutions.



Deterministic, 2 stacks

input alphabet:

$$A = \Sigma_1 = \{-1, 0, +1\} = \{x, y, \bar{x}\}$$

stack alphabet (same for both stacks):

$$Z = \{\gamma, \iota\}$$

output alphabet:

$$B = S_1 \cup \{\mathrm{err}\} = \{\boxed{\leftarrow}, \boxed{\uparrow}, \boxed{\searrow}, \mathrm{err}\}$$

initial stack value (for both):

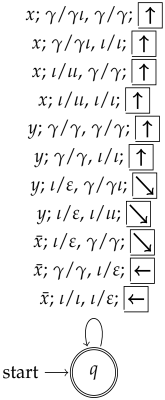
$$\gamma$$

transition $s;\ t_1/T_1, t_2/T_2;\ s'$:

6-uple $\in A \times Z \times Z^* \times Z \times Z^* \times B$

Step of computation based on transition $s;\ t_1/T_1, t_2/T_2;\ s'$:

read s $\rightarrow$ replace the top t_1 of the first stack by T_1; replace the top t_2 of the second stack by T_2 $\rightarrow$ write s'



Deterministic, 2 stacks, simple

input alphabet:

$$A = \Sigma_1 = \{-1, 0, +1\} = \{x, y, \bar{x}\}$$

stack alphabet (same for both stacks):

$$Z = \{\gamma, \iota\}$$

output alphabet:

$$B = S_1 = \{\boxed{\leftarrow}, \boxed{\uparrow}, \boxed{\searrow}\}$$

initial stack value (for both):

$$\gamma$$

transition $s;\ t_1/T_1, t_2/T_2;\ s'$:

$$\text{6-uple} \in A \times Z \times Z^* \times Z \times Z^* \times B$$

Step of computation based on transition $s;\ t_1/T_1, t_2/T_2;\ s'$:

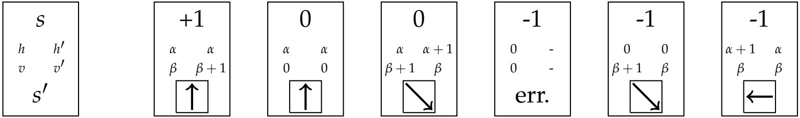
read s $\rightarrow$ replace the top t_1 of the first stack by T_1
replace the top t_2 of the second stack by T_2 $\rightarrow$ write s'

Bicolouring the Motzkin walks explains the 2^n factor (new!)

s	x	y	y	$\bar{x}$	$\bar{x}$	$\bar{x}$
h h'	α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'	β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'	1	1	2	err.	2	3

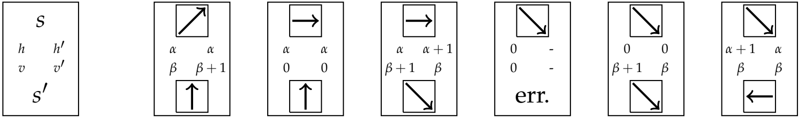
Motzkin words (letters $\{x, y, \bar{x}\}$) $\sim$ Yamanouchi words on $\{1, 2, 3\}$

Bicolouring the Motzkin walks explains the 2^n factor (new!)



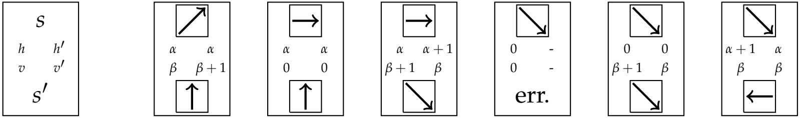
Motzkin words (letters $\{+1, 0, -1\}$) $\sim$ quarter-plane tandem walks

Bicolouring the Motzkin walks explains the 2^n factor (new!)

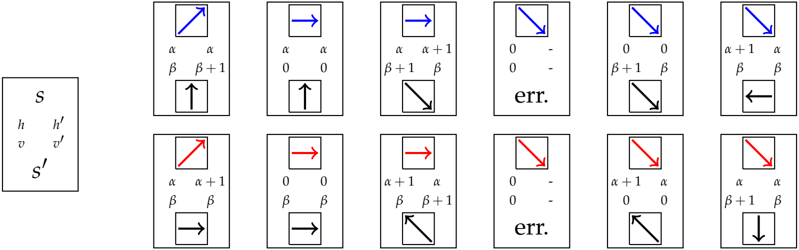


Motzkin walks $\sim$ quarter-plane tandem walks

Bicolouring the Motzkin walks explains the 2^n factor (new!)



