Explicit Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

Journées de Combinatoire de Bordeaux

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Joint work with A. Bostan, M. van Hoeij, M. Kauers, and L. Pech (2017)

Lattice Walks, Why?

Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, . . .)
- statistical physics (Ising model, . . .)
- probability theory (branching processes, games of chance, . . .)
- operations research (queueing theory, . . .)

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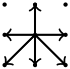
This talk:
Computer Algebra applied to Combinatorics

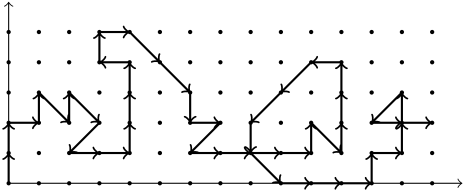
Enumerative Combinatorics of Lattice Walks

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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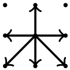
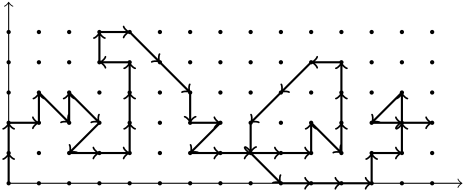
$\mathfrak{S} =$ 



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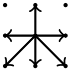
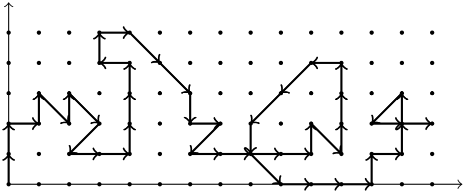
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating Series and Combinatorial Problems

▷ Complete generating series:

$$F(x, y; t) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

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Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(x, y; t)$, resp. $f_{n;i,j}$, and their variants?

- Algebraic nature of F: algebraic? transcendental?
- Explicit form: of F? of f?
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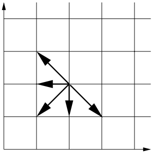
Our goal: Use computer algebra to give computational answers.

Small-Step Models of Interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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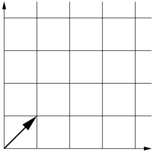

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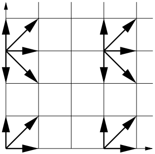

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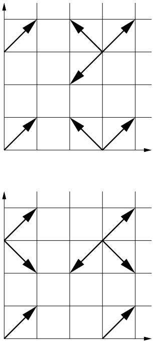

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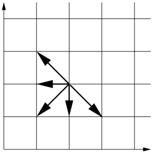

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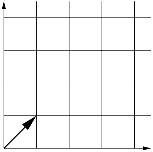

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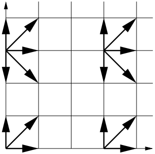

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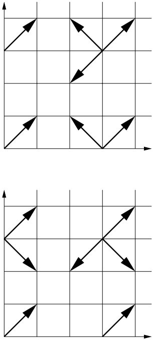

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One is left with 79 interesting distinct models.

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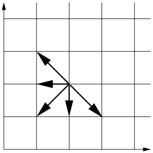

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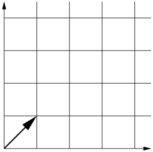

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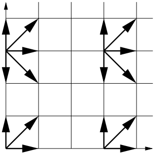

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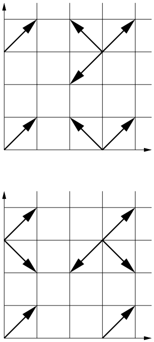

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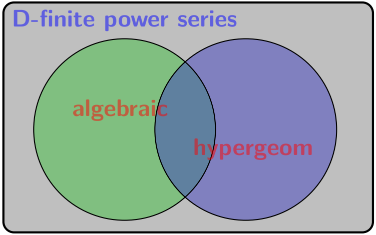
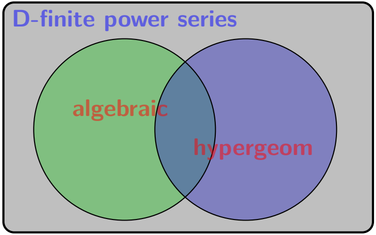
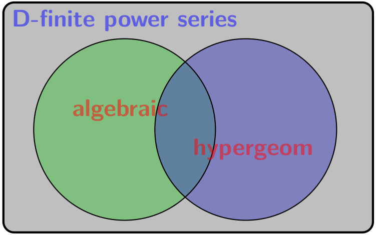
Is any further classification possible?


