Holonomic Summation and Integration

Frédéric Chyzak

Integration, Summation and Special Functions in Quantum Field Theory (July 9–13, 2012)

~~Holonomic Summation and Integration~~ *Parametrised ∂-Finite* Summation and Integration *by Creative Telescoping*

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Inria informatiques mathématiques

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Contents

A Nice Story: Apéry's Proof of Irrationality of $\zeta(3)$

Proof, as explained in (van der Poorten, 1979)

Define:

$$b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2, \quad c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}},$$

$$a_{n,k} = b_{n,k} c_{n,k}, \quad a_n = \sum_{k=0}^{n} a_{n,k}, \quad b_n = \sum_{k=0}^{n} b_{n,k},$$

$$g_n = \gcd(1, \dots, n), \quad p_n = 2 g_n^3 a_n, \quad q_n = 2 g_n^3 b_n.$$

Then, (a_n) and (b_n) satisfy the same 2nd-order recurrence, and:

$$\zeta(3) - \frac{a_n}{b_n} = \mathcal{O}(b_n^{-2}), \quad p_n \in \mathbb{Z}, \quad q_n \in \mathbb{N}, \quad \zeta(3) - \frac{p_n}{q_n} = \mathcal{O}(q_n^{-1.08}).$$

Classical irrationality criterion for $\alpha \in \mathbb{R}$:

$$\left(\forall \epsilon > 0, \ \exists \frac{p}{q}, \ \left| \alpha - \frac{p}{q} \right| < \frac{\epsilon}{q} \right) \implies \alpha \notin \mathbb{Q}.$$

Apéry's Recurrence for (a_n) and (b_n)

Second-order recurrence (Apéry, 1979)

$$(n+1)^3 u_{n+1} - (34n^3 + 3n^2 + 27n + 5)\, u_n + n^3 u_{n-1} = 0$$

Cohen and Zagier's "Creative Telescoping" (van der Poorten, 1979)

"[They] cleverly construct

$$B_{n,k} = 4\,(2n+1)\left(k\,(2k+1) - (2n+1)^2\right) b_{n,k}$$

with the motive that

$$B_{n,k} - B_{n,k-1} = (n+1)^3 b_{n+1,k} - (34n^3 + 51n^2 + 27n + 5) b_{n,k} + n^3 b_{n-1,k}."$$

After summation over k from 0 to $n+1$:

$$\underbrace{B_{n,n+1} - B_{n,-1}}_{0-0=0} = (n+1)^3 b_{n+1} - (34n^3 + 3n^2 + 27n + 5)\, b_n + n^3 b_{n-1}.$$

Differentiating under the Integral Sign

Zeilberger's derivation (1982) of a classical integral

$$f(b) = \int_{-\infty}^{+\infty} e^{-x^2} \cos 2bx \, dx = ?$$

$$f'(b) = \int_{-\infty}^{+\infty} -2xe^{-x^2} \sin 2bx \, dx =$$

$$\left[e^{-x^2} \sin 2bx\right]_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} -2be^{-x^2} \cos 2bx \, dx = -2bf(b)$$

Continuous analogue of creative telescoping:

$$\frac{dF}{dx}(b,x) = \frac{df}{db}(b,x) + 2bf(b,x) \quad \text{for} \quad F(b,x) = -\frac{1}{2x}\frac{df}{db}(b,x)$$

After integration over x from $-\infty$ to $+\infty$:

$$\underbrace{\left[\frac{dF}{dx}(b,x)\right]_{x=-\infty}^{x=+\infty}}_{0-0=0} = f'(b) + 2bf(b)$$

Sums and Integrals (1/5)

Binomial sums

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad \text{(Strehl, 1994)}$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2 \qquad \text{(Blodgelt, 1990)}$$

Integrals of the theory of special functions

Four types of Bessel functions (Glasser & Montaldi, 1994):

$$\int_0^{+\infty} x\, J_1(ax)\, I_1(ax)\, Y_0(x)\, K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

No explicit form, but a 2nd-order linear ODE:

$$\int_0^{\infty} \int_0^{\infty} J_1(x)\, J_1(y)\, J_2(c\sqrt{xy})\, \frac{dx\, dy}{e^{x+y}}$$

Sums and Integrals (2/5)

Extractions of coefficients

Theory of orthogonal polynomials, here, Hermite (Doetsch, 1930):

$$\frac{1}{2\pi i} \oint \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1 + 4y^2)^{\frac{3}{2}}} \, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$

Scalar products involving orthogonal/parametrised families

Chebyshev polynomials, Bessel functions, modified Bessel functions:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1 - x^2}} \, dx = (-1)^n \pi \, I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx) \, dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

Sums and Integrals (3/5)

q-Sums, e.g., from the theory of combinatorial partitions

Finite forms of the Rogers–Ramanujan identities and a generalisation: setting $(q;q)_n = (1-q)\cdots(1-q^n)$,

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \qquad \text{(Andrews, 1974)}$$

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i (q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad \text{(Paule, 1985)}$$

Scalar products in algebraic combinatorics

For $p_1 = x_1 + x_2 + \cdots$ and $p_2 = x_1^2 + x_2^2 + \cdots$:

$$\left\langle \exp\left((p_1^2 - p_2)/2 - p_2^2/4\right) \,\middle|\, \exp\left(t\,(p_1^2 + p_2)/2\right) \right\rangle = \frac{e^{-\frac{1}{4}t\,(t+2)}}{\sqrt{1-t}}$$

Sums and Integrals (4/5)

Combinatorial identities

In the graph-counting sequence k^{k-1}:

$$\sum_{k=0}^{n} \binom{n}{k} i\,(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \qquad \text{(Abel)}$$

In Stirling numbers of the second kind (partitions) and Eulerian numbers (ascents in permutations):

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle \qquad \text{(Frobenius)}$$

In Bernoulli numbers (Taylor expansion of $\tan(x)$):

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \qquad \text{(Gessel, 2003)}$$

Sums and Integrals (5/5)

Identities in more special functions (related, e.g., to number theory)

In Hurwitz's zeta function and the beta function:

$$\int_0^\infty x^{k-1}\zeta(n,\alpha+\beta x)\,dx = \beta^{-k}B(k,n-k)\,\zeta(n-k,\alpha)$$

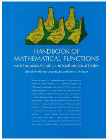
In the polylogarithm functions:

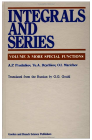
$$\int_0^\infty x^{\alpha-1}\,\mathrm{Li}_n(-xy)\,dx = \frac{\pi\,(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

