

Creative Telescoping for Parametrised Integration and Summation

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A Nice Story: Apéry's Proof of Irrationality of $\zeta(3)$

Proof, as explained in (van der Poorten, 1979)

Define:

$$b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2, \quad c_{n,k} = \sum_{m=1}^n \frac{1}{m^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}},$$

$$a_{n,k} = b_{n,k} c_{n,k}, \quad a_n = \sum_{k=0}^n a_{n,k}, \quad b_n = \sum_{k=0}^n b_{n,k},$$

$$g_n = \gcd(1, \dots, n), \quad p_n = 2g_n^3 a_n, \quad q_n = 2g_n^3 b_n.$$

Then, (a_n) and (b_n) satisfy the same 2nd-order recurrence, and:

$$\zeta(3) - \frac{a_n}{b_n} = \mathcal{O}(b_n^{-2}), \quad p_n \in \mathbb{Z}, \quad q_n \in \mathbb{N}, \quad \zeta(3) - \frac{p_n}{q_n} = \mathcal{O}(q_n^{-1.08}).$$

Classical irrationality criterion for $\alpha \in \mathbb{R}$:

$$\left(\forall \epsilon > 0, \exists \frac{p}{q}, \left| \alpha - \frac{p}{q} \right| < \frac{\epsilon}{q} \right) \implies \alpha \notin \mathbb{Q}.$$

Apéry's Recurrence for (a_n) and (b_n)

Second-order recurrence (Apéry, 1979)

$$(n+1)^3 u_{n+1} - (34n^3 + 3n^2 + 27n + 5) u_n + n^3 u_{n-1} = 0$$

Cohen and Zagier's "[Creative Telescoping](#)" (van der Poorten, 1979)

"[They] cleverly construct

$$B_{n,k} = 4(2n+1)(k(2k+1) - (2n+1)^2) b_{n,k}$$

with the motive that

$$B_{n,k} - B_{n,k-1} = (n+1)^3 b_{n+1,k} - (34n^3 + 51n^2 + 27n + 5) b_{n,k} + n^3 b_{n-1,k}."$$

After summation over k from 0 to $n+1$:

$$\underbrace{B_{n,n+1} - B_{n,-1}}_{0-0=0} = (n+1)^3 b_{n+1} - (34n^3 + 3n^2 + 27n + 5) b_n + n^3 b_{n-1}.$$

Differentiating under the Integral Sign

Zeilberger's derivation (1982) of a classical integral

$$f(b) = \int_{-\infty}^{+\infty} e^{-x^2} \cos 2bx \, dx = ?$$

$$\begin{aligned} f'(b) &= \int_{-\infty}^{+\infty} -2xe^{-x^2} \sin 2bx \, dx = \\ &= \left[e^{-x^2} \sin 2bx \right]_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} -2be^{-x^2} \cos 2bx \, dx = -2bf(b) \end{aligned}$$

Continuous analogue of creative telescoping:

$$\frac{dF}{dx}(b, x) = \frac{df}{db}(b, x) + 2bf(b, x) \quad \text{for} \quad F(b, x) = -\frac{1}{2x} \frac{df}{db}(b, x)$$

After integration over x from $-\infty$ to $+\infty$:

$$\underbrace{\left[\frac{dF}{dx}(b, x) \right]_{x=-\infty}^{x=+\infty}}_{0-0=0} = f'(b) + 2bf(b)$$

Sums and Integrals (1/5)

Binomial sums

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad (\text{Strehl, 1994})$$

$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2 \quad (\text{Blodgelt, 1990})$$

Integrals of the theory of special functions

Four types of **Bessel functions** (Glasser & Montaldi, 1994):

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

No explicit form, but a 2nd-order linear ODE:

$$\int_0^\infty \int_0^\infty J_1(x) J_1(y) J_2(c\sqrt{xy}) \frac{dx dy}{e^{x+y}}$$

Sums and Integrals (2/5)

Extractions of coefficients

Theory of **orthogonal polynomials**, here, Hermite (Doetsch, 1930):

$$\frac{1}{2\pi i} \oint \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2 y^2}{1 + 4y^2}\right)}{y^{n+1}(1 + 4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!}$$

Scalar products involving orthogonal/parametrised families

Chebyshev polynomials, Bessel functions, modified Bessel functions:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}} dx = (-1)^n \pi I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx) dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

Sums and Integrals (3/5)

q -Sums, e.g., from the theory of combinatorial partitions

Finite forms of the Rogers–Ramanujan identities and a generalisation: setting $(q; q)_n = (1 - q) \cdots (1 - q^n)$,

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \quad (\text{Andrews, 1974})$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \quad (\text{Paule, 1985})$$

Scalar products in algebraic combinatorics

For $p_1 = x_1 + x_2 + \cdots$ and $p_2 = x_1^2 + x_2^2 + \cdots$:

$$\left\langle \exp((p_1^2 - p_2)/2 - p_2^2/4) \mid \exp(t(p_1^2 + p_2)/2) \right\rangle = \frac{e^{-\frac{1}{4}t(t+2)}}{\sqrt{1-t}}$$

Sums and Integrals (4/5)

Combinatorial identities

In the graph-counting sequence k^{k-1} :

$$\sum_{k=0}^n \binom{n}{k} i (k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n \quad (\text{Abel})$$

In Stirling numbers of the second kind (partitions) and Eulerian numbers (ascents in permutations):

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle \quad (\text{Frobenius})$$

In Bernoulli numbers (Taylor expansion of $\tan(x)$):

$$\sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^n \binom{n}{k} B_{m+k} \quad (\text{Gessel, 2003})$$

Sums and Integrals (5/5)

Identities in more special functions (related, e.g., to number theory)

In Hurwitz's zeta function and the beta function:

$$\int_0^{\infty} x^{k-1} \zeta(n, \alpha + \beta x) dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$

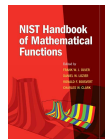
In the polylogarithm functions:

$$\int_0^{\infty} x^{\alpha-1} \text{Li}_n(-xy) dx = \frac{\pi (-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

In the (upper) incomplete Gamma function:

$$\int_0^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy) dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

+ a lot more in:



or in web sites, like Victor Moll's web site on GR

$$7. \int_0^{\infty} x^{-1/2} \sin b \operatorname{erf} x \left\{ \frac{\sin cx}{\cos cx} \right\} J_0(cx) dx = \sqrt{\frac{\pi}{2c}} \cos \left(\frac{b^2}{16c} \pm \frac{\pi}{4} \right) J_0 \left(\frac{b^2}{16c} \right) \quad (b, c > 0).$$

$$8. \int_0^{\infty} x^{-1/2} \cos \sqrt{x} \left(\frac{\sin cx}{\cos cx} \right) J_0(cx) dx =$$

$$= -\frac{1}{2} \sqrt{\frac{\pi}{2c}} \left[\frac{(\cos \sqrt{c})}{(\sin \sqrt{c})} J_0 \left(\frac{b^2}{16c} \right) Y_0 \left(\frac{b^2}{16c} \right) \right]$$

If $= b^2/16c = \pi/4$; $b, c > 0$

2.12.20. Integrals containing $x^a \begin{cases} \sin (bx+a/x) \\ \cos (bx+a/x) \end{cases} J_\nu(cx).$

$$1. \int_0^{\pi} x^{\alpha-1} \left[\frac{\sin(b(x))}{\cos(b(x))} J_{\nu}(cx) dx - 2^{\alpha-1} b^{\alpha} c^{-\alpha} \Gamma\left(\frac{\nu+\alpha-1}{2}\right) \left[\frac{\nu+\alpha-1}{1+\delta+\nu-\alpha/2} \right] \times \right. \\ \left. \times {}_2F_3\left(1-\frac{\nu+\alpha-\delta}{2}, \frac{1}{2}, \frac{1}{2}, 1+\frac{\delta+\nu-\alpha}{2}; \frac{\delta^2\alpha^2}{16}\right) \mp \right. \\ \left. \mp {}_2F_2\left(\frac{\nu}{2}, \frac{1}{2} \left[\frac{\sin(|\alpha+\nu|/2)}{\cos(|\alpha+\nu|/2)} \right] \right) \left[1-\frac{\nu-\alpha}{1+\nu} \right] {}_2F_3\left(1+\frac{\alpha+\nu}{2}, 1+\nu, \frac{1+\alpha+\nu}{2}; \frac{\delta^2\alpha^2}{16}\right) \right] \\ \left. \left\{ b, c, \alpha; 0: -1-\operatorname{Re} \nu < \operatorname{Re} \alpha < \alpha/2+3/2; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right\} \right\}$$

$$2. \int_0^{\infty} \frac{1}{x} \sin bx \left(\frac{\sin ax}{\cos(a/x)} \right) J_\nu(cx) dx =$$

$$= \frac{\pi}{2} J_\nu(z, \sqrt{a}) \left[\frac{\sin \sqrt{a}/2}{\cos(\sqrt{a}/2)} J_\nu(z, \sqrt{a}) \pm \frac{\cos(\sqrt{a}/2)}{\sin(\sqrt{a}/2)} Y_\nu(z, \sqrt{a}) \right] +$$

$$+ \frac{\cos(\sqrt{a}/2)}{\sin(\sqrt{a}/2)} J_\nu(z, \sqrt{a}) K_\nu(z, \sqrt{a}) \quad [0 < c < b; a > 0; z = \sqrt{b^2 + c^2} \pm \sqrt{b^2 - c^2}]$$

$$3. \int_0^{\infty} \frac{1}{x} \cos bx \left\{ \frac{\sin(a/x)}{\sin(a/2x)} \right\} J_\nu(cx) \, x =$$

$$= \pm \frac{\pi}{2} J_\nu(x, V_a) \left\{ \frac{\sin(\sqrt{a} \sqrt{V_a/2})}{\sin(\sqrt{a} \sqrt{V_a})} \right\} J_\nu(x, V_a) \mp \left\{ \frac{\sin(\sqrt{a} \sqrt{V_a/2})}{\cos(\sqrt{a} \sqrt{V_a/2})} \right\} Y_\nu(x, V_a) \mp$$

$$\pm \left\{ \frac{\sin(\sqrt{a} \sqrt{V_a/2})}{J_\nu(x, V_a)} K_\nu(x, V_a) \right\} \quad [0 < c < b; a > 0; x \pm \sqrt{b^2 + c^2} \pm \sqrt{b^2 - c^2}]$$

$$4. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\sin(bx - a/x)}{\cos(bx - a/x)} \right\} J_\nu(cx) dx = 2 \left\{ \frac{\sin(\sqrt{a}/2)}{\cos(\sqrt{a}/2)} \right\} I_\nu(z, \sqrt{a}) K_\nu(z, \sqrt{a})$$