D-finite power series
algebraic
hypergeom



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

Classification of Univariate Power Series



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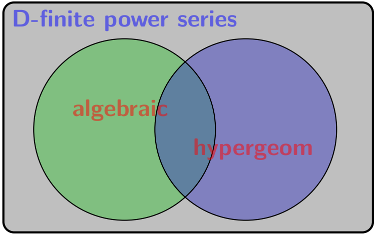

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▷ *Hypergeometric*: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g., Gauss'

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1), $$

$$ t(1-t)S''(t) + \big(c - (a+b+1)t\big)S'(t) - abS(t) = 0. $$

Table of All Conjectured D-Finite $F(1,1;t)$ [Bostan & Kauers, 2009]

	OEIS	$\mathfrak{S}$	alg	ord	equiv		OEIS	$\mathfrak{S}$	alg	ord	equiv
1	A005566		N	3	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	5	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
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3	A151312		N	3	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	5	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	3	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	5	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
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6	A151307		N	5	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	3	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	5	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	4	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	5	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$						
9	A151302		N	5	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y		$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	5	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y		$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	5	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y		$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	5	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y		$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\quad B = 1+\sqrt{3},\quad C = 1+\sqrt{6},\quad \lambda = 7+3\sqrt{6},\quad \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

▷ Computerized discovery of ODE by enumeration + Hermite–Padé.

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12	A151297		N	5	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
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14	A151314		N	5	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
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▷ Computerized discovery of asymptotics by enumeration + LLL/PSLQ.

Further Previous Work

Confirmation of D-finiteness

- ▷ Human proofs for cases 1–22 in [Bousquet-Mélou & Mishna, 2010], but method not adapted to exhibit ODEs.
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Fix of asymptotic formulas (first observed/proved by Melczer)

In fact:

	OEIS	$\mathfrak{S}$	equiv
11	A151261		$\begin{cases} \frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2} & (n=2p) \\ \frac{18}{\pi}\frac{(2\sqrt{3})^n}{n^2} & (n=2p+1) \end{cases}$
13	A151275		$\begin{cases} \frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2} & (n=2p) \\ \frac{144}{\sqrt{5}\pi}\frac{(2\sqrt{6})^n}{n^2} & (n=2p+1) \end{cases}$
15	A151255		$\begin{cases} \frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2} & (n=2p) \\ \frac{32}{\pi}\frac{(2\sqrt{2})^n}{n^2} & (n=2p+1) \end{cases}$

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- ▷ Proof of algebricity, resp. transcendence, of those series.
- ▷ Similar proofs for $F(0,0;t)$, $F(0,1;t)$, and $F(1,0;t)$.

▷ Conjectured asymptotic formulas for the coefficients of $F(0,0;t)$, $F(0,1;t)$, $F(1,0;t)$, since then proved by Melczer & Wilson [2016].

Table of D-Finite $F(x, y; t)$ at $x = y = 0$ [This work]