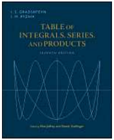
In the (upper) incomplete Gamma function:

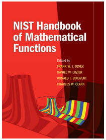
$$\int_0^\infty x^{s-1}\exp(xy)\,\Gamma(a,xy)\,dx = \frac{\pi y^{-s}}{\sin\big((a+s)\,\pi\big)}\,\frac{\Gamma(s)}{\Gamma(1-a)}$$

+ a lot more in:









or in web sites, like Victor Moll's web site on GR

Looking Inside PBM's "Integrals and Series, Vol. 2"

6. $$\int_0^\infty x^{-1/2} \cos b\sqrt{x}\, J_0(cx)\,dx = \frac{\pi b}{8c}\left\{Y_{1/4}(z)\left[Y_{1/4}(z)-J_{1/4}(z)\right]-J_{1/4}(z)\left[J_{1/4}(z)+Y_{1/4}(z)\right]\right\}$$
$[z=b^2/(8c);\ b,\ c>0]$

7. $$\int_0^\infty x^{-1/2} \sin b\sqrt{x} \begin{Bmatrix}\sin cx\\ \cos cx\end{Bmatrix} J_0(cx)\,dx = \sqrt{\frac{\pi}{2c}}\cos\left(\frac{b^2}{16c}\pm\frac{\pi}{4}\right) J_0\left(\frac{b^2}{16c}\right)$$
$[b,\ c>0]$.

8. $$\int_0^\infty x^{-1/2} \cos \sqrt{x} \begin{Bmatrix}\sin cx\\ \cos cx\end{Bmatrix} J_0(cx)\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{2c}}\left[\begin{Bmatrix}\cos t\\ \sin t\end{Bmatrix} J_0\left(\frac{b^2}{16c}\right)\mp\begin{Bmatrix}\sin t\\ \cos t\end{Bmatrix} Y_0\left(\frac{b^2}{16c}\right)\right]$$
$[t=b^2/(16c)-\pi/4;\ b,\ c>0]$.

2.12.20. Integrals containing $x^\alpha \begin{Bmatrix}\sin(bx+a/x)\\ \cos(bx+a/x)\end{Bmatrix} J_\nu(cx)$.

1. $$\int_0^\infty x^{\alpha-1} \begin{Bmatrix}\sin(b/x)\\ \cos(b/x)\end{Bmatrix} J_\nu(cx)\,dx = 2^{\alpha-\delta-1} b^\delta c^{\delta-\alpha}\Gamma\begin{bmatrix}(\nu+\alpha-\delta)/2\\ 1+(\delta+\nu-\alpha)/2\end{bmatrix}\times$$
$$\times {}_0F_3\left(1-\frac{\nu+\alpha-\delta}{2},\ \frac{1}{2}+\delta,\ 1+\frac{\delta+\nu-\alpha}{2};\ \frac{b^2c^2}{16}\right)\mp$$
$$\mp b^{\alpha+\nu}\left(\frac{c}{2}\right)^\nu \begin{Bmatrix}\sin[(\alpha+\nu)\pi/2]\\ \cos[(\alpha+\nu)\pi/2]\end{Bmatrix}\Gamma\begin{bmatrix}-\nu-\alpha\\ 1+\nu\end{bmatrix} {}_0F_3\left(1+\frac{\alpha+\nu}{2},\ 1+\nu,\ \frac{1+\alpha+\nu}{2};\ \frac{b^2c^2}{16}\right)$$
$\left[b,\ c>0;\ -1-\operatorname{Re}\nu<\operatorname{Re}\alpha<\delta+3/2;\ \delta=\begin{Bmatrix}1\\ 0\end{Bmatrix}\right]$.

2. $$\int_0^\infty \frac{1}{x}\sin bx \begin{Bmatrix}\sin(a/x)\\ \cos(a/x)\end{Bmatrix} J_\nu(cx)\,dx = \frac{\pi}{2} J_\nu(z_+\sqrt{a})\left[\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix} J_\nu(z_-\sqrt{a})\pm\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix} Y_\nu(z_-\sqrt{a})\right]+$$
$$+\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix} I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a})$$
$[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}]$.

3. $$\int_0^\infty \frac{1}{x}\cos bx \begin{Bmatrix}\sin(a/x)\\ \cos(a/x)\end{Bmatrix} J_\nu(cx)\,dx = \pm\frac{\pi}{2} J_\nu(z_+\sqrt{a})\left[\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix} J_\nu(z_-\sqrt{a})\mp\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix} Y_\nu(z_-\sqrt{a})\right]\mp$$
$$\mp\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix} I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a})$$
$[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}]$.

4. $$\int_0^\infty \frac{1}{x}\begin{Bmatrix}\sin(bx-a/x)\\ \cos(bx-a/x)\end{Bmatrix} J_\nu(cx)\,dx = 2\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix} I_\nu(z_+\sqrt{a})\,K_\nu(z_-\sqrt{a})$$
$[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}]$.

5. $$\int_0^\infty \frac{1}{x}\begin{Bmatrix}\sin(bx+a/x)\\ \cos(bx+a/x)\end{Bmatrix} J_\nu(cx)\,dx = J_\nu(z_+\sqrt{a})\left[\pm\begin{Bmatrix}\cos(\nu\pi/2)\\ \sin(\nu\pi/2)\end{Bmatrix} J_\nu(z_-\sqrt{a})-\begin{Bmatrix}\sin(\nu\pi/2)\\ \cos(\nu\pi/2)\end{Bmatrix} Y_\nu(z_-\sqrt{a})\right]$$
$[0<c<b;\ a>0;\ z_\pm=\sqrt{b+c}\pm\sqrt{b-c}]$.

2.12.21. Integrals of $A(x)\begin{Bmatrix}\sin(b\sqrt{a^2-x^2})\\ \cos(b\sqrt{a^2-x^2})\end{Bmatrix} J_\nu(cx)$.

1. $$\int_0^a x^{\nu+1}\sin(b\sqrt{a^2-x^2})\,J_\nu(cx)\,dx = \sqrt{\frac{\pi}{2}}\,a^{\nu+3/2} b c^\nu (b^2+c^2)^{-(2\nu+3)/4} J_{\nu+3/2}(a\sqrt{b^2+c^2})$$
$[a>0;\ \operatorname{Re}\nu>-1]$.