$[0 < c < b; a > 0; z_{\pm} = \sqrt{b+c} \pm \sqrt{b-c}]$

2.12.21. Integrals of $A(x) \left\{ \frac{\sin(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} \right\} J_\nu(cx)$.

$$\begin{aligned} \text{I. } \int_0^a x^{v+1} \sin(b\sqrt{a^2-x^2}) J_v(cx) dx = \\ = \sqrt{\frac{\pi}{2}} a^{v+3/2} b c^v (b^2+c^2)^{-(2v+3)/4} J_{v+3/2}(a\sqrt{b^2+c^2}) \end{aligned}$$

$$2. \int_0^a x \sin(b \sqrt{a^2 - x^2}) J_0(cx) dx = \frac{ab}{b^2 + c^2} \left[\frac{\sin(a \sqrt{b^2 + c^2})}{a \sqrt{b^2 + c^2}} - \cos(a \sqrt{b^2 + c^2}) \right] \quad (a > 0).$$

$$J_v \int_0^a \frac{\cos(b \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} J_v(cx) dx = \frac{\pi}{2} J_{v/2} \left(\frac{a \sqrt{b^2 + c^2 + ab}}{2} \right) J_{v/2} \left(\frac{a \sqrt{b^2 + c^2 - ab}}{2} \right) \quad (a > 0; \operatorname{Re} v > -1).$$

$$4. \int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_1(cx) dx = \frac{1}{ac} \cos ab - \frac{1}{ac} \cos(a\sqrt{b^2+c^2})$$

$$5. \int_0^a x^{v+1} \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_v(cx) dx =$$

$$= \sqrt{\frac{\pi}{2}} a^{v+1/2} c^v (b^2+c^2)^{-(2v+1)/4} J_{v+1/2}(a\sqrt{b^2+c^2}) \quad [a>0; \operatorname{Re} v>-1]$$

$$6. \int_0^a x \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_0(cx) dx = \frac{\sin(a\sqrt{b^2+c^2})}{\sqrt{b^2+c^2}} \quad (a>0).$$

2.12.22. Integrals of $A(x) \left\{ \frac{\sin(b\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}} \right\} J_\nu(cx)$.

$$1. \int_a^{\infty} \sin(b\sqrt{x^2 - a^2}) J_1(cx) dx = \frac{b}{c} (c^2 - b^2)^{-1/2} \cos(a\sqrt{c^2 - b^2})$$

[a, b, c > 0; b ≠ c]

$$2. \int_a^{\infty} x^{1-\nu} \sin(b\sqrt{x^2-a^2}) J_\nu(cx) dx =$$

Creative Telescoping for Sums/Integrals

$$U_n = \sum_{k=a}^b u_{n,k} = ?$$

Given a relation $a_r(n)u_{n+r,k} + \cdots + a_0(n)u_{n,k} = v_{n,k+1} - v_{n,k}$, summation leads by “telescoping” to

$$a_r(n)U_{n+r} + \cdots + a_0(n)U_n = v_{n,b+1} - v_{n,a} \stackrel{\text{often}}{=} 0.$$

$$U(x) = \int_a^b u(x,y) dy = ?$$

Given a relation $a_r(x)\frac{\partial^r u}{\partial x^r} + \cdots + a_0(x)u = \frac{\partial}{\partial y}v(x,y)$, integrating leads by “telescoping” to

$$a_r(x)\frac{\partial^r U}{\partial x^r} + \cdots + a_0(x)U = v(x,b) - v(x,a) \stackrel{\text{often}}{=} 0.$$

Adapts easily to $U(x) = \sum_{k=a}^b u_k(x)$ and $U_n = \int_a^b u_n(y) dy$.

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History for Algorithms Based on Fasenmyer's Ansatz

- (Fasenmyer, 1945, 1947, 1949) "Some generalized hypergeometric polynomials", "A note on pure recurrence relations"
- (Rainville, 1960) "Special functions"
- (Verbaeten, 1974, 1976) "The automatic construction of pure recurrence relations", "Rekursiebetrekkingen voor lineaire hypergeometrische funkties"
- (Zeilberger, 1982) "Sister Celine's technique and its generalizations"
- (Lipshitz, 1988) "The diagonal of a D-finite power series is D-finite"
- (Zeilberger, 1990) "A holonomic systems approach to special functions identities"
- (Wilf and Zeilberger, 1992) "An algorithmic proof theory for hypergeometric (ordinary and ' q ') multisum/integral identities"
- (Hornegger, 1992) "Hypergeometrische Summation und polynomiale Rekursion"
- (Wegschaider, 1997) "Computer generated proofs of binomial multi-sum identities"
- (Tefera, 2000, 2002) "Improved algorithms and implementations in the multi-WZ theory", "MultiInt, a MAPLE package for multiple integration by the WZ method"
- (Riese, 2003) "qMultiSum: a package for proving q -hypergeometric multiple summation identities"

Hypergeometric Terms

Definition (hypergeometric terms)

An element h of a $\mathbb{K}(n, k)$ -vector space closed under shifts is *hypergeometric* if the quotients

$$\frac{h_{n+1,k}}{h_{n,k}} \quad \text{and} \quad \frac{h_{n,k+1}}{h_{n,k}}$$

exist and are rational functions.

Basic observation: for all $(i, j) \in \mathbb{Z}^2$,

$$\frac{h_{n+i,k+j}}{h_{n,k}} \in \mathbb{K}(n, k).$$

Generalises to more indices.

An **idealisation** of sequences like: $n!$, $\binom{n}{k}$, falling factorials $n^{\underline{k}} = n(n-1) \cdots (n-k+1)$, etc.

Fasenmyer's Heuristic, as Revisited by Zeilberger

Fasenmyer's ansatz: For a given hypergeometric $h_{n,k}$, solve

$$\sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) h_{n+i,k+j} = 0 \quad \text{where the } c\text{'s don't involve } k.$$

Motivation:

$$\sum_{i=0}^r \left(\sum_{j=0}^s c_{i,j}(n) \right) h_{n+i,k} = g_{n,k+1} - g_{n,k} \quad \text{where} \quad g_{n,k} = \sum_{i=0}^r \sum_{j=0}^{s-1} \tilde{c}_{i,j}(n) h_{n+i,k+j}.$$

Idea: Use the *rational functions* $h_{n+i,k+j}/h_{n,k}$, then **clear denominators**

$$\sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) p_{i,j}(n, k) = 0 \quad \text{for} \quad p_{i,j} \in \mathbb{K}(n)[k],$$

then solve a linear system over $\mathbb{K}(n)$. **If no solution, increase (r, s) .**

Question: possibility to enforce $\max \deg_k p_{i,j} < (r+1)(s+1)$?

Two Motivating Examples

Common denominator C_r of $h_{n+i,k+j}/h_{n,k}$ when $r = s$ increases?

- $\binom{n}{k} = \frac{n!}{k! (n-k)!}:$

$$C_r = (k+1) \cdots (k+r) (n+1-k) \cdots (n+r-k)$$

has **linearly many terms**. The method **succeeds** easily (by finding Pascal's triangle rule).

- $\frac{1}{n^2 + k^2}:$

$$C_r = \prod_{i,j \in \mathbb{N}, i+j \leq r} ((n+i)^2 + (k+j)^2)$$

has **quadratically many terms**. The method seems to **fail** (out of memory).

Wilf and Zeilberger's Proper Hypergeometric Terms

Definition (proper hypergeometric term)

A hypergeometric term h is *proper* if it can be written, for $\epsilon_\ell = \pm 1$,

$$h_{n,k} = \underbrace{P(n,k)}_{P \in \mathbb{K}[n,k]} \underbrace{\zeta^n \xi^k}_{\zeta, \xi \in \mathbb{K}} \prod_{\ell=1}^L \Gamma(\underbrace{a_\ell n + b_\ell k + c_\ell}_{a_\ell, b_\ell \in \mathbb{Z}, c_\ell \in \mathbb{K}})^{\epsilon_\ell}.$$

$$\Gamma(s+1) = s \Gamma(s) \implies \Gamma(s+u)/\Gamma(s) = \text{poly. of degree } u \text{ in } s$$

Key observation

$$h_{n+i,k+j} \rightarrow \Gamma(a_\ell n + b_\ell k + c_\ell + u) \quad \text{where} \quad |u| \leq |a_\ell|r + |b_\ell|s =: \sigma_\ell,$$

$$h_{n+i,k+j} = \zeta^n \xi^k \prod_{\ell=1}^L \Gamma(a_\ell n + b_\ell k + c_\ell - \epsilon_\ell \sigma_\ell)^{\epsilon_\ell} \times \underbrace{p_{i,j}(n,k)}_{\in \mathbb{K}(n)[k]}.$$

Examples: $\binom{n}{k}$, $\frac{1}{n-k}$, n^k .

Counter-example: $\frac{1}{n^2+k^2}$.

Wilf and Zeilberger's Algorithm

Theorem

Fasenmyer's ansatz can be solved for the setting:

$$r = B \quad \text{and} \quad s = (A - 1)B + \deg_k(P) + 1,$$

where

$$A = \sum_{\ell} |a_{\ell}| \quad \text{and} \quad B = \sum_{\ell} |b_{\ell}|.$$

Proof: Observe $\deg_k p_{i,j} \leq \deg_k P + Ar + Bs$, set $r = B$, and enforce

$$\deg_k P + Ar + Bs + 1 < (r + 1)(s + 1).$$

Generalisations:

- to more than 2 indices, with non-explicit bounds;
- to proper q -hypergeometric terms, with similar bounds:

$$h_{n,k} = P(q^n, q^k) \zeta^n \zeta^k q^{\alpha n^2 + \beta nk + \gamma k^2 + \lambda \binom{n}{2} + \mu \binom{k}{2}} \prod_{\ell=1}^L ((q; c_{\ell})_{a_{\ell}n + b_{\ell}k})^{\epsilon_{\ell}}.$$

Really an Algorithm?

What if Wilf and Zeilberger's output is *zero*?

$$\underbrace{\sum_{i=0}^r \underbrace{\left(\sum_{j=0}^s c_{i,j}(n) \right)}_{=0} h_{n+i,k}}_{=0} = g_{n,k+1} - g_{n,k} ?$$

→ Summation over k will deliver nothing, so?

Change of Notation: Recurrence Operators

Sequences:

$$u : (n, k) \mapsto u_{n,k}.$$

Shift operators:

$$\textcolor{red}{S}_n u : (n, k) \mapsto u_{n+1,k} \quad \text{and} \quad \textcolor{red}{S}_k u : (n, k) \mapsto u_{n,k+1}.$$

Multiplication operators:

$$\textcolor{red}{n}u : (n, k) \mapsto \textcolor{red}{n}u_{n,k} \quad \text{and} \quad \textcolor{red}{k}u : (n, k) \mapsto \textcolor{red}{k}u_{n,k}.$$

Operator algebras, e.g., $\mathbb{K}(n)[k]\langle S_n, S_k \rangle$, in which:

$$\textcolor{blue}{S}_n n = (n+1) S_n \quad \text{and} \quad \textcolor{blue}{S}_k k = (k+1) S_k.$$

Wegschaider's Fix

Given $L = \sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) S_n^i S_k^j \in \mathbb{K}(n) \langle S_n, S_k \rangle$ such that $Lh = 0$, find a nonzero $P \in \mathbb{K}(n) \langle S_n \rangle$ such that $Ph = (S_k - 1)(\cdots)$, **in the bad case** $L = (S_k - 1)^m \tilde{L}$.