Motzkin walks ∼ quarter-plane tandem walks



bicoloured Motzkin walks ∼ quarter-plane tandem walks on symmetrized step set

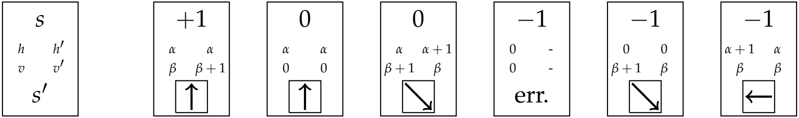
Remind:

s	$+1$	0	0	-1	-1	-1
h h'	α α	α α	α $\alpha+1$	0 -	0 0	$\alpha+1$ α
v v'	β $\beta+1$	0 0	$\beta+1$ β	0 -	$\beta+1$ β	β β
s'	$\uparrow$	$\uparrow$	$\searrow$	err.	$\searrow$	$\leftarrow$

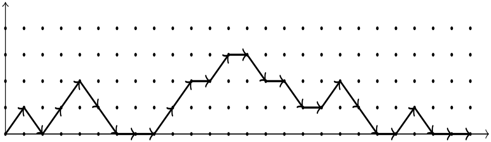
Motzkin words (letters $\{+1, 0, -1\}$) $\sim$ quarter-plane tandem walks

Generalization to p-Łukasiewicz and p-tandem walks (new!)

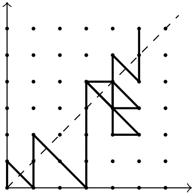
Remind:



Motzkin words (letters $\{+1, 0, -1\}$) $\sim$ quarter-plane tandem walks



$\sim$



$m\,\boxed{\nearrow} + (m+k)\,\boxed{\rightarrow} + m\,\boxed{\searrow}$, with excess $k = 1$

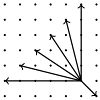
$k = 1$ above diagonal

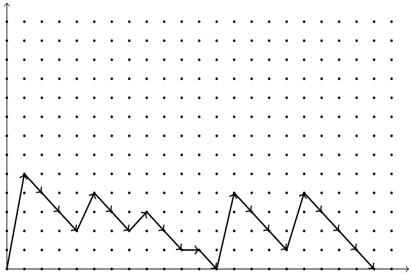
Generalization to p-Łukasiewicz and p-tandem walks (new!)

Does this generalize to long steps for $p > 1$?

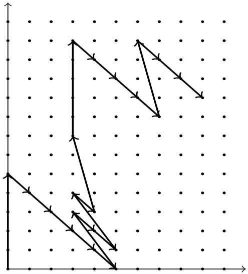
$$\Sigma_p = \{-1, 0, 1, \dots, p\}, \qquad S_p = \{\uparrow, \dots, \nwarrow, \dots, \leftarrow, \searrow\}$$

Example $p = 5$:



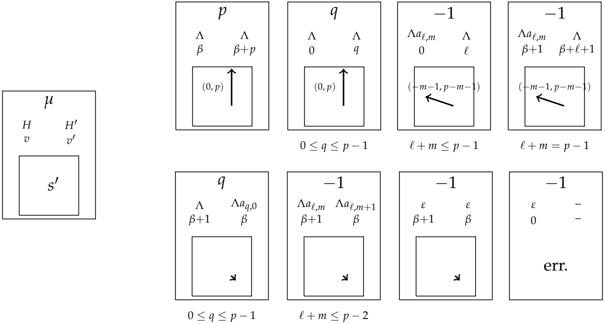


$\sim$

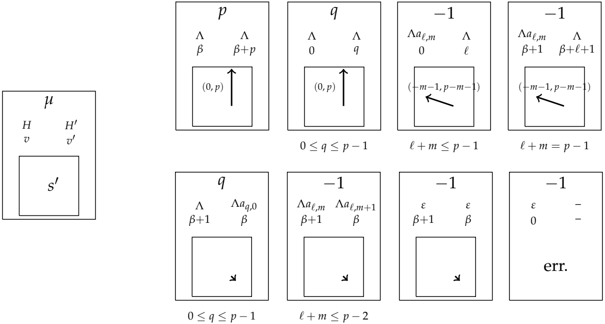


Generalization to p-Łukasiewicz and p-tandem walks (new!)