	OEIS	$\mathfrak{S}$	alg	conj'd equiv
1	A005568		N	$\begin{cases} \frac{32}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
2	A001246		N	$\begin{cases} \frac{8}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151362		N	$\begin{cases} \frac{3\sqrt{6}}{\pi}\frac{6^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
4	A172361		N	$\frac{128}{27\pi}\frac{8^n}{n^3}$
5	A151332		N	$\begin{cases} \frac{16\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ 0 & (n=4p+1,2,3) \end{cases}$
6	A151357		N	$\frac{2A^{3/2}}{\pi}\frac{(2A)^n}{n^3}$
7	A151341		N	$\begin{cases} \frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
8	A151368		N	$\frac{2B^{3/2}}{\pi}\frac{(2B)^n}{n^3}$
9	A151345		N	$\begin{cases} \frac{24\sqrt{30}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
10	A151370		N	$\frac{2\mu^3 C^{3/2}}{\pi}\frac{(2C)^n}{n^3}$
11	A151341		N	$\begin{cases} \frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
12	A151368		N	$\frac{2B^{3/2}}{\pi}\frac{(2B)^n}{n^3}$
13	A151345		N	$\begin{cases} \frac{24\sqrt{30}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
14	A151370		N	$\frac{2\mu^3 C^{3/2}}{\pi}\frac{(2C)^n}{n^3}$
15	A151332		N	$\begin{cases} \frac{16\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ 0 & (n=4p+1,2,3) \end{cases}$
16	A151357		N	$\frac{2A^{3/2}}{\pi}\frac{(2A)^n}{n^3}$
17	A151334		N	$\begin{cases} \frac{81\sqrt{3}}{\pi}\frac{3^n}{n^4} & (n=3p) \\ 0 & (n=3p+1,2) \end{cases}$
18	A151366		N	$\frac{27\sqrt{3}}{\pi}\frac{6^n}{n^4}$
19	A138349		N	$\begin{cases} \frac{768}{\pi}\frac{4^n}{n^5} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$

Table of D-Finite $F(x, y; t)$ at $x = 0$, $y = 1$ [This work]

	OEIS	$\mathfrak{S}$	alg	conj'd equiv
1	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
2	A151392		N	$\begin{cases} \frac{4}{\pi}\frac{4^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151478		N	$\frac{3\sqrt{6}}{2\pi}\frac{6^n}{n^2}$
4	A151496		N	$\frac{32}{9\pi}\frac{8^n}{n^2}$
5	A151380		N	$\frac{3}{4}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151450		N	$\frac{5}{16}\sqrt{\frac{10}{\pi}}\frac{5^n}{n^{3/2}}$
7	A148790		N	$\frac{8}{3\sqrt{\pi}}\frac{4^n}{n^{3/2}}$
8	A151485		N	$\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
9	A151440		N	$\frac{5}{24}\sqrt{\frac{10}{\pi}}\frac{5^n}{n^{3/2}}$
10	A151493		N	$\frac{7}{54}\sqrt{\frac{21}{\pi}}\frac{7^n}{n^{3/2}}$
11	A151394		N	$\begin{cases} \frac{36\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ \frac{54}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p+1) \end{cases}$
12	A151472		N	$\frac{3B^{7/2}}{2\pi}\frac{(2B)^n}{n^3}$
13	A151437		N	$\begin{cases} \frac{72\sqrt{30}}{5\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ \frac{864\sqrt{5}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p+1) \end{cases}$
14	A151492		N	$\frac{6\lambda\mu^3 C^{5/2}}{5\pi}\frac{(2C)^n}{n^3}$
15	A151375		N	$\begin{cases} \frac{448\sqrt{2}}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ \frac{640}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+1) \\ \frac{416\sqrt{2}}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+2) \\ \frac{512}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+3) \end{cases}$
16	A151430		N	$\frac{4A^{7/2}}{\pi}\frac{(2A)^n}{n^3}$
17	A151378		N	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{5/2}}$
18	A151483		Y	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{5/2}}$
19	A005568		N	$\begin{cases} \frac{32}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$

Table of D-Finite $F(x, y; t)$ at $x = 1$, $y = 0$ [This work]