Observe the commutation:

$$(k-a)^\ell (S_k - 1) = (S_k - 1)(k-a-1)^\ell - \ell(k-a-1)^{\ell-1},$$

so that, after iterating,

$$k^m (S_k - 1)^m = (-1)^m m! + (S_k - 1)(\cdots).$$

As $(-1)^m m!^{-1} k^m Lh = 0$, write $\tilde{L} = P + (S_k - 1)Q$ to get:

$$0 = \underbrace{P(n, S_n)}_{\neq 0} h + (S_k - 1) \underbrace{\hat{Q}(n, k, S_n, S_k)}_{\substack{\text{dependency in } k \\ \text{still sums right!}}} h.$$

Verbaeten's Completion (1/2)

$$\text{Ex: } \sum_{k=0}^{2n} \underbrace{(-1)^k \binom{2n}{k}}_{:=h_{n,k}}^3 = (-1)^n \frac{(3n)!}{n!^3} \quad (\text{Dixon, 1891}).$$

$$r_{i,j} := \frac{h_{n-i,k-j}}{h_{n,k}} = (\dots) \left(\frac{k!}{(k-j)!} \frac{(2n-k)!}{(2n-k-(2i-j))!} \right)^3$$

Denom. of $r_{i,j}$ constant on a parallel to Δ .

Common denom. of $r_{i,j}$ over the $\bullet$'s =

$$(k-2n-4)^3 \cdots (k-2n-1)^3 =: C.$$

$$\deg_k C r_{i,j} = 3(2i+4) \leq 30 \text{ over } \bullet\text{'s and } \bullet\text{'s.}$$

of unknowns = # of $\bullet$'s and $\bullet$'s = 36

equations = 1 + maximal degree in k = 31

$\implies$ success after completion!

Verbaeten's Completion (2/2)

- Verbaeten could already solve Fasenmyer's ansatz for a class of special proper hypergeometric terms;
- Elegant geometric interpretation by Hornegger (volumes of polytopes);
- Nicely put to practice, also in the multisum case, by Wegschaider for sums, resp. Riese for q -sums.

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- 1 Introduction: Early Days and Examples
- 2 Fasenmyer's (a.k.a. "*k*-Free") Ansatz
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Differentiably Finite Functions and Series

Definition (D-finite term; hyperexponential term)

An element f of a finite-dimensional $\mathbb{K}(x, y)$ -vector space closed under derivations is called *differentiably finite*, in short *D-finite*.

$$\dim \operatorname{span} \sum_{i,j \geq 0} \mathbb{K}(x, y) D_x^i D_y^j f < \infty.$$

The 1-dimensional case is called *hyperexponential*.

Vast closure properties, under: derivation, addition, product, and ...

Lipshitz's Closure under Diagonals

$$\text{Series } f = \sum_{n,m \in \mathbb{N}} c_{n,m} x^n y^m \in \mathbb{Q}[[x, y]] \rightarrow \text{diagonal } \Delta f = \sum_{n \in \mathbb{N}} c_{n,n} x^n \in \mathbb{Q}[[x]].$$

Theorem (Lipshitz, 1989)

The diagonal of a D-finite series is D-finite.

To prepare for the proof, introduce

$$g = \frac{1}{s} f\left(s, \frac{x}{s}\right) \in \bigcup_{m \in \mathbb{Z}} \left\{ \sum_{(p,q) \in \mathbb{Z}^2, p+q \geq m} \phi_{p,q} s^p x^q \right\},$$

so that $\Delta f = \text{res}_s g$, where

$$\text{res}_s \sum_{(p,q) \in \mathbb{Z}^2, p+q \geq m} \phi_{p,q} s^p x^q = \sum_{q=m+1}^{\infty} \phi_{-1,q} x^q.$$

Differential Operators

Functions:

$$u : (x, y) \mapsto u(x, y).$$

Derivation operators:

$$D_x u : (x, y) \mapsto \frac{\partial u}{\partial x}(x, y) \quad \text{and} \quad D_y u : (x, y) \mapsto \frac{\partial u}{\partial y}(x, y).$$

Multiplication operators:

$$xu : (x, y) \mapsto xu(x, y) \quad \text{and} \quad yu : (x, y) \mapsto yu(x, y).$$

Operator algebras, e.g., $\mathbb{K}(x)[y]\langle D_x, D_y \rangle$, in which:

$$D_x x = x D_x + 1 \quad \text{and} \quad D_y y = y D_y + 1.$$

Remark: if $p \in \mathbb{K}[x, y]$,

$$D_x^a D_y^b p = p D_x^a D_y^b + (\text{terms of total order less than } a + b).$$

Lipshitz's Elementary Proof

By D-finiteness, g annihilated by:

$$\begin{aligned} A(s, x, D_s) &= \lambda(s, x) D_s^m + (\text{lower order terms in } D_s \text{ only}), \\ B(s, x, D_x) &= \lambda(s, x) D_x^{m'} + (\text{lower order terms in } D_x \text{ only}), \\ &\text{for } \lambda \in \mathbb{K}[s, x] \quad \text{and} \quad \text{all degrees} \leq h. \end{aligned}$$

$$\begin{aligned} \text{By induction using } \lambda D_s^a D_x^b g &= \sum_{0 \leq i+j \leq a+b-1} (\deg \leq h) D_s^i D_x^j g, \\ \lambda^{a+b} x^c D_s^a D_x^b g &= \sum_{0 \leq i < m, 0 \leq j < m'} (\deg \leq (a+b)h + c) D_s^i D_x^j g. \end{aligned}$$

Dimension analysis: for $a + b + c \leq N$,

$$\dim. \binom{N+3}{3} \simeq \Theta(N^3) \longrightarrow \dim. \binom{(h+1)N+2}{2} \simeq \Theta(N^2).$$

g killed by $L(x, D_x, D_s) = P(x, D_x) D_s^v + (\text{higher-order deriv. w.r.t. } s)$,
so that $P (\neq 0!)$ kills $\text{res}_s g = \Delta f$.

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History for Zeilberger's Fast Algorithm(s)

- (Gosper, 1978) "Decision procedure for indefinite hypergeometric summation"
- (Almkvist and Zeilberger, 1990) "The method of differentiating under the integral sign"
- (Zeilberger, 1991) "The method of creative telescoping"
- (Chyzak, 2000) "An extension of Zeilberger's fast algorithm to general holonomic functions"
- (Koutschan, 2010) "A fast approach to creative telescoping"

Gosper's Algorithm (1/2)

Specifications

INPUT: a hypergeometric term f_k .

OUTPUT: a rational function $R(k)$ such that Rf is an **indefinite sum** w.r.t. k of f , or $\nexists$ ("a proof that no such R exists").

Simplified variant (basing on Abramov's algorithm)

- ① Rewrite the equation

$$f_k = R(k+1)f_{k+1} - R(k)f_k$$

in a rational function R into

$$1 = R(k+1)\rho(k) - R(k).$$

- ② Solve by **Abramov's decision algorithm**: finding no R is a **proof** that none exists.

Gosper's Algorithm (2/2)

Original variant

- 1 Write the ratio $\rho(k) = f_{k+1}/f_k$ in the (unique) form

$$\rho(k) = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$$

so that $\gcd(p, q) = \gcd(p, r) = \gcd(q(k), r(k+h)) = 1$ for all integer $h > 0$.

- 2 The change of variables $R = \frac{rS}{p}$ yields an equation in a polynomial S :

$$p(k) = q(k) S(k+1) - r(k) S(k).$$

- 3 Solve (using **explicit bounds**). If an S is found, return $R = rS/p$, else, this is a proof that no R exists.

Both variants reduce to linear-system solving.

Example of Use of Gosper's Algorithm

$$\sum_{k=0}^{n-1} \boxed{\frac{(-1)^k (4k+1) \binom{2k+1}{k}}{4^k (4k^2-1)}} = -\frac{2(n+1)}{4n+1} \boxed{\frac{(-1)^n (4n+1) \binom{2n+1}{n}}{4^n (4n^2-1)}} - 2.$$

The hypergeometric summand is given by:

$$\rho = -\frac{1}{2} \frac{(4k+5)(2k-1)}{(4k+1)(k+2)},$$

$$p(k) = 4k+1, \quad q(k) = \frac{1}{2} - k, \quad r(k) = k+2.$$

Parametrised Gosper Algorithm

Specifications

INPUT: a hypergeometric f_k and rational functions $s_0(k), \dots, s_m(k)$.

OUTPUT: a rational function $R(k)$ and constants $\eta_0, \dots, \eta_m$ such that Rf is an **indefinite sum** w.r.t. k of

$$(\eta_0 s_0(k) + \dots + \eta_m s_m(k)) f_k,$$

or $\nexists$ (“a proof that no such R exists for any family $\{\eta_i\}$ ”).

Sketch of algorithm

η_i involved only **linearly** and **in inhomogeneous side** of equation $\rightarrow$ simply linear solving for $S(k)$ with additional unknowns $\eta_0, \dots, \eta_m$.

Zeilberger's ("Fast") Algorithm

Specifications

INPUT: hypergeometric term $f_{n,k}$.

OUTPUT: rational functions $\eta_0(n), \dots, \eta_r(n), \phi(n, k)$ for minimal $r \in \mathbb{N}$ such that

$$\eta_r(n)f_{n+r,k} + \dots + \eta_0(n)f_{n,k} = \phi(n, k+1)f_{n,k+1} - \phi(n, k)f_{n,k}.$$

*Termination not guaranteed in general, but it is in "holonomic" case.
Explicit criterion due to Abramov.*

Sketch

For increasing values $r = 0, 1, \dots$:

- Compute the rational functions $s_i(n, k) = f_{n+i,k}/f_{n,k}$.
- Appeal to the parametrised Gosper algorithm.
- If $(\phi, \{\eta_i\})$ is found, return it (else loop).

A q -analogue exists for q -hypergeometric terms.