For fixed p:



For fixed p:



input alphabet: $A = \{-1, 0, 1, \dots, p\}$

stack alphabet for stack 1: $Z_1 = \{\gamma\} \cup \{a_{\ell,m} : 0 \leq \ell + m \leq p - 1\}$

stack alphabet for stack 2: $Z_2 = \{\gamma, \iota\}$

output alphabet: $B = \{$ ↑, …, ↖, …, ←, ↘, err $\}$

initial stack value (for both): γ

transition μ; t_1/T_1,t_2/T_2; s': 6-uple $\in A \times Z_1 \times Z_2^* \times Z_1 \times Z_2^* \times B$

Sketch of proof

Forward transducer provides a partial function $\Phi_p : \Sigma_p^n \to S_p^n$

- Image stays in the quarter plane.
- err. not reached if applied to a p-Łukasiewicz prefix.
- (H, v) goes back to $(\varepsilon, 0)$ if applied to a p-Łukasiewicz word.

Backward transducer provides a total function $\Psi_p : S_p^n \to \Sigma_p^n$

- Image stays in the upper half plane.
- (H, v) goes back to $(\varepsilon, 0)$ if applied to a p-tandem word.

Revertibility

- For $w \in \Sigma_p^n$, if $\Phi_p(w)$ is well defined and its computation terminates with $(H, v) = (\varepsilon, 0)$, then $\Psi_p(\Phi_p(w)) = w$.
- For $w' \in S_p^n$, if the computation of $\Psi_p(w')$ terminates with $(H, v) = (\varepsilon, 0)$, then $\Phi_p(\Psi_p(w')) = w'$.

Parameters on letters

$$\mathrm{wt}_\ell(a_{\ell,m}) = \ell + 1, \qquad \mathrm{wt}_m(a_{\ell,m}) = m + 1, \qquad \kappa(\boxed{\searrow}) = \kappa(-1) = -2,$$

$$\kappa(\boxed{\nwarrow}) = \kappa(j) = p \text{ if } 0 \leq j \leq p \qquad (\text{here: } \boxed{\nwarrow} = (-j, p - j))$$

Parameters

Parameters on letters

$$\mathrm{wt}_\ell(a_{\ell,m}) = \ell + 1, \qquad \mathrm{wt}_m(a_{\ell,m}) = m + 1, \qquad \kappa(\boxed{\searrow}) = \kappa(-1) = -2,$$

$$\kappa(\boxed{\nwarrow}) = \kappa(j) = p \text{ if } 0 \le j \le p \qquad (\text{here: } \boxed{\nwarrow} = (-j, p - j))$$

p-Łukasiewicz: $w = \mu_1 \ldots \mu_n \longleftrightarrow$ p-tandem: $w' = s'_1 \ldots s'_n$

stacks: $H = \lambda_1 \ldots \lambda_h$ and $v \in \mathbb{N}$

Parameters along the computation of the automaton

$$\mathrm{wt}_\ell(H) = \sum_{j=1}^{h} \mathrm{wt}_\ell(\lambda_j), \qquad \mathrm{wt}_m(H) = \sum_{j=1}^{h} \mathrm{wt}_m(\lambda_j),$$

$$z_i = \sum_{1 \le j \le i} \mu_j, \qquad (x_i, y_i) = \sum_{1 \le j \le i} w'_j, \qquad k_i = \sum_{1 \le j \le i} \kappa(\mu_j), \qquad k'_i = \sum_{1 \le j \le i} \kappa(w'_j),$$

$$r_i = x_i - \mathrm{wt}_m(H_i), \qquad s_i = y_i - v_i \qquad (\text{for stacks at step } i)$$