2. $$\int_0^a x\sin(b\sqrt{a^2-x^2})\,J_0(cx)\,dx = \frac{ab}{b^2+c^2}\left[\frac{\sin(a\sqrt{b^2+c^2})}{a\sqrt{b^2+c^2}}-\cos(a\sqrt{b^2+c^2})\right]$$
$[a>0]$.

3. $$\int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_\nu(cx)\,dx = \frac{\pi}{2} J_{\nu/2}\left(\frac{a\sqrt{b^2+c^2}+ab}{2}\right) J_{\nu/2}\left(\frac{a\sqrt{b^2+c^2}-ab}{2}\right)$$
$[a>0;\ \operatorname{Re}\nu>-1]$.

4. $$\int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_1(cx)\,dx = \frac{1}{ac}\cos ab-\frac{1}{ac}\cos(a\sqrt{b^2+c^2})$$
$[a>0]$.

5. $$\int_0^a x^{\nu+1}\frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_\nu(cx)\,dx = \sqrt{\frac{\pi}{2}}\,a^{\nu+1/2} c^\nu (b^2+c^2)^{-(2\nu+1)/4} J_{\nu+1/2}(a\sqrt{b^2+c^2})$$
$[a>0;\ \operatorname{Re}\nu>-1]$.

6. $$\int_0^a x\frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_0(cx)\,dx = \frac{\sin(a\sqrt{b^2+c^2})}{\sqrt{b^2+c^2}}$$
$[a>0]$.

2.12.22. Integrals of $A(x)\begin{Bmatrix}\sin(b\sqrt{x^2-a^2})\\ \cos(b\sqrt{x^2-a^2})\end{Bmatrix} J_\nu(cx)$.

1. $$\int_a^\infty \sin(b\sqrt{x^2-a^2})\,J_1(cx)\,dx = \frac{b}{c}(c^2-b^2)_+^{-1/2}\cos(a\sqrt{c^2-b^2})$$
$[a,\ b,\ c>0;\ b\neq c]$.

2. $$\int_a^\infty x^{1-\nu}\sin(b\sqrt{x^2-a^2})\,J_\nu(cx)\,dx = \sqrt{\frac{\pi}{2}}\,a^{3/2-\nu} b c^{-\nu}(c^2-b^2)_+^{(2\nu-3)/4} J_{\nu-3/2}(a\sqrt{c^2-b^2})$$
$[a>0;\ b\neq c;\ \operatorname{Re}\nu>-1/2]$.

Creative Telescoping for Sums/Integrals

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

Given a relation $a_r(n)u_{n+r,k} + \cdots + a_0(n)u_{n,k} = v_{n,k+1} - v_{n,k}$, summation leads by "telescoping" to

$$a_r(n)U_{n+r} + \cdots + a_0(n)U_n = v_{n,b+1} - v_{n,a} \overset{\text{often}}{=} 0.$$

$$U(x) = \int_a^b u(x,y)\,dy = ?$$

Given a relation $a_r(x)\dfrac{\partial^r u}{\partial x^r} + \cdots + a_0(a)u = \dfrac{\partial}{\partial y}v(x,y)$, integrating leads by "telescoping" to

$$a_r(x)\frac{\partial^r U}{\partial x^r} + \cdots + a_0(x)U = v(x,b) - v(x,a) \overset{\text{often}}{=} 0.$$

Adapts easily to $U(x) = \sum_{k=a}^{b} u_k(x)$ and $U_n = \int_a^b u_n(y)\,dy$.

Contents

History for Algorithms Based on Fasenmyer's Ansatz

- (Fasenmyer, 1945, 1947, 1949) "Some generalized hypergeometric polynomials", "A note on pure recurrence relations"
- (Rainville, 1960) "Special functions"
- (Verbaeten, 1974, 1976) "The automatic construction of pure recurrence relations", "Rekursiebetrekkingen voor lineaire hypergeometrische funkties"
- (Zeilberger, 1982) "Sister Celine's technique and its generalizations"
- (Lipshitz, 1988) "The diagonal of a D-finite power series is D-finite"
- (Zeilberger, 1990) "A holonomic systems approach to special functions identities"
- (Wilf and Zeilberger, 1992) "An algorithmic proof theory for hypergeometric (ordinary and 'q') multisum/integral identities"
- (Hornegger, 1992) "Hypergeometrische Summation und polynomiale Rekursion"
- (Wegschaider, 1997) "Computer generated proofs of binomial multi-sum identities"
- (Tefera, 2000, 2002) "Improved algorithms and implementations in the multi-WZ theory", "MultInt, a MAPLE package for multiple integration by the WZ method"
- (Riese, 2003) "qMultiSum: a package for proving q-hypergeometric multiple summation identities"

Hypergeometric Terms

Definition (hypergeometric terms)

An element h of a $\mathbb{K}(n,k)$-vector space closed under shifts is *hypergeometric* if the quotients

$$\frac{h_{n+1,k}}{h_{n,k}} \quad \text{and} \quad \frac{h_{n,k+1}}{h_{n,k}}$$

exist and are rational functions.

Basic observation: for all $(i,j) \in \mathbb{Z}^2$,

$$\frac{h_{n+i,k+j}}{h_{n,k}} \in \mathbb{K}(n,k).$$

Generalises to more indices.

An idealisation of sequences like: $n!$, $\binom{n}{k}$, falling factorials $n^{\underline{k}} = n\,(n-1)\cdots(n-k+1)$, etc.

Fasenmyer's Heuristic, as Revisited by Zeilberger

Fasenmyer's ansatz: For a given hypergeometric $h_{n,k}$, solve

$$\sum_{i=0}^{r}\sum_{j=0}^{s} c_{i,j}(n)\, h_{n+i,k+j} = 0 \qquad \text{where the } c\text{'s don't involve } k.$$

Motivation:

$$\sum_{i=0}^{r}\left(\sum_{j=0}^{s} c_{i,j}(n)\right) h_{n+i,k} = g_{n,k+1} - g_{n,k} \quad \text{where} \quad g_{n,k} = \sum_{i=0}^{r}\sum_{j=0}^{s-1} \tilde{c}_{i,j}(n)\, h_{n+i,k+j}.$$

Idea: Use the *rational functions* $h_{n+i,k+j}/h_{n,k}$, then clear denominators

$$\sum_{i=0}^{r}\sum_{j=0}^{s} c_{i,j}(n)\, p_{i,j}(n,k) = 0 \quad \text{for} \quad p_{i,j} \in \mathbb{K}(n)[k],$$

then solve a linear system over $\mathbb{K}(n)$. If no solution, increase (r,s).

