Homework exercises for 2020-10-12

Formal power series, D-finite series

The computations in the following exercises are feasible by hand, and we encourage the students to perform all of them in order to practice for the exam. In addition (but this is optional), we recommend to those of you having access to a computer algebra system to also conduct the computations using a computer. This might be useful not only for checking the results obtained ‘by hand’, but also in order to get a feeling on the concrete applicability of the algorithms in their implemented form.

Exercice 1
1. Describe a Newton iteration that directly computes a square root, without appealing to successive logarithm and exponential computations.

2. Estimate the complexity of this algorithm.

Exercice 2
1. Compute a differential equation satisfied by $F := (\cos X)^2 + (\sin X)^2$ by the algorithms for closures of D-finite series.

2. Deduce that $F$ is equal to 1.

Exercice 3
(This exercise may require a few more computations.)
Define the formal power series

$$A := \exp \left( \frac{1 - \sqrt{1 - 4X}}{2} \right).$$

1. Justify that this series is well defined.

2. By considering approximations to $O(X^3)$, show that

$$A = 1 + X + \frac{3}{2} X^2 + O(X^3).$$
3. Compute a linear differential equation with polynomial coefficients satisfied by \( A \), and check it at \( X = 0 \).

4. Convert this linear differential equation to a linear recurrence on its coefficients sequence \( (a_n)_{n \in \mathbb{N}} \) [hint: the quantities \( n_0 \) and \( n_1 \) defined on the slides of the lecture are \( n_0 = 0 \) and \( n_1 = 2 \)], and check it by verifying that \( a_3 = 19/6 \).