	OEIS	$\mathfrak{S}$	alg	conj'd equiv
1	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
2	A151392		N	$\begin{cases} \frac{4}{\pi}\frac{4^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151471		N	$\begin{cases} \frac{2\sqrt{6}}{\pi}\frac{6^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
4	A151496		N	$\frac{32}{9\pi}\frac{8^n}{n^2}$
5	A151379		N	$\begin{cases} \frac{4\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
6	A148934		N	$\frac{\sqrt{2}A^{3/2}}{\pi}\frac{(2A)^n}{n^2}$
7	A151410		N	$\begin{cases} \frac{4\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
8	A151464		N	$\frac{2B^{3/2}\sqrt{3}}{3\pi}\frac{(2B)^n}{n^2}$
9	A151423		N	$\begin{cases} \frac{4\sqrt{30}}{5\pi}\frac{(2\sqrt{6})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
10	A151490		N	$\frac{\sqrt{6}\mu C^{3/2}}{3\pi}\frac{(2C)^n}{n^2}$
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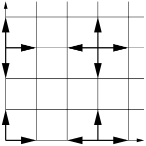
16	A148934		N	$\frac{\sqrt{2}A^{3/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A151497		N	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{5/2}}$
18	A151483		Y	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{5/2}}$
19	A005817		N	$\frac{32}{\pi}\frac{4^n}{n^3}$

The Kernel Equation [≤ Knuth, 1968]: an Example, ✥



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walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$

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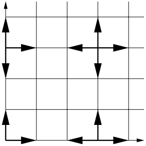

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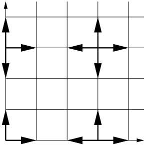

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Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!]\, f_{n;i,j-1} + [\![0 < i]\!]\, f_{n;i-1,j} + f_{n;i,j+1}.$$

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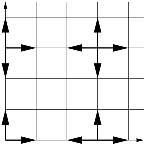

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$$(1 - t\,(x + \bar{x} + y + \bar{y}))\, F(x, y; t) = -\bar{y} t F(x, 0; t) - \bar{x} t F(0, y; t) + 1.$$

(Notation: $\bar{x} = 1/x$, $\bar{y} = 1/y$.)

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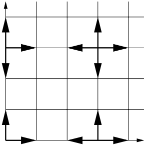

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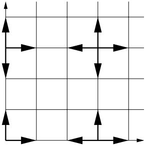
Remarks:

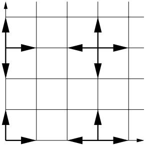
- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.



$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}) \quad .$$

D-Finiteness via the Finite Group: an Example



$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of (x, y) into any element of

$$\mathcal{G} = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}.$$

Kernel equation:

$$\begin{aligned}
J(x, y; t)xyF(x, y; t) &= -txF(x, 0; t) - tyF(0, y; t) + xy,\\
-J(x, y; t)\bar{x}yF(\bar{x}, y; t) &= t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,\\
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-J(x, y; t)x\bar{y}F(x, \bar{y}; t) &= txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
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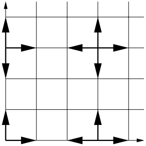

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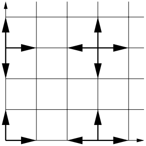
Adding together yields:

$$\sum_{g\in\mathcal{G}} \operatorname{sign}(g)\, g\big(xy\, F(x,y;t)\big) = \qquad \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x,y;t)}.$$

D-Finiteness via the Finite Group: an Example, ✥



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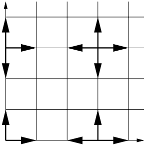

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Cases 1–19 are D-Finite

$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j$ $\longrightarrow$ a group $\mathcal{G}$ of birational transformations

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the group $\mathcal{G}$ is finite and:

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In particular, $F(x,y;t)$ is D-finite w.r.t. x, y, and t.

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▷ Remark: The formula provides no direct information for $x = y = 1$.

From Positive Parts to Residues and Integration

By Lipshitz's approach via diagonals

$$\begin{aligned}[x^> y^>]R(x,y,t) &= S(x,y,t) \odot R(x,y,t) \\ &= \Delta_{x,u}\,\Delta_{y,v}\,\Delta_{t,w}\,S(x,y,t)R(u,v,w) \\ &= [u^{-1}v^{-1}w^{-1}]\frac{1}{uvw}S\left(\frac{x}{u},\frac{y}{v},\frac{t}{w}\right)R(u,v,w) \\ &\qquad \text{where} \quad S(x,y,t) = \frac{x}{1-x}\,\frac{y}{1-y}\,\frac{1}{1-t}\end{aligned}$$

$+$ noncommutative elimination technique, from 12 to 9 variables!