Question: possibility to enforce $\max \deg_k p_{i,j} < (r+1)\,(s+1)$?

Two Motivating Examples

Common denominator C_r of $h_{n+i,k+j}/h_{n,k}$ when $r = s$ increases?

- $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$:

  $$C_r = (k+1)\cdots(k+r)\,(n+1-k)\cdots(n+r-k)$$

  has linearly many terms. The method succeeds easily (by finding Pascal's triangle rule).

- $\frac{1}{n^2+k^2}$:

  $$C_r = \prod_{i,j\in\mathbb{N},\ i+j\leqslant r} \left((n+i)^2 + (k+j)^2\right)$$

  has quadratically many terms. The method seems to fail (out of memory).

Wilf and Zeilberger's Proper Hypergeometric Terms

Definition (proper hypergeometric term)

A hypergeometric term h is *proper* if it can be written, for $\epsilon_\ell = \pm 1$,

$$h_{n,k} = \underbrace{P(n,k)}_{P \in \mathbb{K}[n,k]} \underbrace{\zeta^n \xi^k}_{\zeta,\xi \in \mathbb{K}} \prod_{\ell=1}^{L} \Gamma(\underbrace{a_\ell n + b_\ell k + c_\ell}_{a_\ell, b_\ell \in \mathbb{Z},\ c_\ell \in \mathbb{K}})^{\epsilon_\ell}.$$

$$\Gamma(s+1) = s\,\Gamma(s) \implies \Gamma(s+u)/\Gamma(s) = \text{poly. of degree } u \text{ in } s$$

Key observation

$$h_{n+i,k+j} \quad \to \quad \Gamma(a_\ell n + b_\ell k + c_\ell + u) \quad \text{where} \quad |u| \leqslant |a_\ell| r + |b_\ell| s =: \sigma_\ell,$$

$$h_{n+i,k+j} = \zeta^n \xi^k \prod_{\ell=1}^{L} \Gamma(a_\ell n + b_\ell k + c_\ell - \epsilon_\ell \sigma_\ell)^{\epsilon_\ell} \times \underbrace{p_{i,j}(n,k)}_{\in \mathbb{K}(n)[k]}.$$

Examples: $\binom{n}{k}$, $\frac{1}{n-k}$, $n^{\underline{k}}$. Counter-example: $\frac{1}{n^2+k^2}$.

Wilf and Zeilberger's Algorithm

Theorem

Fasenmyer's ansatz can be solved for the setting:

$$r = B \quad \text{and} \quad s = (A-1)\,B + \deg_k(P) + 1,$$

where

$$A = \sum_\ell |a_\ell| \quad \text{and} \quad B = \sum_\ell |b_\ell|.$$

Proof: Observe $\deg_k p_{i,j} \leqslant \deg_k P + Ar + Bs$, set $r = B$, and enforce

$$\deg_k P + Ar + Bs + 1 < (r+1)\,(s+1).$$

Generalisations:

- to more than 2 indices, with non-explicit bounds;
- to proper q-hypergeometric terms, with similar bounds:

$$h_{n,k} = P(q^n, q^k)\,\zeta^n \xi^k q^{\alpha n^2 + \beta nk + \gamma k^2 + \lambda\binom{n}{2} + \mu\binom{k}{2}} \prod_{\ell=1}^{L} \left((q; c_\ell)_{a_\ell n + b_\ell k}\right)^{\epsilon_\ell}.$$

Really an Algorithm?

What if Wilf and Zeilberger's output is *zero*?

$$\underbrace{\sum_{i=0}^{r} \underbrace{\left(\sum_{j=0}^{s} c_{i,j}(n)\right)}_{=0} h_{n+i,k}}_{=0} = g_{n,k+1} - g_{n,k} \;?$$

$\longrightarrow$ Summation over k will deliver nothing, so?

Change of Notation: Recurrence Operators

Sequences:

$$u : (n, k) \mapsto u_{n,k}.$$

Shift operators:

$$S_n u : (n, k) \mapsto u_{n+1,k} \quad \text{and} \quad S_k u : (n, k) \mapsto u_{n,k+1}.$$

Multiplication operators:

$$n u : (n, k) \mapsto n u_{n,k} \quad \text{and} \quad k u : (n, k) \mapsto k u_{n,k}.$$

Operator algebras, e.g., $\mathbb{K}(n)[k]\langle S_n, S_k \rangle$, in which:

$$S_n n = (n + 1)\, S_n \quad \text{and} \quad S_k k = (k + 1)\, S_k.$$

Wegschaider's Fix

Given $L = \sum_{i=0}^{r} \sum_{j=0}^{s} c_{i,j}(n)\, S_n^i S_k^j \in \mathbb{K}(n)\langle S_n, S_k \rangle$ such that $Lh = 0$, find a nonzero $P \in \mathbb{K}(n)\langle S_n \rangle$ such that $Ph = (S_k - 1)(\cdots)$, in the bad case $L = (S_k - 1)^m \tilde{L}$.

Observe the commutation:

$$(k-a)^{\underline{\ell}}(S_k - 1) = (S_k - 1)(k - a - 1)^{\underline{\ell}} - \ell(k - a - 1)^{\underline{\ell - 1}},$$

so that, after iterating,

$$k^{\underline{m}}(S_k - 1)^m = (-1)^m m! + (S_k - 1)\,(\cdots).$$

As $(-1)^m m!^{-1} k^{\underline{m}} Lh = 0$, write $\tilde{L} = P + (S_k - 1)Q$ to get:

$$0 = \underbrace{P(n, S_n)}_{\neq 0}\, h + \underbrace{(S_k - 1)\, \underbrace{\hat{Q}(n, k, S_n, S_k)}_{\text{dependency in } k}\, h}_{\text{still sums right!}}.$$

Contents

Differentiably Finite Functions and Series

Definition (D-finite term; hyperexponential term)

An element f of a finite-dimensional $\mathbb{K}(x,y)$-vector space closed under derivations is called *differentiably finite*, in short *D-finite*.

$$\dim \operatorname{span} \sum_{i,j \geqslant 0} \mathbb{K}(x,y)\, D_x^i D_y^j f < \infty.$$

The 1-dimensional case is called *hyperexponential*.

Vast closure properties, under: derivation, addition, product, and ...