Example of Use of Zeilberger's Algorithm (1/5)

Jacobi's **orthogonal polynomials** $P_n^{(\alpha,\beta)}(x)$ can be expressed in terms of Gauss's **hypergeometric function** ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} \middle| z\right)$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{smallmatrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{smallmatrix} \middle| \frac{1-x}{2}\right),$$

for

$${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

Zeilberger's algorithm provides **recurrences** in n , α , or β , like:

$$\begin{aligned} 0 &= 2(n+2)(n+\alpha+\beta+2)(2n+\alpha+\beta+2)P_{n+2}^{(\alpha,\beta)}(x) \\ &\quad - ((2n+\alpha+\beta+2)_3x + (2n+\alpha+\beta+3)(\alpha-\beta)(\alpha+\beta))P_{n+1}^{(\alpha,\beta)}(x) \\ &\quad + 2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+4)P_n^{(\alpha,\beta)}(x). \end{aligned}$$

Slight extension: mixed recurrences, contiguity relations.

Example of Use of Zeilberger's Algorithm (2/5)

The summand is

$$f_k = \frac{(\alpha + 1)_n}{n!} \frac{(-n)_k}{(\alpha + 1)_k} \frac{(n + \alpha + \beta + 1)_k}{k!} \frac{(1 - x)^k}{2^k}.$$

The ratio f_{k+1}/f_k is

$$\rho = \frac{1}{2} \frac{(n - k) (n + k + \alpha + \beta + 1) (x - 1)}{(k + 1) (k + \alpha + 1)}.$$

Example of Use of Zeilberger's Algorithm (3/5)

The parametrised recurrence to solve for rational solutions is

$$\begin{aligned}
 & (n+2-k)(n+1-k)(n-k)(n+\alpha+\beta+2) \\
 & \quad (n+\alpha+\beta+1)(n+k+\alpha+\beta+1)(x-1) R(k+1) \\
 & - 2(k+1)(k+\alpha+1)(n+\alpha+\beta+1) \\
 & \quad (n+2-k)(n+1-k)(n+\alpha+\beta+2) R(k) \\
 & \quad = 2\eta_0(k+1)(k+\alpha+1)(n+\alpha+\beta+1) \\
 & \quad \quad (n+1-k)(n+2-k)(n+\alpha+\beta+2) \\
 & \quad + 2\eta_1(n+k+\alpha+\beta+1)(n+\alpha+1)(k+1) \\
 & \quad \quad (k+\alpha+1)(n+2-k)(n+\alpha+\beta+2) \\
 & \quad + 2\eta_2(n+\alpha+2)(n+\alpha+1)(n+\alpha+\beta+2+k) \\
 & \quad \quad (n+k+\alpha+\beta+1)(k+1)(k+\alpha+1).
 \end{aligned}$$

Example of Use of Zeilberger's Algorithm (4/5)

A multiple of the denominators of all solutions is

$$(n + 2 - k)(n + 1 - k).$$

The recurrence rewrites

$$\begin{aligned} & \left(-(n + \alpha + \beta + 2)(n + \alpha + \beta + 1)(x - 1)k^2 + \cdots \right) S(k + 1) \\ & + \left(-2(n + \alpha + \beta + 2)(n + \alpha + \beta + 1)k^2 + \cdots \right) S(k) \\ & = (c_0 k^4 + \cdots) \eta_0 + (c_1 k^4 + \cdots) \eta_1 + (c_2 k^4 + \cdots) \eta_2. \end{aligned}$$

Therefore, any solution $S(k)$ has degree at most 2 in k .

Example of Use of Zeilberger's Algorithm (5/5)

Solving yields $P(n, k) = Q(n, k + 1) - Q(n, k)$ where

$$\begin{aligned} P(n, k) = & 2(n + \beta + 1)(n + \alpha + 1)(2n + \alpha + \beta + 4)u(n, k) \\ & - (2n + \alpha + \beta + 3)(\alpha^2 + \alpha^2x + 4\alpha nx + 2\alpha\beta x + 6\alpha x + 12nx \\ & + \beta^2x + 4nx\beta + 4n^2x + 6\beta x + 8x - \beta^2)u(n + 1, k) \\ & + 2(n + 2)(n + \alpha + \beta + 2)(2n + \alpha + \beta + 2)u(n + 2, k), \end{aligned}$$

$$Q(n, k) = \frac{N(n, k)}{D(n, k)}u(n, k) \quad \text{for}$$

$$\begin{aligned} N(n, k) = & -2(2n + \alpha + \beta + 4)(2n + \alpha + \beta + 3)(2n + \alpha + \beta + 2) \\ & \times (n + \alpha + 1)(\alpha + k)k \\ D(n, k) = & (n + \alpha + \beta + 1)(n + 1 - k)(n + 2 - k). \end{aligned}$$

The announced recurrence is obtained after summation.

Almkvist and Zeilberger's Algorithm

Strict analogue for bivariate hyperexponential functions:

$$\frac{D_x f(x, y)}{f(x, y)} = R(x, y) \in \mathbb{K}(x, y) \quad \text{and} \quad \frac{D_y f(x, y)}{f(x, y)} = S(x, y) \in \mathbb{K}(x, y).$$

Example:

$$F(x) = \int_{-\infty}^{+\infty} f(x, y) dy \quad \text{for} \quad f(x, y) = \exp\left(-\frac{x^2}{y^2} - y^2\right).$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right)f = D_y\left(\frac{2(3y^2 - 2x^2)}{y^3}f\right),$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right)F = 0,$$

$$F(x) = \sqrt{\pi} \exp(-2x).$$

Termination always guaranteed by “holonomy”!

Functions versus Equations versus Vector Space

Special function or combinatorial sequence f :

$$f(n, z) = J_n(z) \quad (\text{Bessel function}).$$

[Also: algebraic, trigonometric, elementary, transcendental, hypergeometric functions; binomial coefficients, harmonic numbers, hypergeometric sequences; orthogonal polynomials; q -analogues.]

Linear functional system (+ initial conditions):

$$\begin{aligned} z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) &= 0, & z J_n'(z) + z J_{n+1}(z) - n J_n(z) &= 0, \\ z J_{n+2}(z) - 2(n+1) J_{n+1}(z) + z J_n(z) &= 0. \end{aligned}$$

Vector space closed under D_z and S_n , here **finite-dimensional**:

$$V = \mathbb{C}(n, z) J_n(z) \oplus \mathbb{C}(n, z) J_{n+1}(z) = \mathbb{C}(n, z) J_n(z) \oplus \mathbb{C}(n, z) J_n'(z).$$

∂ -Finite Terms

$$\partial_i = D_i \text{ or } S_i$$

t is ∂ -finite w.r.t. the operator algebra

$$\mathbb{K}(x_1, \dots, x_m) \langle \partial_1, \dots, \partial_m \rangle$$

$$\Leftrightarrow$$

the $\partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} t$'s span a

finite-dimensional $\mathbb{K}(x_1, \dots, x_m)$ -vector space:

$$\dim_{\mathbb{K}(x_1, \dots, x_m)} \left(\mathbb{K}(x_1, \dots, x_m) \langle \partial_1, \dots, \partial_m \rangle t \right) < +\infty$$

t is described by

higher-order linear functional equations.

Algorithmic closures under $+$, $\times$, the ∂_i 's, integration, summation
 $\Rightarrow$ simplification and zero test of ∂ -finite expressions.

Extended Gosper Decision Algorithm

ALGORITHM: $T = \text{Indefinite}(t, (b_i)_{i=1,\dots,d}, A)$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } t \in V = \bigoplus_{i=1}^d \mathbb{K}(x) b_i, \\ \text{an operator algebra } A = \mathbb{K}(x)\langle\partial\rangle, \\ \text{the action of } A \text{ on } V. \end{cases}$

OUTPUT: $T \in V$ such that $\partial T = t$; or $\nexists$.

- ① let $T = \phi_1 b_1 + \dots + \phi_d b_d$ for **undetermined coefficients** $\phi_i \in \mathbb{K}(x)$;
- ② extract the coefficients of $\partial T - t$ in the b_i 's to obtain a **first-order functional system** in the ϕ_i 's;
- ③ solve for **rational solutions** (uncoupling + **Abramov's decision algorithms**);
- ④ if solvable return T ; otherwise return $\nexists$.

Example of Indefinite ∂ -Finite Summation

$$\text{Input: } \begin{cases} \sum_{j=1}^k t_j \text{ for } t_k = \binom{k}{p} H_k, & H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \\ (\cdots) t_{k+2} + (\cdots) t_{k+1} + (\cdots) t_k = 0. \end{cases}$$

$$\text{Algorithmic reformulation: } A = \mathbb{Q}(p, k) \langle S_k \rangle, \quad T = \phi_0 t_k + \phi_1 t_{k+1}.$$

$$\begin{aligned} \left\{ \begin{aligned} (k+2-p) \phi_0(k+1) + (2k+3) \phi_1(k+1) - (k+2-p) \phi_1(k) &= 0, \\ (k+2-p)(k+1-p) \phi_0(k) + (k+1)^2 \phi_1(k+1) &= \\ &- (k+2-p)(k+1-p). \end{aligned} \right. \\ \longrightarrow (\cdots) \phi_1(k+2) + (\cdots) \phi_1(k+1) + (\cdots) \phi_1(k) = (\cdots). \end{aligned}$$

$$\text{Output: } \begin{cases} \phi_0(k) = \frac{(k-p)(k+p+2)}{(p+1)^2}, \quad \phi_1(k) = -\frac{(k-p)(k-p+1)}{(p+1)^2}, \\ T = \sum t_k \delta k = \binom{k}{p} \frac{k-p}{(p+1)^2} ((p+1)H_k - 1). \end{cases}$$

Example of Indefinite ∂ -Finite Integration

$$\text{Input: } \begin{cases} \int \text{Ci}(z) \, dz \text{ for } t(z) = \text{Ci}(z) = \int_0^z \frac{\cos(t) - 1}{t} dt, \\ (\cdots) t'''(z) + (\cdots) t''(z) + (\cdots) t'(z) = 0. \end{cases}$$

$$\text{Algorithmic reformulation: } A = \mathbb{Q}(z)\langle D_z \rangle, \quad T = \phi_0 t + \phi_1 t' + \phi_2 t''.$$

$$\left\{ \begin{array}{l} z\phi_1 + z\phi_2' - 2\phi_2 = 0, \\ z^2\phi_0 + z^2\phi_1' - z\phi_1 - z\phi_2' + (2 - z^2)\phi_2 = 0, \\ \phi_0' = 0. \end{array} \right.$$

$$\longrightarrow z^3\phi_2''' - 2z^2\phi_2'' + (z^3 + 4z)\phi_2' - 4\phi_2 = 0.$$

$$\text{Output: } \begin{cases} \phi_0(z) = z, & \phi_1(z) = 1, & \phi_2(z) = z, \\ T = \int \text{Ci}(z) \, dz = z \text{Ci}(z) - \sin(z). \end{cases}$$

Extended Zeilberger (a.k.a. Chyzak's) Algorithm

ALGORITHM: $(P, Q) = \text{Definite}(u, (b_i)_{i=1,\dots,d}, A)$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } A = \mathbb{K}(x, y)\langle \partial_x, \partial_y \rangle, \\ \text{a finite } \mathbb{K}(x, y)\text{-basis } (b_i)_{i=1,\dots,d} \text{ of } Au. \end{cases}$

OUTPUT: $\begin{cases} P \in \mathbb{K}(x, y)\langle \partial_x \rangle, \\ Q \in A \text{ such that } Pu = \partial_y Qu, \text{ resp. } Pu = (\partial_y - 1)Qu. \end{cases}$

For increasing values of r :

- ① let $P = \sum_{i=0}^r \eta_i \partial_x^i$ and $t = Pu$ for **undetermined coefficients** $\eta_i(x)$;
- ② let $(T, (\eta_i)) = \text{ParamIndefinite}(t, (b_i), A, \{\eta_0, \dots, \eta_r\})$.
- ③ if $T \neq \#$ return (P, Q) for $Qu = T$.