Parameters

Parameters on letters

$$\mathrm{wt}_\ell(a_{\ell,m}) = \ell + 1, \qquad \mathrm{wt}_m(a_{\ell,m}) = m + 1, \qquad \kappa(\boxed{\searrow}) = \kappa(-1) = -2,$$

$$\kappa(\boxed{\nwarrow}) = \kappa(j) = p \text{ if } 0 \leq j \leq p \qquad (\text{here: } \boxed{\nwarrow} = (-j, p - j))$$

$$p\text{-Łukasiewicz: } w = \mu_1 \dots \mu_n \longleftrightarrow p\text{-tandem: } w' = s'_1 \dots s'_n$$

$$\text{stacks: } H = \lambda_1 \dots \lambda_h \text{ and } v \in \mathbb{N}$$

Parameters along the computation of the automaton

$$\mathrm{wt}_\ell(H) = \sum_{j=1}^{h} \mathrm{wt}_\ell(\lambda_j), \qquad \mathrm{wt}_m(H) = \sum_{j=1}^{h} \mathrm{wt}_m(\lambda_j),$$

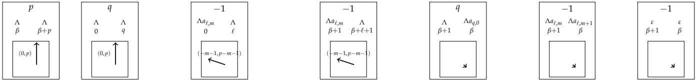
$$z_i = \sum_{1 \leq j \leq i} \mu_j, \qquad (x_i, y_i) = \sum_{1 \leq j \leq i} w'_j, \qquad k_i = \sum_{1 \leq j \leq i} \kappa(\mu_j), \qquad k'_i = \sum_{1 \leq j \leq i} \kappa(w'_j),$$

$$r_i = x_i - \mathrm{wt}_m(H_i), \qquad s_i = y_i - v_i \qquad (\text{for stacks at step } i)$$

Easy observations:

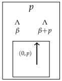
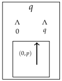
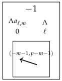
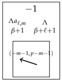
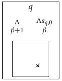
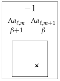
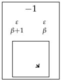
$$\mathrm{wt}_\ell(H) = 0 \Leftrightarrow \mathrm{wt}_m(H) = 0 \Leftrightarrow |H| = 0 \Leftrightarrow H = \varepsilon, \qquad k'_i = y_i - x_i$$

Variations in parameters accross transitions and walks



transition		$0 \le q \le p-1$	$\ell+m \le p-1$	$\ell+m = p-1$	$0 \le q \le p-1$	$\ell+m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta \mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta \mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

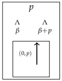
Variations in parameters accross transitions and walks

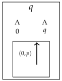
transition		 $0 \le q \le p-1$	 $\ell+m \le p-1$	 $\ell+m = p-1$	 $0 \le q \le p-1$	 $\ell+m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\|H\|$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

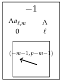
Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

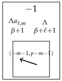
- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta|H|.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon)$.
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0)$.
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

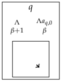
Variations in parameters accross transitions and walks

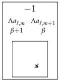


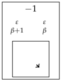










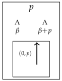
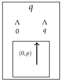
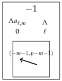
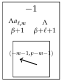
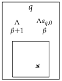
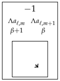
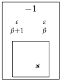


transition							
		$0 \le q \le p-1$	$\ell+m \le p-1$	$\ell+m = p-1$	$0 \le q \le p-1$	$\ell+m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta \mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta \mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta\lvert H\rvert.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon).$
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0).$
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

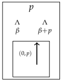
Variations in parameters accross transitions and walks

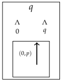
transition		 $0\le q\le p-1$	 $\ell+m\le p-1$	 $\ell+m=p-1$	 $0\le q\le p-1$	 $\ell+m\le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1\ge q$	$p-m-1\ge\ell$	$p-m-1=\ell$	$p-1\ge q$	$p-m-2\ge\ell$	

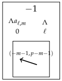
Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

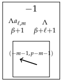
- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v$, $\quad \Delta k - \Delta k' = (p+2)\Delta\lvert H\rvert$.
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon)$.
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0)$.
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H)$, $\quad \Delta y = \Delta s + \Delta v$.