Lipshitz's Closure under Diagonals

Series $f = \sum_{n,m \in \mathbb{N}} c_{n,m} x^n y^m \in \mathbb{Q}[[x,y]] \to$ *diagonal* $\Delta f = \sum_{n \in \mathbb{N}} c_{n,n} x^n \in \mathbb{Q}[[x]]$.

Theorem (Lipshitz, 1989)

The diagonal of a D-finite series is D-finite.

To prepare for the proof, introduce

$$g = \frac{1}{s} f\left(s, \frac{x}{s}\right) \in \bigcup_{m \in \mathbb{Z}} \left\{ \sum_{(p,q) \in \mathbb{Z}^2,\ p+q \geqslant m} \phi_{p,q} s^p x^q \right\},$$

so that $\Delta f = \operatorname{res}_s g$, where

$$\operatorname{res}_s \sum_{(p,q) \in \mathbb{Z}^2,\ p+q \geqslant m} \phi_{p,q} s^p x^q = \sum_{q=m+1}^{\infty} \phi_{-1,q} x^q.$$

Differential Operators

Functions:

$$u : (x, y) \mapsto u(x, y).$$

Derivation operators:

$$D_x u : (x, y) \mapsto \frac{\partial u}{\partial x}(x, y) \quad \text{and} \quad D_y u : (x, y) \mapsto \frac{\partial u}{\partial y}(x, y).$$

Multiplication operators:

$$xu : (x, y) \mapsto x\, u(x, y) \quad \text{and} \quad yu : (x, y) \mapsto y\, u(x, y).$$

Operator algebras, e.g., $\mathbb{K}(x)[y]\langle D_x, D_y\rangle$, in which:

$$D_x x = x D_x + 1 \quad \text{and} \quad D_y y = y D_y + 1.$$

Remark: if $p \in \mathbb{K}[x, y]$,

$$D_x^a D_y^b p = p D_x^a D_y^b + (\text{terms of total order less than } a + b).$$

Lipshitz's Elementary Proof

By D-finiteness, g annihilated by:

$$A(s,x,D_s) = \lambda(s,x)\,D_s^m + (\text{lower order terms in } D_s \text{ only}),$$
$$B(s,x,D_x) = \lambda(s,x)\,D_x^{m'} + (\text{lower order terms in } D_x \text{ only}),$$
$$\text{for } \lambda \in \mathbb{K}[s,x] \quad \text{and} \quad \text{all degrees} \leqslant h.$$

By induction using $\lambda D_s^a D_x^b g = \sum_{0 \leqslant i+j \leqslant a+b-1} (\deg \leqslant h)\, D_s^i D_x^j g$,

$$\lambda^{a+b} x^c D_s^a D_x^b g = \sum_{0\leqslant i<m,\ 0\leqslant j<m'} (\deg \leqslant (a+b)\,h + c)\, D_s^i D_x^j g.$$

Dimension analysis: for $a+b+c \leqslant N$,

$$\text{dim. } \binom{N+3}{3} \simeq \Theta(N^3) \longrightarrow \text{dim. } \binom{(h+1)\,N+2}{2} \simeq \Theta(N^2).$$

g killed by $L(x,D_x,D_s) = P(x,D_x)\,D_s^v + (\text{higher-order deriv. w.r.t. } s)$, so that P ($\neq 0$!) kills $\operatorname{res}_s g = \Delta f$.

Contents

History for Zeilberger's Fast Algorithm(s)

- (Gosper, 1978) "Decision procedure for indefinite hypergeometric summation"
- (Almkvist and Zeilberger, 1990) "The method of differentiating under the integral sign"
- (Zeilberger, 1991) "The method of creative telescoping"
- (Chyzak, 2000) "An extension of Zeilberger's fast algorithm to general holonomic functions"
- (Koutschan, 2010) "A fast approach to creative telescoping"

Gosper's Algorithm (1/2)

Specifications

INPUT: a hypergeometric term f_k.
OUTPUT: a rational function $R(k)$ such that Rf is an indefinite sum w.r.t. k of f, or $\nexists$ ("a proof that no such R exists").

Simplified variant (basing on Abramov's algorithm)

1. Rewrite the equation

$$f_k = R(k+1)\,f_{k+1} - R(k)\,f_k$$

in a rational function R into

$$1 = R(k+1)\,\rho(k) - R(k).$$

2. Solve by Abramov's decision algorithm: finding no R is a proof that none exists.

Gosper's Algorithm (2/2)

Original variant

1. Write the ratio $\rho(k) = f_{k+1}/f_k$ in the (unique) form

$$\rho(k) = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$$

so that $\gcd(p,q) = \gcd(p,r) = \gcd\big(q(k), r(k+h)\big) = 1$ for all integer $h > 0$.

2. The change of variables $R = \frac{rS}{p}$ yields an equation in a polynomial S:

$$p(k) = q(k)\,S(k+1) - r(k)\,S(k).$$

3. Solve (using explicit bounds). If an S is found, return $R = rS/p$, else, this is a proof that no R exists.

Both variants reduce to linear-system solving.

Example of Use of Gosper's Algorithm

$$\sum_{k=0}^{n-1} \boxed{\frac{(-1)^k(4k+1)\binom{2k+1}{k}}{4^k(4k^2-1)}} = -\frac{2(n+1)}{4n+1} \boxed{\frac{(-1)^n(4n+1)\binom{2n+1}{n}}{4^n(4n^2-1)}} - 2.$$

The hypergeometric summand is given by:

$$\rho = -\frac{1}{2}\frac{(4k+5)(2k-1)}{(4k+1)(k+2)},$$

$$p(k) = 4k+1, \qquad q(k) = \frac{1}{2} - k, \qquad r(k) = k+2.$$

Parametrised Gosper Algorithm

Specifications

INPUT: a hypergeometric f_k and rational functions $s_0(k), \dots, s_m(k)$.
OUTPUT: a rational function $R(k)$ and constants $\eta_0, \dots, \eta_m$ such that Rf is an indefinite sum w.r.t. k of

$$\big(\eta_0\, s_0(k) + \cdots + \eta_m\, s_m(k)\big)\, f_k,$$

or $\nexists$ ("a proof that no such R exists for any family $\{\eta_i\}$").

Sketch of algorithm

η_i involved only linearly and in inhomogeneous side of equation $\rightarrow$ simply linear solving for $S(k)$ with additional unknowns $\eta_0, \dots, \eta_m$.