A ∂ -Finite Integral: $\frac{2}{\pi} \int_0^1 \frac{\cos(z t)}{\sqrt{1-t^2}} dt = J_0(z)$

$$A = \mathbb{Q}(z, t) \langle D_z, D_t \rangle, \quad f = \frac{\cos(z t)}{\sqrt{1-t^2}} \leftrightarrow \text{basis} = \left(f, D_z f = -t \frac{\sin(z t)}{\sqrt{1-t^2}} \right)$$

$$\left\{ \begin{array}{l} D_z^2 + t^2, \quad t(1-t^2) D_t - (1-t^2) z D_z - t^2. \end{array} \right.$$

$$P = z D_z^2 + D_z + z, \quad Q = \frac{1-t^2}{t} D_z, \quad P \int_0^1 \frac{\cos(z t)}{\sqrt{1-t^2}} + \left[Q \frac{\cos(z t)}{\sqrt{1-t^2}} \right]_0^1 = 0:$$

$$(z D_z^2 + D_z + z) \frac{2}{\pi} \int_0^1 \frac{\cos(z t)}{\sqrt{1-t^2}} dt = 0 \quad + \quad 2 \text{ initial conditions.}$$

A Neumann Addition Theorem $J_0(z)^2 + 2 \sum_{k=1}^{\infty} J_k(z)^2 = 1$

① $A = \mathbb{Q}(z, k) \langle D_z, S_k \rangle, \quad J_k(z) \leftrightarrow \text{basis} = (J_k(z), D_z J_k(z)):$

$$\left\{ \begin{array}{l} z^2 D_z^2 + z D_z + z^2 - k^2, \quad z S_k + z D_z - k. \end{array} \right.$$

② By closure, $J_k(z)^2 \leftrightarrow \text{basis} = (J_k(z)^2, S_k J_k(z)^2, D_z J_k(z)^2):$

$$\left\{ \begin{array}{l} z D_z^2 + (-2k+1) D_z - 2z S_k + 2z, \\ z D_z S_k + z D_z + (2k+2) S_k - 2k, \\ z^2 S_k^2 - 4(k+1)^2 S_k - 2z(k+1) D_z + 4k(k+1) - z^2. \end{array} \right.$$

③ $P = D_z, Q = \frac{k}{z} + \frac{1}{2} D_z, \quad P \sum_{k=0}^{\infty} J_k(z)^2 + \left[Q J_k(z)^2 \right]_{k=0}^{\infty} = 0:$

$$D_z \left(2 \sum_{k=1}^{\infty} J_k(z)^2 + J_0(z)^2 - 1 \right) = 0 \quad + \text{ initial condition at 0.}$$

Koutschan's Heuristics

Chyzak's fast algorithm can be **slow**:

- ① **uncouple** the system for the ϕ 's (non-commutative Gauss elim.)
- ② **algorithmically** bound denominators (Abramov's bound)
- ③ **algorithmically** bound numerator degrees (indicial equation)

Handles multiple sums/integrals iteratively.

Always finds a solution (at least theoretically).

Koutschan's **fast** heuristics

- ① **don't uncouple!**
- ② **heuristically** set denominator exponents (in accordance to Pu)
- ③ **heuristically** set numerator degrees (prop. to the denom. degs)

Simultaneous multiple sums/integrals possible, at times faster.

May fail to find a solution.

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History of Gröbner Bases for Creative Telescoping

- (Galligo, 1985) “Some algorithmic questions on ideals of differential operators”
- (Takayama, 1989) “Gröbner basis and the problem of contiguous relations”
- (Kandri-Rody and Weispfenning, 1990) “Noncommutative Gröbner bases in algebras of solvable type”
- (Takayama, 1990) “Gröbner basis, integration and transcendental functions”, “An algorithm of constructing the integral of a module: an infinite dimensional analog of Gröbner basis”
- (Kredel, 1993) “Solvable polynomial rings”
- (Chyzak and Salvy, 1998) “Non-commutative elimination in Ore algebras proves multivariate identities”
- (Oaku and Takayama, 1998) “Algorithms for D -modules: restriction, tensor product, localization, and local cohomology groups”
- (Saito, Sturmfels, and Takayama, 2000) “Gröbner deformations of hypergeometric differential equations”

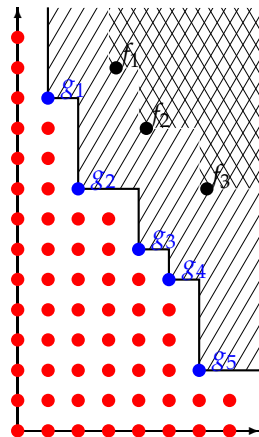
Crash Course on Gröbner Bases

Generalize Euclidean division and the Euclidean (gcd) algorithm to ideals in polynomial rings $\mathbb{K}[x_1, \dots, x_m]$.

- ① **Monomial order(ing)**: total order $<$ on the monomials, compatible with the product, with minimal element 1
- ② **Gröbner basis** of a ideal I w.r.t. $<$: generators **matching the stairs** of I

$$\text{GB}(f_1, f_2, f_3) = (g_1, \dots, g_5)$$

- ③ **Quotient**: vector basis **below the stairs**; a ring in the commutative case
- ④ **Reduction** of P modulo I : unique remainder written on this basis
- ⑤ **Elimination**: procedure for computing $I \cap \mathbb{K}[x_1, \dots, x_s]$, $s < m$



Annihilating Ideals

For an operator algebra $A = \mathbb{K}(x_1, \dots, n_1, \dots) \langle D_1, \dots, S_1, \dots \rangle$:

- **left ideal**: $\text{ann} f = \{ P \in A \mid Pf = 0 \} \leftrightarrow$ “all equations on f ”
- **left module**: $A / \text{ann} f \simeq Af \leftrightarrow$ “space of shifted derivatives of f ”

Calculations modulo $\text{ann} f$.

Example of Bessel functions, $J_n(z)$:

$$\text{ann} J = A (z^2 D_z^2 + z D_z + (z^2 - n^2), z D_z + z S_n - n, z S_n^2 - 2(n+1)S_n + z).$$

$$V = AJ \simeq A / \text{ann} J$$

$$(J_n(z), J_{n+1}(z), J'_n(z)) \leftrightarrow (1 + \text{ann} J, S_n + \text{ann} J, D_z + \text{ann} J)$$

Also for $\mathbb{K}[x_1, \dots, n_1, \dots] \langle D_1, \dots, S_1, \dots \rangle$ and intermediate cases.

Skew Gröbner Bases

Buchberger's algorithm extends to left ideals.

Gröbner bases are still **finite** (noetherianity).

(Galligo, 1985; Takayama, 1989; Kandry-Rody and Weispfenning, 2000; Kredel, 1993)

Direct extensions to Gröbner bases for left modules.

Technical restriction on commutations:

$$\partial_i x_j = (a_{i,j} x_j + b_{i,j}) \partial_i + c_{i,j}(x_1, \dots, x_m) \quad \text{with } a_{i,j} \neq 0.$$

Restriction on orders allowed: $c_{i,j}(x_1, \dots, x_m) < x_j \partial_i$.

Both OK for $\partial = D$ and $\partial = S$, as $Dx = xD + 1$ and $Sx = xS + S$.

Closures of ∂ -Finite Functions under $+$, $\times$, ∂

Finite dimensions make algorithms terminate.

Example: the product $h = T_n(x) \times \frac{e^{-ux}}{\sqrt{1-x^2}}$.

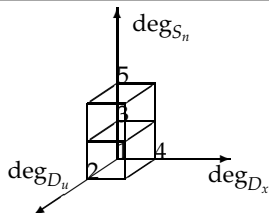
$$A = \mathbb{Q}(x, n, u) \langle D_x, S_n, D_u \rangle, \quad \text{Total degree in } D_x > S_n > D_u.$$

A Gröbner basis for $\text{ann} f$, $f = T_n(x)$, provides **normal forms** in Af :

$$D_u, \quad (x^2 - 1) D_x - n S_n + n x, \quad S_n^2 - 2x S_n + 1.$$

A Gröbner basis for $\text{ann} f$, $g = \frac{e^{-ux}}{\sqrt{1-x^2}}$, provides **normal forms** in Ag :

$$D_u + x, \quad S_n - 1, \quad (x^2 - 1) D_x + u x^2 + x - u.$$



$$1 < \boxed{D_u} < S_n < \boxed{D_x} < D_u^2 < D_u S_n < D_u D_x \\ < \boxed{S_n^2} < S_n D_x < D_x^2 < \dots$$

Normal forms in $Af \otimes_{Q(x,n,u)} Ag$:

$$\begin{aligned} 1(f \otimes g) &= f \otimes g, \\ D_u(f \otimes g) &= \textcolor{blue}{D_u} f \otimes g + f \otimes \textcolor{teal}{D_u} g = -x(f \otimes g), \\ S_n(f \otimes g) &= \textcolor{blue}{S_n} f \otimes \textcolor{teal}{S_n} g = S_n f \otimes g, \\ D_x(f \otimes g) &= D_x f \otimes g + f \otimes \textcolor{teal}{D_x} g \\ &= (x^2 - 1)^{-1} \left(n S_n f \otimes g - (ux^2 + (n+1)x - u)(f \otimes g) \right), \\ S_n^2(f \otimes g) &= \textcolor{blue}{S_n^2} f \otimes \textcolor{teal}{S_n^2} g = 2x S_n f \otimes g - f \otimes g. \end{aligned}$$

Gröbner basis **computed** for $f \otimes g$:

$$\boxed{D_u} + x, \quad (x^2 - 1)\boxed{D_x} - nS_n + ux^2 + (n+1)x - u, \quad \boxed{S_n^2} - 2xS_n + 1.$$

An Integral: $\int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}} dx = \pi (-1)^n I_n(u)$

Now, in $\mathbb{Q}[n, u, \mathbf{x}] \langle D_x, S_n, D_u \rangle$, **heuristically eliminating $\mathbf{x}$** by the order

$$\mathbf{x} \gg D_x > S_n > D_u > n > u$$

yields another Gröbner basis:

$$\underbrace{S_n^2 + 2S_n D_u + 1, \quad D_x(D_u^2 - 1) - nS_n + uD_u^2 - (n-1)D_u - u, \quad \mathbf{x} + D_u}_{\text{non-trivial by chance!}}$$

The integral $H = \int_{-1}^{+1} h dx$ satisfies

$$(S_n^2 + 2S_n D_u + 1) H = 0,$$

$$(nS_n - uD_u^2 + (n-1)D_u + u) H = \left[(D_u^2 - 1) h \right]_{\mathbf{x}=-1}^{\mathbf{x}+1} = 0.$$

Takayama's Approach

(Chyzak and Salvy's variant)

Optimisation dedicated to integrals/sums over natural boundaries

Compute the P 's such that, for some Q ,

$$P(n, u, D_x, S_n, D_u) - D_x Q(x, n, u, D_x, S_n, D_u) \in \text{ann } h$$

without computing the Q 's.