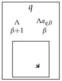
Variations in parameters accross transitions and walks

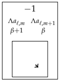


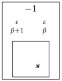










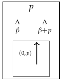
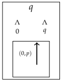
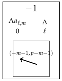
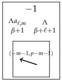
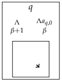
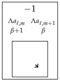
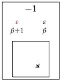


transition	(1)	(2) $0\le q\le p-1$	(3) $\ell+m\le p-1$	(4) $\ell+m=p-1$	(5) $0\le q\le p-1$	(6) $\ell+m\le p-2$	(7)
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1\ge q$	$p-m-1\ge\ell$	$p-m-1=\ell$	$p-1\ge q$	$p-m-2\ge\ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta\lvert H\rvert.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon).$
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0).$
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

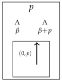
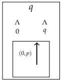
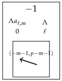
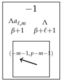
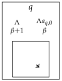
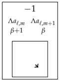
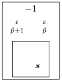
Variations in parameters accross transitions and walks

transition		 $0 \le q \le p-1$	 $\ell + m \le p-1$	 $\ell + m = p-1$	 $0 \le q \le p-1$	 $\ell + m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta \lvert H \rvert$	0	0	-1	-1	1	0	0
$\Delta \mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta \mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta\, \mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta \lvert H \rvert.$
- $\Delta x \ge \Delta\, \mathrm{wt}_m(H)$ and $(\Delta x > \Delta\, \mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon)$.
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0)$.
- $\Delta x = \Delta r + \Delta\, \mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

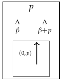
Variations in parameters accross transitions and walks

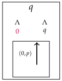
transition		 $0 \le q \le p-1$	 $\ell+m \le p-1$	 $\ell+m = p-1$	 $0 \le q \le p-1$	 $\ell+m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\|H\|$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

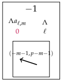
Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

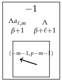
- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta|H|.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon).$
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0).$
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

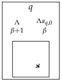
Variations in parameters accross transitions and walks

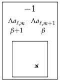


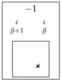










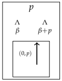
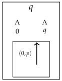
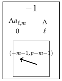
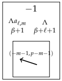
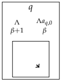
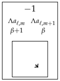
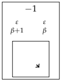


transition	(1)	(2)	(3)	(4)	(5)	(6)	(7)
		$0\le q\le p-1$	$\ell+m\le p-1$	$\ell+m=p-1$	$0\le q\le p-1$	$\ell+m\le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1\ge q$	$p-m-1\ge\ell$	$p-m-1=\ell$	$p-1\ge q$	$p-m-2\ge\ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta\lvert H\rvert.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon)$.
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0)$.
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

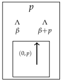
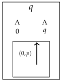
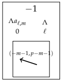
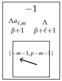
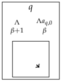
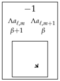
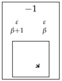
Variations in parameters accross transitions and walks

transition		 $0 \le q \le p-1$	 $\ell+m \le p-1$	 $\ell+m = p-1$	 $0 \le q \le p-1$	 $\ell+m \le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\|H\|$	0	0	-1	-1	1	0	0
$\Delta \mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta \mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1 \ge q$	$p-m-1 \ge \ell$	$p-m-1 = \ell$	$p-1 \ge q$	$p-m-2 \ge \ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta \mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta|H|.$
- $\Delta x \ge \Delta \mathrm{wt}_m(H)$ and $(\Delta x > \Delta \mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon).$
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0).$
- $\Delta x = \Delta r + \Delta \mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