Zeilberger's ("Fast") Algorithm

Specifications

INPUT: hypergeometric term $f_{n,k}$.
OUTPUT: rational functions $\eta_0(n), \dots, \eta_r(n), \phi(n,k)$ for minimal $r \in \mathbb{N}$ such that

$$\eta_r(n) f_{n+r,k} + \cdots + \eta_0(n) f_{n,k} = \phi(n, k+1) f_{n,k+1} - \phi(n,k) f_{n,k}.$$

Termination not guaranteed in general, but it is in "holonomic" case. Explicit criterion due to Abramov.

Sketch

For increasing values $r = 0, 1, \dots$:

- Compute the rational functions $s_i(n,k) = f_{n+i,k}/f_{n,k}$.
- Appeal to the parametrised Gosper algorithm.
- If $(\phi, \{\eta_i\})$ is found, return it (else loop).

A q-analogue exists for q-hypergeometric terms.

Example of Use of Zeilberger's Algorithm (1/5)

Jacobi's orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed in terms of Gauss's hypergeometric function ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\middle|\frac{1-x}{2}\right),$$

for

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k.$$

Zeilberger's algorithm provides recurrences in n, α, or β, like:

$$\begin{aligned}0 &= 2(n+2)(n+\alpha+\beta+2)(2n+\alpha+\beta+2)\,P_{n+2}^{(\alpha,\beta)}(x)\\ &\quad - \left((2n+\alpha+\beta+2)_3 x + (2n+\alpha+\beta+3)(\alpha-\beta)(\alpha+\beta)\right)P_{n+1}^{(\alpha,\beta)}(x)\\ &\quad + 2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+4)\,P_n^{(\alpha,\beta)}(x).\end{aligned}$$

Slight extension: mixed recurrences, contiguity relations.

Example of Use of Zeilberger's Algorithm (2/5)

The summand is

$$f_k = \frac{(\alpha+1)_n}{n!} \frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!} \frac{(1-x)^k}{2^k}.$$

The ratio f_{k+1}/f_k is

$$\rho = \frac{1}{2} \frac{(n-k)\,(n+k+\alpha+\beta+1)\,(x-1)}{(k+1)\,(k+\alpha+1)}.$$

Example of Use of Zeilberger's Algorithm (3/5)

The parametrised recurrence to solve for rational solutions is

$$
\begin{aligned}
&(n+2-k)\,(n+1-k)\,(n-k)\,(n+\alpha+\beta+2) \\
&\qquad (n+\alpha+\beta+1)\,(n+k+\alpha+\beta+1)\,(x-1)\,R(k+1) \\
&-2(k+1)\,(k+\alpha+1)\,(n+\alpha+\beta+1) \\
&\qquad (n+2-k)\,(n+1-k)\,(n+\alpha+\beta+2)\,R(k) \\
&\qquad = 2\eta_0\,(k+1)\,(k+\alpha+1)\,(n+\alpha+\beta+1) \\
&\qquad\qquad (n+1-k)\,(n+2-k)\,(n+\alpha+\beta+2) \\
&\qquad + 2\eta_1\,(n+k+\alpha+\beta+1)\,(n+\alpha+1)\,(k+1) \\
&\qquad\qquad (k+\alpha+1)\,(n+2-k)\,(n+\alpha+\beta+2) \\
&\qquad + 2\eta_2\,(n+\alpha+2)\,(n+\alpha+1)\,(n+\alpha+\beta+2+k) \\
&\qquad\qquad (n+k+\alpha+\beta+1)\,(k+1)\,(k+\alpha+1).
\end{aligned}
$$

Example of Use of Zeilberger's Algorithm (4/5)

A multiple of the denominators of all solutions is

$$(n+2-k)\,(n+1-k).$$

The recurrence rewrites

$$\begin{aligned}\big(-(n+\alpha+\beta+2)\,(n+\alpha+\beta+1)\,(x-1)\,k^2+\cdots\big)\,S(k+1)\\ +\big(-2(n+\alpha+\beta+2)\,(n+\alpha+\beta+1)\,k^2+\cdots\big)\,S(k)\\ =(c_0k^4+\cdots)\,\eta_0+(c_1k^4+\cdots)\,\eta_1+(c_2k^4+\cdots)\,\eta_2.\end{aligned}$$

Therefore, any solution $S(k)$ has degree at most 2 in k.

Example of Use of Zeilberger's Algorithm (5/5)

Solving yields $P(n,k) = Q(n,k+1) - Q(n,k)$ where

$$\begin{aligned}
P(n,k) = {} & 2(n+\beta+1)\,(n+\alpha+1)\,(2n+\alpha+\beta+4)\,u(n,k) \\
& - (2n+\alpha+\beta+3)\,(\alpha^2+\alpha^2x+4\alpha nx+2\alpha\beta x+6\alpha x+12nx \\
& + \beta^2x+4nx\beta+4n^2x+6\beta x+8x-\beta^2)\,u(n+1,k) \\
& + 2(n+2)\,(n+\alpha+\beta+2)\,(2n+\alpha+\beta+2)\,u(n+2,k),
\end{aligned}$$

$$Q(n,k) = \frac{N(n,k)}{D(n,k)}u(n,k) \quad \text{for}$$

$$\begin{aligned}
N(n,k) &= -2(2n+\alpha+\beta+4)\,(2n+\alpha+\beta+3)\,(2n+\alpha+\beta+2) \\
&\quad \times (n+\alpha+1)\,(\alpha+k)\,k \\
D(n,k) &= (n+\alpha+\beta+1)\,(n+1-k)\,(n+2-k).
\end{aligned}$$

The announced recurrence is obtained after summation.

Almkvist and Zeilberger's Algorithm

Strict analogue for bivariate hyperexponential functions:

$$\frac{D_x f(x,y)}{f(x,y)} = R(x,y) \in \mathbb{K}(x,y) \quad \text{and} \quad \frac{D_y f(x,y)}{f(x,y)} = S(x,y) \in \mathbb{K}(x,y).$$

Example:

$$F(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy \quad \text{for} \quad f(x,y) = \exp\left(-\frac{x^2}{y^2} - y^2\right).$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right) f = D_y\left(\frac{2(3y^2 - 2x^2)}{y^3} f\right),$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right) F = 0,$$

$$F(x) = \sqrt{\pi}\exp(-2x).$$

Termination always garanteed by "holonomy"!

Functions versus Equations versus Vector Space

Special function or combinatorial sequence f:

$$f(n,z) = J_n(z) \qquad \text{(Bessel function).}$$

[Also: algebraic, trigonometric, elementary, transcendental, hypergeometric functions; binomial coefficients, harmonic numbers, hypergeometric sequences; orthogonal polynomials; q-analogues.]