Idea: View ideal $\text{ann } h$ as a module

$$\sum_{i=1}^{\ell} Q[n, u, x] \langle D_x, S_n, D_u \rangle g_i = \lim_{N \rightarrow \infty} \sum_{i=1}^{\ell} \sum_{j=0}^N Q[n, u] \langle D_x, S_n, D_u \rangle x^j g_i.$$

Adapts to mixed simultaneous integrations/summations.

Example: Same Integral

- For $N = 3$:

$$\begin{aligned} & 1D_u + x, \quad xD_u + x^2, \quad x^2D_u + x^3, \\ & 1(-D_x - nS_n - u) + x(n+1) + x^2(u + D_x), \quad x(-D_x - nS_n - u) + x^2(n+1) + x^3(u + D_x), \\ & 1(S_n^2 + 1) - x2S_n, \quad x(S_n^2 + 1) - x^22S_n, \quad x^2(S_n^2 + 1) - x^32S_n. \end{aligned}$$

- Moding out D_x on the left, by $p(x) D_x = D_x p(x) - p'(x)$, yields

$$\begin{aligned} & D_u 1 + x, \quad D_u x + x^2, \quad D_u x^2 + x^3, \\ & (-nS_n - u)1 + (n-1)x + ux^2, \quad 1 + (-nS_n - u)x + (n-2)x^2 + ux^3, \\ & (S_n^2 + 1)1 - 2S_n x, \quad (S_n^2 + 1)x - 2S_n x^2, \quad (S_n^2 + 1)x^2 - 2S_n x^3. \end{aligned}$$

- Gröbner basis for $x \gg S_n > D_u$:

$$\begin{aligned} & \boxed{(-u - nS_n - (n-1)D_u + uD_u^2)1}, \quad \boxed{(S_n^2 + 1 + 2D_u S_n)1}, \\ & D_u 1 + x, \quad (-u - nS_n - (n-1)D_u)1 + ux^2, \\ & (unS_n D_u + n(n-2)S_n - u + (u^2 + n^2 - 3n + 2)D_u + nu)1 + u^2 x^3. \end{aligned}$$

Dimension of Ideals

$$M_k(I) := \langle m : m \text{ is below the stairs and of total degree } \leq s \rangle$$

Theorem (Hilbert)

For any I , there is an integer $\delta(I)$ such that $\#M_k(I) = \Theta(s^{\delta(I)})$.

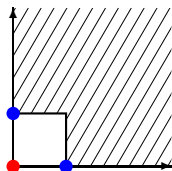
- $\delta(I)$ is the (Hilbert) dimension of I .
- Finite measure of infinite-dimensional vector-spaces.
- Also measures capacity for elimination.
- Can be obtained by a Gröbner-basis calculation.
- $I \subset J \Rightarrow \delta(I) \geq \delta(J)$

Caution!

dimension in $\mathbb{K}(\underline{x})\langle\underline{\partial}\rangle \neq$ dimension in $\mathbb{K}[\underline{x}]\langle\underline{\partial}\rangle$

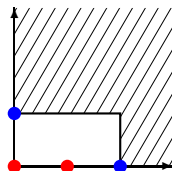
Examples of Dimensions in $\mathbb{K}(\underline{x})\langle \underline{\partial} \rangle$

Binomial coeffs $\binom{n}{k}$ w.r.t. S_n, S_k ;
Hypergeometric sequences:



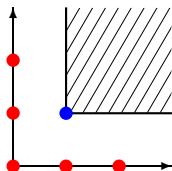
$$\delta(I) = 0, \dim S/I = 1$$

Bessel $J_\nu(x)$ w.r.t. S_ν, D_x ;
Orthogonal polys w.r.t. S_n, D_x :



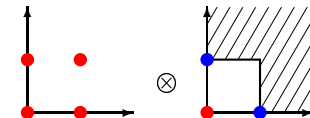
$$\delta(I) = 0, \dim S/I = 2$$

Stirling nbs w.r.t. S_n, S_k :



$$\delta(I) = 1, \dim S/I = \infty$$

Abel-type w.r.t. S_m, S_k, S_r, S_s ,
 $h(m,k) (k+r)^k (m-k+s)^{m-k} \frac{r}{k+r}$:



$$\delta(I) = 2 \text{ in space of dimension } 4$$

Sample Dimensions in $\mathbb{K}(\underline{x})\langle\underline{\partial}\rangle$ vs in $\mathbb{K}[\underline{x}]\langle\underline{\partial}\rangle$

Experimentation “shows”:

Differential case

algebra \ function	1	$\exp(x+y)$	$\Gamma(x+y)$	$\Gamma(x)\Gamma(y)$
$\mathbb{C}(x,y)\langle D_x, D_y \rangle$	0	0	1	2
$\mathbb{C}[x,y]\langle D_x, D_y \rangle$	2	2	3	4

Recurrence case

algebra \ function	1	$\frac{1}{nk}$	$\frac{1}{1+nk}$	$\binom{n}{k}$	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
$\mathbb{C}(n,k)\langle S_n, S_k \rangle$	0	0	0	0	1
$\mathbb{C}[n,k]\langle S_n, S_k \rangle$	2	2	3	2	3

Elimination for Holonomic Functions

In the **differential setting**: $D_i x_j = x_j D_i + \delta_{i,j}$.

Holonomic function

a function of $x_1, \dots, x_m$ is **holonomic**



the $x_1^{\alpha_1} \dots x_m^{\alpha_m} D_1^{\beta_1} \dots D_m^{\beta_m} f$ for $\alpha_1 + \dots + \alpha_m + \beta_1 + \dots + \beta_m \leq N$
span a $\mathbb{K}$ -vector space of dimension $\mathcal{O}(N^m)$.

Elimination lemma

For a holonomic function f and **any choice**

$$\{u_1, \dots, u_{m+1}\} \subset \{x_1, \dots, x_m, D_1, \dots, D_m\},$$

there is a **non-zero** $P(u_1, \dots, u_{m+1})$ such that $Pf = 0$.

Corollary 1: A holonomic function is D-finite.

Corollary 2: Creative telescoping succeeds for holonomic functions.

D-Finite Functions are Holonomic

Finite-dimensional confinement of derivatives:

$$f \in V = \bigoplus_{j=1}^d \mathbb{K}(x_1, \dots, x_m) b_j, \quad D_s b_i = \sum_{j=1}^d \frac{a_{s,i,j}}{w} b_j, \quad a_{s,i,j}, w \in \mathbb{K}[x_1, \dots, x_m]$$

Action of x 's and D 's on degrees: for $\mu = 1 + \max \{ \deg w, \deg a_{s,i,j} \}$,

$$D_s \left(\frac{p}{w^N} b_i \right) = \frac{D_s p}{w^N} b_i - N \frac{p D_s w}{w^{N+1}} b_i + \sum_{j=1}^d \frac{a_{s,i,j} p}{w^{N+1}} b_j, \quad x_s \left(\frac{p}{w^N} b_i \right) = \frac{x_s w p}{w^{N+1}}.$$

$$\frac{(\deg \leq \alpha)}{w^N} \rightarrow \frac{(\deg \leq \alpha + \mu)}{w^{N+1}}$$

Dimension analysis: for $\alpha_1 + \dots + \alpha_m + \beta_1 + \dots + \beta_m \leq N$,

$$x_1^{\alpha_1} \dots x_m^{\alpha_m} D_1^{\beta_1} \dots D_m^{\beta_m} f = \frac{(\deg \leq \mu N)}{w^N} \rightarrow \dim. \binom{\mu N + m}{m} \simeq \Theta(N^m).$$

Other Techniques by Homogenisation

For $F(x) = \int_a^b f(x, y) dy$. Common themes:

- reduce by weights -1 for y , $+1$ for ∂_y ;
- “homogenise” to recover termination.

Saito, Sturmfels, Takayama

- Homogenise the algebra by $\partial_y y = y \partial_y + h^2$.
- Compute Gröbner basis for homogenised generators of $\text{ann} f$, then dehomogenise the output.

Oaku’s “homogenisation” for the V-filtration

- Add a variable v to the algebra.
- Mark monomials in generators by v raised to the V-degree.
- Compute Gröbner basis that eliminates v , then set $v = 1$.

Weyl Closure

A problem: $f = p^{-1}$ for $p = y^2 - y + x$ (Chyzak and Salvy, 1998)

$$g_1 = pD_x + 1 \quad \text{and} \quad g_2 = pD_y + 2y - 1$$

In $A = \mathbb{Q}(x, y)\langle D_x, D_y \rangle$: $\text{ann}_A f = Ag_1 + Ag_2$.

In $W = \mathbb{Q}[x, y]\langle D_x, D_y \rangle$: $Wg_1 + Wg_2$ has Hilbert dimension 3 (!)
 $\rightarrow$ elimination of y from $\{g_1, g_2\}$ fails

$$g_3 = pD_x D_y + 2D_y \in W \cap \text{ann}_A f$$

$Wg_1 + Wg_2 + Wg_3$ has Hilbert dimension 2

$\rightarrow$ elimination of y from $\{g_1, g_2, g_3\}$ finds $(1 - 4x) D_x^2 - 6D_x - D_y^2$

Problem: GB, Takayama, etc., require $\text{ann}_W f$ as inputs, not $\text{ann}_A f$.

Completion of $\text{ann}_A f$ to $\text{ann}_W f$ by Tsai's Weyl-closure algorithms (2000, 2002)

(Differential case only.)