Variations in parameters accross transitions and walks

transition							
		$0\le q\le p-1$	$\ell+m\le p-1$	$\ell+m=p-1$	$0\le q\le p-1$	$\ell+m\le p-2$	
Δx	0	0	$-m-1$	$-m-1$	1	1	1
Δy	p	p	$p-m-1$	$p-m-1$	-1	-1	-1
Δz	p	q	-1	-1	q	-1	-1
Δv	p	q	ℓ	ℓ	-1	-1	-1
$\Delta\lvert H\rvert$	0	0	-1	-1	1	0	0
$\Delta\,\mathrm{wt}_\ell(H)$	0	0	$-\ell-1$	$-\ell-1$	$q+1$	0	0
$\Delta\,\mathrm{wt}_m(H)$	0	0	$-m-1$	$-m-1$	1	1	0
Δk	p	p	-2	-2	p	-2	-2
$\Delta k'$	p	p	p	p	-2	-2	-2
Δr	0	0	0	0	0	0	1
Δs	0	$p-q$	$p-m-1-\ell$	0	0	0	0
hypothesis		$p-1\ge q$	$p-m-1\ge\ell$	$p-m-1=\ell$	$p-1\ge q$	$p-m-2\ge\ell$	

Observation: accross a transition or a fragment of walk, for $\Delta = \text{right} - \text{left}$

- $\Delta z = \Delta\,\mathrm{wt}_\ell(H) + \Delta v, \quad \Delta k - \Delta k' = (p+2)\Delta\lvert H\rvert.$
- $\Delta x \ge \Delta\,\mathrm{wt}_m(H)$ and $(\Delta x > \Delta\,\mathrm{wt}_m(H) \implies H_{\text{left}} = \varepsilon).$
- $\Delta y \ge \Delta v$ and $(\Delta y > \Delta v \implies v_{\text{left}} = 0).$
- $\Delta x = \Delta r + \Delta\,\mathrm{wt}_m(H), \quad \Delta y = \Delta s + \Delta v.$

Conclusion (I): Combinatorics

Bijections explained by deterministic 2-stack push-down transducers

- p-tandem walks in the upper half plane returning to $y = 0$ $\sim$ p-Łukasiewicz walks $\sim$ p-tandem walks in the quarter plane

  Endpoints on same shifted diagonal. Same numbers of $(1, -1)$.

- bicoloured Motzkin walks $\sim$ tandem walks with symmetrized step set in the quarter plane

Conclusion (I): Combinatorics

Bijections explained by deterministic 2-stack push-down transducers

- p-tandem walks in the upper half plane returning to $y = 0$ $\sim$ p-Łukasiewicz walks $\sim$ p-tandem walks in the quarter plane

  Endpoints on same shifted diagonal. Same numbers of $(1, -1)$.

- bicoloured Motzkin walks $\sim$ tandem walks with symmetrized step set in the quarter plane

Comparisons

- simpler, shorter and faster implementation (quadratic/linear time): fast generation?
- concurrent approach by oriented maps finds a different bijection

Conclusion (I): Combinatorics

Bijections explained by deterministic 2-stack push-down transducers

- p-tandem walks in the upper half plane returning to $y = 0$ $\sim$ p-Łukasiewicz walks $\sim$ p-tandem walks in the quarter plane

  Endpoints on same shifted diagonal. Same numbers of $(1, -1)$.
- bicoloured Motzkin walks $\sim$ tandem walks with symmetrized step set in the quarter plane

Comparisons

- simpler, shorter and faster implementation (quadratic/linear time): fast generation?
- concurrent approach by oriented maps finds a different bijection

Further generalizations?

- symmetrized step step for length $p > 1$?
- several p simultaneously?

Conclusion (II): Formal Proofs

Formal Theorem (in coq): $\mathcal{H}(n,0) \sim \mathcal{Q}(n,0)$.

Conclusion (II): Formal Proofs

Formal Theorem (in coq): $\mathcal{H}(n,0) \sim \mathcal{Q}(n,0)$.

Today: Formal proofs induce a new proof pattern

pursuit of complete proofs + type-theoretic context

$\Longrightarrow$ highlights computational contents of proofs

$\Longrightarrow$ leads to purely functional description, a program as a proof object

Conclusion (II): Formal Proofs

Formal Theorem (in coq): $\mathcal{H}(n,0) \sim \mathcal{Q}(n,0)$.

Today: Formal proofs induce a new proof pattern

pursuit of complete proofs + type-theoretic context

$\Longrightarrow$ highlights computational contents of proofs

$\Longrightarrow$ leads to purely functional description, a program as a proof object

Hope: Formal proofs for experimental mathematics

growing libraries of mathematical theories

$\Longrightarrow$ "live" proofs/programs used to experiment, generalize, etc