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An intuition by Cauchy integrals

$$[x^> y^>]R(x,y,t) = [u^{-1}v^{-1}]\frac{R(u,v,t)}{uv(x-u)(y-v)}$$
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Remark: Residue formulas provide information for $x = y = 1$.

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Goal: compute $F(t) := \oint \oint H(u, v, t) du\, dv$ for $H \in \mathbb{Q}(u, v, t)$.

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Suppose you could find (algorithmically?) $a_r, \dots, a_0$ in $\mathbb{Q}(t)$ and $U(u, v, t)$, $V(u, v, t)$ in $\mathbb{Q}(u, v, t)$ and prove:

$$a_r(t)\frac{\partial^r H(u, v, t)}{\partial t^r} + \cdots + a_0(t)H(u, v, t) = \frac{\partial U(u, v, t)}{\partial u} + \frac{\partial V(u, v, t)}{\partial v}.$$

Creative Telescoping for Residue Integrals of Rational Functions

Goal: compute $F(t) := \oint \oint H(u, v, t) du\, dv$ for $H \in \mathbb{Q}(u, v, t)$.

Suppose you could find (algorithmically?) $a_r, \dots, a_0$ in $\mathbb{Q}(t)$ and $U(u, v, t), V(u, v, t)$ in $\mathbb{Q}(u, v, t)$ and prove:

$$a_r(t)\frac{\partial^r H(u, v, t)}{\partial t^r} + \cdots + a_0(t)H(u, v, t) = \frac{\partial U(u, v, t)}{\partial u} + \frac{\partial V(u, v, t)}{\partial v}.$$

Then, integrating over closed contours yields:

$$a_r(t)\frac{\partial^r F(t)}{\partial t^r} + \cdots + a_0(t)F(t) = 0.$$

Explicit Expressions for the Cases 1–19

Theorem [This work]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(x, y; t)$ is expressible using iterated integrals of ${}_2F_1$ functions.

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Example: King walks in the quarter plane (A025595, )

$$F(1,1;t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Proved by deriving and solving:

$$t^2(4t+1)(8t-1)(2t-1)(t+1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' + (1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$

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▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum_g \frac{\text{sign}(g)\, g(xy)}{J(x,y;t)}$, then $F = \frac{1}{xy}[x^> y^>]R = [u^{-1}v^{-1}]H$, for $H = \frac{R(1/u,1/v;t)}{(1-xu)(1-yv)}$.

2. If $L \in \mathbb{Q}(x, y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x, y, u, v, t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(x, y; t)) = 0$ after integration w.r.t. u and v over closed contours. Use creative telescoping to find L (as well as U and V).

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3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order ≤ 2 and the P_i have order 1.
   THIS IS A MIRACLE!

4. Solve L_2 in terms of ${}_2F_1$s and deduce F.

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   Use creative telescoping to find L (as well as U and V).
   OK in practice with early evaluation $(x, y) = (1, 1)$, but not for symbolic (x, y).
   Works also for $(0, 0)$, $(x, 0)$, and $(0, y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order ≤ 2 and the P_i have order 1.
   THIS IS A MIRACLE!
4. Solve L_2 in terms of ${}_2F_1$s and deduce F.
5. For $F(x, y; t)$, run whole process for $F(0, 0; t)$, $F(x, 0; t)$, and $F(0, y; t)$, then substitute into kernel equation!