Linear functional system (+ initial conditions):

$$z^2 J_n''(z) + zJ_n'(z) + (z^2 - n^2) J_n(z) = 0, \quad zJ_n'(z) + zJ_{n+1}(z) - nJ_n(z) = 0,$$
$$zJ_{n+2}(z) - 2(n+1) J_{n+1}(z) + zJ_n(z) = 0.$$

Vector space closed under D_z and S_n, here **finite-dimensional**:

$$V = \mathbb{C}(n,z) J_n(z) \oplus \mathbb{C}(n,z) J_{n+1}(z) = \mathbb{C}(n,z) J_n(z) \oplus \mathbb{C}(n,z) J_n'(z).$$

∂-Finite Terms

$$\partial_i = D_i \text{ or } S_i$$

t is ∂-finite w.r.t. the operator algebra
$\mathbb{K}(x_1, \dots, x_m)\langle \partial_1, \dots, \partial_m \rangle$

$\Updownarrow$

the $\partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} t$'s span a
finite-dimensional $\mathbb{K}(x_1, \dots, x_m)$-vector space:

$$\dim_{\mathbb{K}(x_1,\dots,x_m)} \left(\mathbb{K}(x_1, \dots, x_m)\langle \partial_1, \dots, \partial_m \rangle t \right) < +\infty$$

t is described by
higher-order linear functional equations.

Algorithmic closures under $+$, $\times$, the ∂_i's, integration, summation
$\Longrightarrow$ simplification and zero test of ∂-finite expressions.

Extended Gosper Decision Algorithm

ALGORITHM: $T = \textbf{Indefinite}\big(t, (b_i)_{i=1,\dots,d}, A\big)$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } t \in V = \bigoplus_{i=1}^{d} \mathbb{K}(x)\, b_i, \\ \text{an operator algebra } A = \mathbb{K}(x)\langle\partial\rangle, \\ \text{the action of } A \text{ on } V. \end{cases}$

OUTPUT: $T \in V$ such that $\partial T = t$; or $\nexists$.

1. let $T = \phi_1 b_1 + \cdots + \phi_d b_d$ for undetermined coefficients $\phi_i \in \mathbb{K}(x)$;
2. extract the coefficients of $\partial T - t$ in the b_i's to obtain a first-order functional system in the ϕ_i's;
3. solve for rational solutions (uncoupling + Abramov's decision algorithms);
4. if solvable return T; otherwise return $\nexists$.

Example of Indefinite ∂-Finite Summation

Input: $\begin{cases} \sum_{j=1}^{k} t_j \text{ for } t_k = \binom{k}{p} H_k, \quad H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \\ (\cdots)\, t_{k+2} + (\cdots)\, t_{k+1} + (\cdots)\, t_k = 0. \end{cases}$

Algorithmic reformulation: $A = \mathbb{Q}(p,k)\langle S_k \rangle, \quad T = \phi_0 t_k + \phi_1 t_{k+1}.$

$$(k+2-p)\,\phi_0(k+1) + (2k+3)\,\phi_1(k+1) - (k+2-p)\,\phi_1(k) = 0,$$

$$(k+2-p)\,(k+1-p)\,\phi_0(k) + (k+1)^2\,\phi_1(k+1) = -(k+2-p)\,(k+1-p).$$

$$\longrightarrow \quad (\cdots)\,\phi_1(k+2) + (\cdots)\,\phi_1(k+1) + (\cdots)\,\phi_1(k) = (\cdots).$$

Output: $\begin{cases} \phi_0(k) = \frac{(k-p)(k+p+2)}{(p+1)^2}, \; \phi_1(k) = -\frac{(k-p)(k-p+1)}{(p+1)^2}, \\ T = \sum t_k\, \delta k = \binom{k}{p} \frac{k-p}{(p+1)^2} \left((p+1)H_k - 1\right). \end{cases}$

Example of Indefinite ∂-Finite Integration

Input: $\begin{cases} \int \mathrm{Ci}(z)\,dz \text{ for } t(z) = \mathrm{Ci}(z) = \int_0^z \frac{\cos(t)-1}{t}\,dt, \\ (\cdots)\,t'''(z) + (\cdots)\,t''(z) + (\cdots)\,t'(z) = 0. \end{cases}$

Algorithmic reformulation: $A = \mathbb{Q}(z)\langle D_z\rangle, \quad T = \phi_0 t + \phi_1 t' + \phi_2 t''.$

$$\begin{cases} z\phi_1 + z\phi_2' - 2\phi_2 = 0, \\ z^2\phi_0 + z^2\phi_1' - z\phi_1 - z\phi_2' + (2-z^2)\,\phi_2 = 0, \\ \phi_0' = 0. \end{cases}$$

$$\longrightarrow \quad z^3\phi_2''' - 2z^2\phi_2'' + (z^3+4z)\,\phi_2' - 4\phi_2 = 0.$$

Output: $\begin{cases} \phi_0(z) = z, \qquad \phi_1(z) = 1, \qquad \phi_2(z) = z, \\ T = \int \mathrm{Ci}(z)\,dz = z\,\mathrm{Ci}(z) - \sin(z). \end{cases}$

Extended Zeilberger (a.k.a. Chyzak's) Algorithm

ALGORITHM: $(P, Q) = \textbf{Definite}\big(u, (b_i)_{i=1,\dots,d}, A\big)$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } A = \mathbb{K}(x,y)\langle \partial_x, \partial_y \rangle, \\ \text{a finite } \mathbb{K}(x,y)\text{-basis } (b_i)_{i=1,\dots,d} \text{ of } Au. \end{cases}$

OUTPUT: $\begin{cases} P \in \mathbb{K}(x,y)\langle \partial_x \rangle, \\ Q \in A \text{ such that } Pu = \partial_y Qu, \text{ resp. } Pu = (\partial_y - 1)Qu. \end{cases}$

For increasing values of r:

1. let $P = \sum_{i=0}^{r} \eta_i \partial_x^i$ and $t = Pu$ for undetermined coefficients $\eta_i(x)$;
2. let $\big(T, (\eta_i)\big) = \textbf{ParamIndefinite}\big(t, (b_i), A, \{\eta_0, \dots, \eta_r\}\big)$.
3. if $T \neq \nexists$ return (P, Q) for $Qu = T$.