Contents

- 1 Introduction: Early Days and Examples
- 2 Fasenmyer's (a.k.a. "*k*-Free") Ansatz
- 3 Lipshitz's Diagonals
- 4 Zeilberger's Ansatz
- 5 Gröbner Bases: Closures and Elimination
- 6 Beyond Holonomy
- 7 Termination, Bounds, Complexity
- 8 Conclusions

Examples of Non- ∂ -Finite Terms

- Stirling numbers of the second kind, $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ = numbers of partitions of a set of n objects into k non-empty subsets:

$$\left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\} \quad (k > 0)$$

with initial conditions $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ and $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right\} = 0$, ($n > 0$).

- Hurwitz' zeta function, $\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$:

$$\frac{\partial \zeta}{\partial x} = -s \zeta(s+1, x)$$

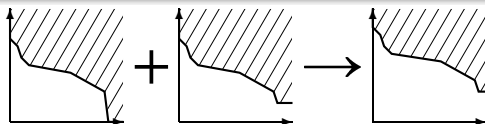
with initial conditions $\zeta(0, x) = 1/2 - x$ and $\zeta(s, 1) = \zeta(s)$ (Riemann's zeta function).

$$\text{ann} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = A_{n,k} (S_n S_k - (k+1) S_k - 1), \quad \text{ann} \zeta = A_{x,s} (D_x + s S_s).$$

History for Algorithms in Positive Dimension

- (Majewicz, 1996) “WZ-style certification and Sister Celine’s technique for Abel-type sums”
- (Kauers, 2007) “Summation algorithms for Stirling number identities”
- (Chen, Sun, 2009) “Extended Zeilberger’s algorithm for identities on Bernoulli and Euler polynomials”
- (Chyzak, Kauers, Salvy, 2009) “A non-holonomic systems approach to special function identities”

Algebraic Closure Properties in Positive Dimension



Proposition

δ of sum $\leq \max$. of δ 's, δ of product $\leq$ sum of δ 's, δ of der. $\leq \delta$.

Algorithm (for a product fg): for a *graded* ordering,

For $s = 0, 1, 2, \dots$, **until** $\delta(I) \leq \text{bound}$:

for each $|\underline{\alpha}| \leq s$, **reduce** $\partial^{\underline{\alpha}}(fg)$ to a sum $\sum u_{\alpha, \beta, \gamma}(\underline{x}) \partial^{\underline{\beta}} f \partial^{\underline{\gamma}} g$

over $\beta \in M_s(\text{ann } f), \gamma \in M_s(\text{ann } g)$ (M_s : "monomials under the stairs")

search for $\mathbb{Q}(x)$ -linear relations, set I to the ideal they generate

return I , a subideal of $\text{ann } fg$

Example (Stirling numbers of the second kind): $\{n \atop k\}, \delta(I) = 1$,

1st-order rec. $\xrightarrow{s=3} \{n \atop k\} \{m \atop k\}, \delta(I) = 2$, 2nd-order rec.

A Slightly Different View of Fasenmyer's Technique

Rediscovering Pascal's Triangle Rule

Reduce all monomials of degree $\leq s = 2$:

$$\begin{aligned}
 1 &\rightarrow \mathbf{1}, & S_n &\rightarrow \frac{n+1}{n+1-k} \mathbf{1}, & S_k &\rightarrow \frac{n-k}{k+1} \mathbf{1} \\
 S_n^2 &\rightarrow \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)} \mathbf{1}, & S_k^2 &\rightarrow \frac{(n-k-1)(n-k)}{(k+2)(k+1)} \mathbf{1}, & S_n S_k &\rightarrow \frac{n+1}{k+1} \mathbf{1}.
 \end{aligned}$$



Common denominator: $C_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$C_2, C_2 S_n, C_2 S_k, C_2 S_n^2, C_2 S_k^2, C_2 S_n S_k$ **confined** in

$\langle \mathbf{1}, k \mathbf{1}, k^2 \mathbf{1}, k^3 \mathbf{1}, k^4 \mathbf{1} \rangle$ over $\mathbb{K}(n)$

$$\rightarrow C_2 (S_n S_k - S_k - 1) \in \text{ann} \binom{n}{k}.$$

$\deg C_s = \mathcal{O}(s) \Rightarrow$ this **had to happen** for some s .

More Examples from this Perspective

- Proper hypergeometric terms: essentially the same situation.

- $\frac{1}{n^2 + k^2}$: confinement in a space of dimension $\mathcal{O}(s^2)$,
no elimination of k succeeds.

- $f = \frac{a(x, y_1, \dots, y_r)}{b(x, y_1, \dots, y_r)}$: $C_s = b^s$,
confinement in a space of dimension $\mathcal{O}(s^r)$ over $\mathbb{K}(x)$,
elimination of $y_1, \dots, y_r$ has to succeed.
Base case of proof that D-finite functions are holonomic.

Adding a Left Part and a Right Part

Specification of a Creative-Telescoping Algorithm

Input: generators of (a subideal of) $\text{ann} f$

Output: all (A, B) such that:

- $A + \partial_y B \in \text{ann} f$, resp. $A + (\partial_y - 1)B \in \text{ann} f$,
- A is free of y and ∂_y .

Definition (telescoping ideal of I w.r.t. y)

$$T_y(I) := (I + \partial_y S_{x,y}) \cap S_x \quad \text{where}$$

$$S_{x,y} := \mathbb{K}(x, y) \langle \partial_x, \partial_y \rangle \text{ and } S_x := \mathbb{K}(x) \langle \partial_x \rangle.$$

When $I = S_{x,y}G_1 + \cdots + S_{x,y}G_\ell$, this involves

$$(S_{x,y}G_1 + \cdots + S_{x,y}G_\ell) + \partial_y S_{x,y}.$$

Polynomial Growth and Creative Telescoping when $\delta > 0$

Definition (polynomial growth p)

An ideal has polynomial growth p if there exists a sequence of polynomials $P_s(x, y)$, s.t. if $|a| + b \leq s$,

$P_s \partial_{x_1}^{a_1} \dots \partial_{x_k}^{a_k} \partial_y^b$ reduces to polys of degree $\mathcal{O}(s^p)$ in y .

Theorem (Chyzak, Kauers & Salvy, 2009)

$$\delta(T_y(I)) \leq \max(\delta(I) + p - 1, 0).$$

Proof:

$$P_s \partial_x^a \partial_y^b - \sum_{\beta \in M_s(I)} (\deg. \text{ in } y \leq \mathcal{O}(s^p)) \partial^\beta \in I$$

$\Rightarrow$ any choice of $(\delta(I) + p - 1) + 1$ generators among $\{\partial_{x_1}, \dots, \partial_{x_k}, \partial_y\}$ is algebraically dependent modulo I .

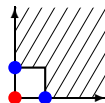
Corollary (sufficient condition for creative telescoping)

$\delta(I) + p - 1 < k \Rightarrow$ identities exist for the sum/int. w.r.t. y .

Examples with Polynomial Growth $p = 1$

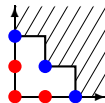
- Proper hypergeometric (Wilf & Zeilberger, 1992):

$$Q(n, k) \zeta^k \frac{\prod_{i=1}^u (a_i n + b_i k + c_i)!}{\prod_{i=1}^v (u_i n + v_i k + w_i)!},$$

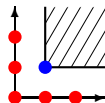


Q polynomial, a_i, b_i, u_i, v_i **integers**.

- Differential D-finite (special case of holonomy).
- Stirling: $\delta = 1 \rightarrow$ for ≥ 3 vars, e.g., Frobenius' id.:



$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle.$$



- Abel type: $\delta = 2 \rightarrow$ for ≥ 4 vars, e.g., Abel's id.:

$$\sum_{k=0}^n \binom{n}{k} i(k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n.$$

Algorithm I: Fasenmyer's Style

Polynomial growth + linear algebra $\rightarrow J := I \cap S_x[\partial_y]$.

Algorithm

For increasing values of s , until $\delta(J) \leq \text{bound}$:

- ① Reduce all $\partial_x^a \partial_y^b$ with $a + b \leq s$;
- ② Normalize to a common denominator;
- ③ Set up a linear system to cancel the positive powers of y ;
- ④ If a non-zero solution is found, it has the form $A(x, \partial_x) + \partial_y B(y, \partial_x, \partial_y)$; return it.

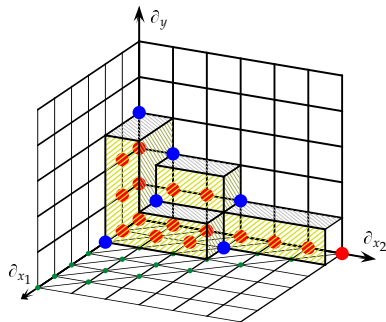
This computes A in

$$(J + \partial_y S_x[\partial_y]) \cap S_x, \quad \text{not in } T_t(I) := (I + \partial_y S_{x,y}) \cap S_x.$$

$$S_x = \mathbb{K}(x) \langle \partial_x \rangle \subset S_x[\partial_y] \subset S_{x,y} := \mathbb{K}(x, y) \langle \partial_x, \partial_y \rangle.$$

Algorithm II. Zeilberger's Style Extended to $\text{Dim} > 0$

Compute A in $T_t(I) := (I + \partial_y S_{x,y}) \cap S_x \leftrightarrow$ faster, more precise.



For $s = 0, 1, 2, \dots$, until $\delta(J) \leq \text{bound}$:

set $A := \sum_{|\alpha| \leq s} \eta_\alpha \partial^\alpha$

for undetermined coeffs $\eta_\alpha(y) \in \mathbb{K}(x)$

set $B := \sum_{\beta \in M_s(I)} \phi_\beta(y) \partial^\beta$

for undetermined coeffs $\phi_\beta(y) \in \mathbb{K}(x, y)$

reduce $A - \partial_y B$

onto the basis $M_{s+1}(I)$

extract coeffs to form a linear system

of first order w.r.t. ∂_y

solve and set J to the ideal of the A 's

return the pairs (A, B)

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History for Termination, Bounds, and Complexity

- (Wilf and Zeilberger, 1992) "An algorithmic proof theory for hypergeometric (ordinary and 'q') multisum/integral identities"
- (Yen, 1996, 1997) "A two-line algorithm for proving terminating hypergeometric identities", "A two-line algorithm for proving q -hypergeometric identities"
- (Abramov, 2002, 2003) "Applicability of Zeilberger's algorithm to hypergeometric terms", "When does Zeilberger's algorithm succeed?"
- (Zhang, 2003) "A new elementary algorithm for proving q -hypergeometric identities"
- (Apagodu and Zeilberger, 2005, 2006) "Sharp upper bounds for the orders of the recurrences output by the Zeilberger and q -Zeilberger algorithms", "Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory"
- (Bostan, Cluzeau, and Salvy, 2005) "Fast algorithms for polynomial solutions of linear differential equations"
- (Bostan, Chyzak, Salvy, and Cluzeau, 2006) "Low complexity algorithms for linear recurrences"
- (Guo, Hou, Sun, 2008) "Proving hypergeometric identities by numerical verifications"
- (Bostan, Chen, Chyzak, and Li, 2010) "Complexity of creative telescoping for bivariate rational functions"
- (Chen, 2011) "Some applications of differential-difference algebra to creative telescoping"
- (Chen, Chyzak, Feng, and Li) "On the existence of telescopers for hyperexponential-hypergeometric sequences" (in preparation)

