Exercise to prepare for 2021-01-25

Ore Algebras, Gröbner Bases, \(\partial\)-Finite Functions

Recall that a series \(s \in \mathbb{Q}[[X,Y]]\) is called algebraic (over \(\mathbb{Q}(X,Y)\)) if it satisfies \(P(X,Y,s(X,Y)) = 0\) for some nonzero \(P \in \mathbb{Q}[X,Y,F]\). Define the Ore algebras

\[
\begin{align*}
A_{X,Y} &:= \mathbb{Q}(X,Y)(\partial_X, \partial_Y; id, id, d/dX, d/dY), \\
A_{Z} &:= \mathbb{Q}(Z)(\partial_Z; id, id, d/dZ).
\end{align*}
\]

**Exercise 1.** Devise an algorithm that, given a nonzero polynomial \(P \in \mathbb{Q}[X,Y,F]\) whose partial derivative \(P_F\) satisfies \(\gcd(P, P_F) = 1\), outputs a Gröbner basis \(G\) of a left ideal \(I\) in \(A_{X,Y}\) such that \(I \subset \text{ann} s\), \(\dim_{\mathbb{Q}(X,Y)}(A_{X,Y}/I) < \infty\) whenever \(P(X,Y,s(X,Y)) = 0\). [Hint: You may want to interpret some \(s\) as a cyclic vector under the action of \(A_{X,Y}\).]

**Exercise 2.** Assume \(P, G, I\) are as in the first exercise. Further, let \(f\) be some nonconstant function that is \(\partial\)-finite with respect to \(A_{Z}\), and consider a monic skew polynomial \(L \in A_{Z}\) satisfying \(L \cdot f = 0\). Devise an algorithm that outputs a Gröbner basis \(H\) of a left ideal \(J\) in \(A_{X,Y}\) such that 

\[
J \subset \text{ann} g, \quad \dim_{\mathbb{Q}(X,Y)}(A_{X,Y}/J) < \infty
\]

whenever \(I \subset \text{ann} s\) and \(g(X,Y) = f(s(X,Y)) = 0\). [Hint: You may want to interpret \(g\) as a cyclic vector under the action of \(A_{X,Y}\).]