A ∂-Finite Integral: $\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = J_0(z)$

$$A = \mathbb{Q}(z,t)\langle D_z, D_t\rangle,\ f = \frac{\cos(zt)}{\sqrt{1-t^2}} \leftrightarrow \text{basis} = \left(f, D_z f = -t\frac{\sin(zt)}{\sqrt{1-t^2}}\right)$$

$$\wr\quad D_z^2 + t^2, \qquad t\,(1-t^2)\,D_t - (1-t^2)\,zD_z - t^2.$$

$$P = zD_z^2 + D_z + z, \quad Q = \frac{1-t^2}{t}D_z, \quad P\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} + \left[Q\frac{\cos(zt)}{\sqrt{1-t^2}}\right]_0^1 = 0:$$

$$(zD_z^2 + D_z + z)\,\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = 0 \qquad +\ 2 \text{ initial conditions.}$$

A Neumann Addition Theorem $J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1$

1. $A = \mathbb{Q}(z,k)\langle D_z, S_k\rangle$, $\quad J_k(z) \leftrightarrow$ basis $= \big(J_k(z), D_z J_k(z)\big)$:

$$\left\{ z^2 D_z^2 + z D_z + z^2 - k^2, \qquad z S_k + z D_z - k. \right.$$

2. By closure, $J_k(z)^2 \leftrightarrow$ basis $= \big(J_k(z)^2, S_k J_k(z)^2, D_z J_k(z)^2\big)$:

$$\left\{ \begin{array}{l} z D_z^2 + (-2k+1)\, D_z - 2z S_k + 2z, \\ z D_z S_k + z D_z + (2k+2)\, S_k - 2k, \\ z^2 S_k^2 - 4(k+1)^2 S_k - 2z\,(k+1)\, D_z + 4k\,(k+1) - z^2. \end{array} \right.$$

3. $P = D_z$, $Q = \frac{k}{z} + \frac{1}{2} D_z$, $\quad P \sum_{k=0}^{\infty} J_k(z)^2 + \Big[Q\, J_k(z)^2\Big]_{k=0}^{\infty} = 0$:

$$D_z \left(2 \sum_{k=1}^{\infty} J_k(z)^2 + J_0(z)^2 - 1 \right) = 0 \qquad + \text{ initial condition at } 0.$$

Koutschan's Heuristics

Chyzak's fast algorithm can be slow:

1. uncouple the system for the ϕ's (non-commutative Gauss elim.)
2. algorithmically bound denominators (Abramov's bound)
3. algorithmically bound numerator degrees (indicial equation)

Handles multiple sums/integrals iteratively.
Always finds a solution (at least theoretically).

Koutschan's fast heuristics

1. don't uncouple!
2. heuristically set denominator exponents (in accordance to Pu)
3. heuristically set numerator degrees (prop. to the denom. degs)

Simultaneous multiple sums/integrals possible, at times faster.
May fail to find a solution.

Contents

Non-Minimality of Computed Operators

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{pk}{n} = (-p)^n \qquad \text{(order } p-1 \text{ instead of 1),}$$

$$\sum_{k=0}^{n} \frac{1}{pk+1} \binom{pk+1}{k} \binom{p(n-k)}{n-k} = \binom{pn+1}{n}$$

(order 2 instead of 1 for $p = 3$).

This non-minimality is not understood well.

$$\eta_r(n)\, u_{n+r,k} + \cdots + \eta_0(n)\, u_{n,k} = v_{n,k+1} - v_{n,k}$$

The existence of v constrains d!

This relation does not take the summation range into account!

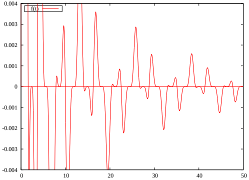
An Integral Not Amenable to Creative Telescoping?

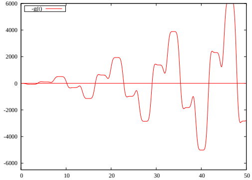
Borrowed from (Borwein, Straub, Wan, Zudilin, 2011).

$$F(x) = \int_0^\infty \underbrace{xt\, J_0(xt)\, J_0(t)^4}_{=f(x,t)}\, dt \quad \to \quad Pf = D_t g \quad \text{for} \quad g = Qf.$$

$$P = (x-4)(x-2)(x+2)(x+4)x^3D_x^3 + 6(x^2-10)x^4D_x^2 + (7x^4-32x^2+64)xD_x + (x^2-8)(x^2+8)$$

$$\begin{aligned} Q = {} & 5x^3t^2D_xD_t^3 - x^2t^3D_t^4 - t^{-1}x^2(5x^4t^4 - 60x^2t^4 + 64t^4 - 28t^2 - 4) - 7x^2t^2D_t^3 \\ & + x^2t(10x^2t^2 - 20t^2 - 1)D_t^2 - 5x^3(2x^2t^2 - 12t^2 - 1)D_xD_t \\ & + 4x^2(5x^2t^2 - 15t^2 - 1)D_t + 5t^{-1}x^3(2x^2t^2 - 12t^2 - 1)D_x \end{aligned}$$



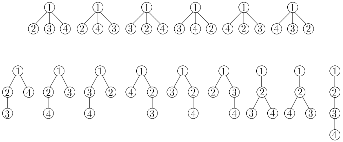


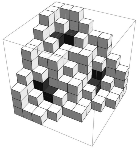
$D_x^i f|_{x=3/2} = \Theta\big(\cos(3\pi/2 + \pi/4)\ \sin(t + \pi/4)\ t^{i-3/2}\big)$! No limit of g at ∞ !

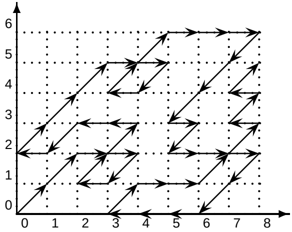
For $i > 0$, no limit of $\int_0^T D_x^i f|_{x=3/2}\, dt$ at ∞ !

Successful Applications of Creative Telescoping

- *Descendants in heap-ordered trees* (Prodinger, 1996): Zeilberger's algorithm ($2\times$)
- *Totally symmetric plane partition* (Kauers, Koutschan & Zeilberger, 2011): q-Koutschan ($n\times$)
- Lattice walks confined in a quadrant, in particular *Gessel's conjecture for* $\leftarrow, \rightarrow, \swarrow, \nearrow$ (Kauers, Koutschan & Zeilberger, 2009): variant of Takayama for specialisation ($n \leftrightarrow S_n$)







All were first proofs!