Example: King Walks Continued (Creative Telescoping)

$$F(x, y; t) = [x^> y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})}$$

$$= \oint \oint \frac{(1+u)(1+v)}{uv - t(1 + u + v + u^2 + v^2 + u^2v + uv^2 + u^2v^2)} \frac{(1-u)(1-v)}{(1-ux)(1-vy)} \frac{du\,dv}{(2i\pi)^2}$$

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At $x = y = 1$:

$$t^2(4t+1)(8t-1)(2t-1)(t+1)\frac{\partial^3 H(1,1,u,v;t)}{\partial t^3} + (576t^5 + \cdots)\frac{\partial^2 H(1,1,u,v;t)}{\partial t^2}$$
$$+ (1152t^4 + \cdots)\frac{\partial H(1,1,u,v;t)}{\partial t} + (384t^3 + \cdots)H(1,1,u,v;t)$$
$$= \frac{\partial}{\partial u}\left(\frac{\text{tdeg} = 17,\ \text{nterms} = 146}{\text{tdeg} = 18,\ \text{nterms} = 156}\right) + \frac{\partial}{\partial v}\left(\frac{\text{tdeg} = 29,\ \text{nterms} = 630}{\text{tdeg} = 33,\ \text{nterms} = 596}\right).$$

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At generic x and $y = 0$:

$$(t^{21} + \cdots [79 \text{ terms}])\frac{\partial^5 H(x,0,u,v;t)}{\partial t^5} + \cdots + (t^{16} + \cdots [61 \text{ terms}])H(x,0,u,v;t)$$

$$= \frac{\partial}{\partial u}\left(\frac{\text{tdeg} = 44,\ \text{nterms} = 6378}{\text{tdeg} = 34,\ \text{nterms} = 731}\right) + \frac{\partial}{\partial v}\left(\frac{\text{tdeg} = 65,\ \text{nterms} = 35110}{\text{tdeg} = 57,\ \text{nterms} = 5856}\right).$$

Example: King Walks Continued (Creative Telescoping)

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Integrating w.r.t. u and v yields:

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Example: King Walks Continued (Differential Factorization)

$$t^2(4t+1)(8t-1)(2t-1)(t+1)y''' + t(576t^4+200t^3-252t^2-33t+5)y''$$
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$$\updownarrow$$
$$L = t^2(4t+1)(8t-1)(2t-1)(t+1)\partial_t^3 + t(576t^4+200t^3-252t^2-33t+5)\partial_t^2$$
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and $z = ty' + y$

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$$+(384t^3-72t^2-144t-12)z = 0$$ and $y = t^{-1}\int z$

Example: King Walks Continued (Summary)

$$\begin{aligned}
F(1,1;t) &= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + \cdots \\
&= \left([x^> y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})} \right)_{x=y=1} \\
&= \oint \oint \frac{(1+u)(1+v)}{uv - t(1 + u + v + u^2 + v^2 + u^2v + uv^2 + u^2v^2)} \frac{du\,dv}{(2i\pi)^2} \\
&= \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} \; \frac{3}{2} \\ 2 \end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2} \right) dx
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\end{aligned}
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Remark: Theory of boundary-value problems + Conformal gluing functions $\to$ a different integral representation.

Hypergeometric Series Occurring in Explicit Expressions for $F(x, y; t)$

	$\mathfrak{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$
2		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$
3		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2}{(12t^2+1)^2}$
4		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$
5		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$64t^4$
6		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
8		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
9		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
10		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
11		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
12		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
13		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
14		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
15		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$64t^4$
16		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
17		${}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ 1\end{matrix}\middle\vert\, w\right)$	$27t^3$
18		${}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ 1\end{matrix}\middle\vert\, w\right)$	$27t^2(2t+1)$
19		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$

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6		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
8		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
9		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
10		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
11		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
12		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
13		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
14		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
15		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$64t^4$
16		${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ & 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
17		${}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\middle\vert\, w\right)$	$27t^3$
18		${}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\middle\vert\, w\right)$	$27t^2(2t+1)$
19		${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}\middle\vert\, w\right)$	$16t^2$

Observation: Related to complete elliptic integrals, $E(\sqrt{w})$ and $K(\sqrt{w})$.