Order Bounds

For proper hypergeometric terms $P(n, k) \zeta^n \zeta^k \prod_{\ell=1}^L \Gamma(a_\ell n + b_\ell k + c_\ell)^{\epsilon_\ell}$:

$$r \leq \sum_{\ell=1}^L |b_\ell| \quad (\text{Wilf and Zeilberger, 1992})$$

$$r \leq \sum_{\ell=1, \epsilon=+1}^L b_\ell^+ + \sum_{\ell=1, \epsilon=-1}^L (-b_\ell)^+ + \left(- \sum_{\ell=1, \epsilon=\pm 1}^L \epsilon b_\ell \right)^+ \quad (\text{Yen, 1993})$$

For hyperexponential terms $p(x, y) \exp\left(\frac{a(x, y)}{b(x, y)}\right) \prod_{s \in S} s(x, y)^{\alpha_s}$:

(Apagodu and Zeilberger, 2006), if $a \neq 0$, $\alpha_s \notin \bar{\mathbb{Q}}$:

$$r \leq \deg_y(b) + \max(\deg_y(a), \deg_y(b)) + \left(\sum_{s \in S} \deg_y(s) \right) - 1.$$

Proving Summation Identities by Evaluations

Linear-algebraic system to encode output recurrence $\rightarrow$ Bounds on **orders, degrees, and heights** (Cramer's rules) $\rightarrow$ Bound on **maximal integer singularity** of recurrence

(Yen, 1996)

To prove $\sum_{k=0}^{\infty} P(n, k) \zeta^n \zeta^k \prod_{\ell=1}^L \Gamma(a_{\ell} n + b_{\ell} k + c_{\ell})^{\epsilon_{\ell}} = 1$ for all n , it is sufficient to verify it at $n = 0, \dots, N$, where $N = ((3\sigma)^{3\beta} \delta^5 h)^{\beta} + 1$.

$$\lambda_+ = \#\{\ell \mid \epsilon_{\ell} = +1\}, \quad \lambda_- = \#\{\ell \mid \epsilon_{\ell} = -1\},$$

$$\sigma = \max(\lambda_+, \lambda_-) \max\{|a_{\ell}|, |b_{\ell}|, |c_{\ell}|\},$$

$$\delta = 1 + \max(\deg_n P, \deg_k P), \quad \beta = (\delta + 1)(2\sigma)^3,$$

$$h = \text{height}(P) = \text{maximal absolute value of coeffs.}$$

Proof: For the linear-algebra formulation, bound Fasenmyer's ansatz by Wilf and Zeilberger, and translate it to Zeilberger's ansatz.

Proving Summation Identities by Fewer Evaluations

(Guo, Hou, and Sun, 2008)

To reduce N :

- directly use Zeilberger's ansatz with tighter order bound and explicit denominator bound;
- perform explicit majorisations on the concrete matrix, rather than use a generic bound;
- simplify the matrix by algorithmic manipulations.

→ new N given by a procedure, not a formula.

Examples:

	$\sum_k \binom{n}{k} = 2^n$	$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$
Yen	10^{11}	10^{115}
Guo, Hou, Sun	4	12091

Definite Integration of Bivariate Rational Functions

First step toward complexity analysis of creative telescoping

$$f \in \mathbb{K}(x, y) \text{ with } \deg_x = \deg_y = \delta \longrightarrow L(x, D_x)f = D_y g \text{ for } g \in \mathbb{K}(x, y)$$

(Bostan, Chen, Chyzak, Li, 2010): **Complexity is polynomial!**

New algorithm based on Hermite reduction + Analyses:

	Method	$\deg_{D_x} L$	$\deg_x L$	$\deg_x g$	$\deg_y g$	Complexity
Minimal	Hermite	$\leq \delta$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^2)$	$\tilde{\mathcal{O}}(\delta^{\omega+4})$ L.V.
Telescoper	A. & Z.	$\leq \delta$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^2)$	$\tilde{\mathcal{O}}(\delta^{2\omega+3})$ L.V.
Nonminimal	Lipshitz	$\leq 6(\delta+1)^2$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^{6\omega})$ det.
Telescoper	Cubic	$\leq 6\delta$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^{4\omega})$ det.

Hermite-based: matrix size = δ , entries of degree δ^3 .

A. & Z.: matrix size = δ^2 , entries of degree δ^2 .

Lipshitz: total-degree filtration at degree = $\delta \rightarrow$ matrix size = δ^6 .

Cubic: bi-filtration at (degree, order) = $(\delta^2, \delta) \rightarrow$ matrix size = δ^4 .

Hermite Reduction for Indefinite Integration in $\mathbb{K}(y)$

One step, given a Bézout relation $1 = sq' + tq$ for irreducible q :

$$\int \frac{p}{q^j} = \int \frac{-sp}{j-1} \cdot \frac{-(j-1)q'}{q^j} + \int \frac{tp}{q^{j-1}} = \frac{-sp}{(j-1)q^{j-1}} + \int \frac{tp + (sp)'/(j-1)}{q^{j-1}}.$$

By iterating:

$$\int \sum_{j=1}^v \frac{p_j}{q^j} = \sum_{j=1}^{v-1} \frac{a_j}{q^j} + \int \frac{u}{q} \quad \text{where} \quad \deg_y u < \deg_y q.$$

Viewed differently:

$$\sum_{j=1}^v \frac{p_j}{q^j} = \left(\sum_{j=1}^{v-1} \frac{a_j}{q^j} \right)' + \frac{u}{q} \quad \text{where} \quad \deg_y u < \deg_y q.$$

Complexity for general $\frac{p}{q}$: $\tilde{O}(\max(\deg p, \deg q))$.

An Algorithm for Definite Integration in $\mathbb{K}(x, y)$

Given $f = \frac{p}{q} \in \mathbb{K}(x)(y)$, to compute $\int_a^b f(x, y) dy$:

$$f(x, y) = D_y \left(\frac{\cdots}{\dots} \right) + \frac{u_0}{q^*}, \quad \dots, \quad D_x^r f(x, y) = D_y \left(\frac{\cdots}{\dots} \right) + \frac{u_r}{q^*},$$

where:

- ① $q^* = q_1 \cdots q_m$ is the squarefree part of $q = q_1^1 q_2^2 \cdots q_m^m$;
- ② $\deg_y u_i < \deg_y q^*$ for $0 \leq i \leq r$.

Linear algebra over $\mathbb{K}(x)$: for some $r \leq \deg_y q^*$,

$$\sum_{i=0}^r \eta_i(x) D_x^i f(x, y) = D_y \left(\frac{\cdots}{\dots} \right) + \frac{0}{q^*}.$$

Matrix size ruled by $\deg_y u$, not by $\deg_y g$!

Complexity:

- bivariate Hermite by evaluation/interpolation;
- linear algebra over $\mathbb{K}[x]$ by Storjohann–Villard.

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Non-Minimality of Computed Operators

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{pk}{n} = (-p)^n \quad (\text{order } p-1 \text{ instead of } 1),$$

$$\sum_{k=0}^n \frac{1}{pk+1} \binom{pk+1}{k} \binom{p(n-k)}{n-k} = \binom{pn+1}{n}$$

(order 2 instead of 1 for $p=3$).

This non-minimality is not understood well.

$$\eta_r(n) u_{n+r,k} + \cdots + \eta_0(n) u_{n,k} = v_{n,k+1} - v_{n,k}$$

The existence of v constrains d !

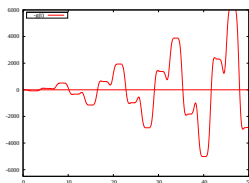
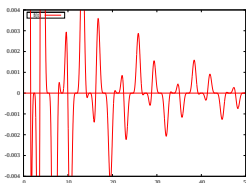
This relation does not take the summation range into account!

An Integral Not Amenable to Creative Telescoping?

Borrowed from (Borwein, Straub, Wan, Zudilin, 2011).

$$F(x) = \int_0^\infty \underbrace{xt J_0(xt) J_0(t)^4}_{=f(x,t)} dt \quad \rightarrow \quad Pf = D_t g \quad \text{for} \quad g = Qf.$$

$$\begin{aligned} P &= (x-4)(x-2)(x+2)(x+4)x^3D_x^3 + 6(x^2-10)x^4D_x^2 + (7x^4-32x^2+64)x D_x + (x^2-8)(x^2+8) \\ Q &= 5x^3t^2D_xD_t^3 - x^2t^3D_t^4 - t^{-1}x^2(5x^4t^4 - 60x^2t^4 + 64t^4 - 28t^2 - 4) - 7x^2t^2D_t^3 \\ &\quad + x^2t(10x^2t^2 - 20t^2 - 1)D_t^2 - 5x^3(2x^2t^2 - 12t^2 - 1)D_xD_t \\ &\quad + 4x^2(5x^2t^2 - 15t^2 - 1)D_t + 5t^{-1}x^3(2x^2t^2 - 12t^2 - 1)D_x \end{aligned}$$

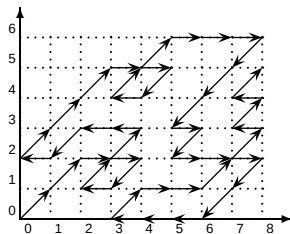
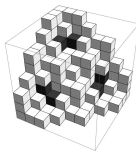
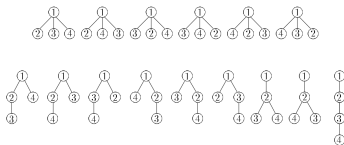


$$D_x^i f|_{x=3/2} = \Theta(\cos(3\pi/2 + \pi/4) \sin(t + \pi/4) t^{i-3/2}) ! \quad \text{No limit of } g \text{ at } \infty !$$

$$\text{For } i > 0, \text{ no limit of } \int_0^T D_x^i f|_{x=3/2} dt \text{ at } \infty !$$

Successful Applications of Creative Telescoping

- *Descendants in heap-ordered trees* (Prodinger, 1996): Zeilberger's algorithm ($2 \times$)
- *Totally symmetric plane partition* (Kauers, Koutschan & Zeilberger, 2011): q -Koutschan ($n \times$)
- Lattice walks confined in a quadrant, in particular *Gessel's conjecture* for $\leftarrow, \rightarrow, \swarrow, \nearrow$ (Kauers, Koutschan & Zeilberger, 2009): variant of Takayama for specialisation ($n \leftrightarrow S_n$)



All were first proofs!