Computer Algebra Ingredients (Steps 2 to 4)

Well-studied algorithms

- Creative telescoping: [Zeilberger, 1990], [Lipshitz, 1988], [Almkvist & Zeilberger, 1990], [Takayama, 1990], [Wilf & Zeilberger, 1990] [Chyzak, 2000], [Koutschan, 2010], [Chen, Kauers, & Singer, 2012], [Bostan, Lairez, & Salvy, 2013], [Lairez, 2015], ..., [Bostan, Chyzak, Lairez, & Salvy, 2018], [van der Hoeven, 2017–], ...
- Factorization of ODE: [Beke, 1894], [Schwarz, 1989], [Grigor'ev, 1990], [Singer, 1996], [van Hoeij, 1997]
- Solving with 2F1: [Fang, van Hoeij, 2011], [Kunwar, van Hoeij, 2013], [Kunwar, 2014], [van Hoeij, Vidunas, 2015], [van Hoeij, Imamoglu, 2015]

Already combined for a simpler problem: Diagonal 3D Rook Paths [Bostan, Chyzak, van Hoeij, & Pech, 2011]

Problem: Determine the number a_n of paths from $(0,0,0)$ to (n,n,n) that use positive multiples of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Solution: $G(x) = 1 + 6 \cdot \int_0^x \frac{{}_2F_1\left(\begin{matrix}1/3 \; 2/3\\ 2\end{matrix} \,\middle|\, \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\, dw.$

Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \stackrel{?}{=} \frac{1}{1-w} \stackrel{?}{=} 1 + w + w^2 + \cdots$$

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New formula

$$F(a,b;t) = [x^{-1}y^{-1}]\left[\frac{\bar{x}\bar{y}R(x,y;t)}{(x-a)(y-b)}\right]_{\Gamma_1} = [x^{-1}y^{-1}]\left[\frac{R(\bar{x},\bar{y};t)}{(1-ax)(1-by)}\right]_{\Gamma_2}.$$

Interpretation [Aparicio-Monforte & Kauers, 2013]

- $[x^{-1}y^{-1}]$ is linear on the vector space $\mathbb{Q}^{\mathbb{Z}^2}$;
- the rational functions $R(x,y;t)$ and $(x-a)^{-1}(y-b)^{-1}$ are expanded as a series with support in the cone $\Gamma_1 = \{x^iy^jt^n : i, |j| \le n \ge 0\}$;
- the rational functions $R(\bar{x},\bar{y};t)$ and $(1-ax)^{-1}(1-by)^{-1}$ are expanded as a series with support the cone $\Gamma_2 = \{x^iy^jt^n : -i, |j| \le n \ge 0\}$;
- a theory of series with support in a cone legitimates the product.

Link with creative telescoping [This work]

$$L(H) = \partial_u U + \partial_v V \implies L([H]_\Gamma) = 0$$

provided H, U, V admit expansions with respect to the same cone Γ.

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Moreover, *some admissible Γ makes $[H]_\Gamma$ be the wanted combinatorial series*.

Proofs of Algebraicity/Transcendence of $F(x, y; t)$ and $F(1, 1; t)$

Theorem

- In cases 1–19, $F(x, y; t)$ is transcendental since $F(0, 0; t)$ is.
- In cases 1–16 and 19, $F(1, 1; t)$ is transcendental.
- Specific simplifications prove algebraicity of $F(1, 1; t)$ in cases 17–18.

Proof: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- F is algebraic $\Longrightarrow$ G is algebraic.
- Computing a few coefficients of G shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to L_2 (order 2) or just computing exponential solutions (order 1) decides whether L_2 has nonzero algebraic solutions.

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In the transcendental cases of the theorem, $G \neq 0$ and
L_2 is proved to have no nonzero algebraic solution.

Conclusions

A succession of functional equations of several types

rec. relation on $f_{n;i,j}$ $\rightarrow$ kernel equation on $F(x, y; t)$ $\rightarrow$ ODE on $F(1, 1; t)$

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Wanted

Better understanding of the systematic emergence of elliptic integrals