Efficient Algorithms in Computer Algebra (2021–2022)

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Chapter “Hypergeometric summation and hypergeometric solutions”
Part I

Last time’s exercises
For a linear differential equation with polynomial coefficients
\[ q_mA^{(m)} + \cdots + q_0A = \sum_{i,j \geq 0} q_{i,j} x^i A^{(j)} = 0, \]
introduce the integers
\[ n_0 = \max \{ i - j : q_{i,j} \neq 0 \} = \max_{0 \leq j \leq m} (\deg q_j - j), \]
\[ n_1 = \max \{ j - i : q_{i,j} \neq 0 \} = \max_{0 \leq j \leq m} (j - \text{val } q_j). \]
Write direct formulas for the indicial polynomial at 0 in terms of \( n_1 \) and the \( q_{i,j} \) and for the indicial polynomial at \( \infty \) in terms of \( n_0 \) and the \( q_{i,j} \).
Exercise 1

For a linear differential equation with polynomial coefficients

\[ q_m A^{(m)} + \cdots + q_0 A = \sum_{i,j \geq 0} q_{i,j} x^i A^{(j)} = 0, \]

introduce the integers

\[ n_0 = \max\{i - j : q_{i,j} \neq 0\} = \max_{0 \leq j \leq m} (\deg q_j - j), \]
\[ n_1 = \max\{j - i : q_{i,j} \neq 0\} = \max_{0 \leq j \leq m} (j - \val q_j). \]

Write direct formulas for the indicial polynomial at 0 in terms of \( n_1 \) and the \( q_{i,j} \) and for the indicial polynomial at \( \infty \) in terms of \( n_0 \) and the \( q_{i,j} \).

As in the lecture, we have

\[ \forall n \geq \max(0, n_0), \sum_{i,j \geq 0} q_{i,j} (n - i + j) \cdots (n - i + 1) a_{n-i+j} = \sum_{k=-n_0}^{n_1} \tilde{p}_k(n) a_{n+k} = 0. \]

So, directly, using the notation of falling factorials,

\[ I_0(\alpha) = \sum_{i,j \geq 0, j-i=n_1} q_{i,j} \alpha^j, \quad I_\infty(\alpha) = \sum_{i,j \geq 0, i-j=n_0} q_{i,j} \alpha^j. \]
Exercise 2

Consider the differential equation

\[(x - 1)(x^2 - 2)y''' + 2x(x^2 - x - 1)y' + 4(x - 2)y = 0.\]

Using algorithms of the course:

1. Show that this equation has no nonzero polynomial solution.
2. Show that all its rational solutions have at most a pole of order 1 at \(x = 1\) and no pole elsewhere.
3. Determine all its rational solutions.
Exercise 2

Consider the differential equation

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3. Determine all its rational solutions.

Polynomial solving:

- \(n_0 = \max\{3 - 2, 3 - 1, 1 - 0\} = 2, \ I_\infty(\alpha) = 2\alpha,\)
- degree must be 0, but no nonzero constant is a solution.
Consider the differential equation

\[(x - 1)(x^2 - 2)y'' + 2x(x^2 - x - 1)y' + 4(x - 2)y = 0.\]

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The only possible poles are at 1 and \(\pm \sqrt{2}\).
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The only possible poles are at 1 and $\pm \sqrt{2}$.

- Shifting by $x \rightarrow x + 1$ yields:

$$x(x^2 + 2x - 1)\tilde{y}''' + 2(x + 1)(x^2 + x - 1)\tilde{y}' + 4(x - 1)\tilde{y} = 0.$$ 

Now, $n_1 = \max\{2 - 1, 1 - 0, 0 - 0\} = 1$ and

$$I_0(\alpha) = -\alpha(\alpha - 1) - 2\alpha = -\alpha(\alpha + 1)$$

(this is $I_1$ of the original equation). The possible valuations are 0 and $-1$: the original equation may have a pole of order 1 at 1.
Consider the differential equation

\[(x - 1)(x^2 - 2)y'' + 2x(x^2 - x - 1)y' + 4(x - 2)y = 0.\]

Using algorithms of the course:

1. Show that this equation has no nonzero polynomial solution.
2. Show that all its rational solutions have at most a pole of order 1 at \(x = 1\) and no pole elsewhere.
3. Determine all its rational solutions.

The only possible poles are at 1 and \(\pm \sqrt{2}\).

- Shifting by \(x \rightarrow x + \sqrt{2}\) yields:
  
  \[(x+\sqrt{2}−1)x(x+2\sqrt{2})\tilde{y}'' + 2(x+\sqrt{2})(x^2+(2\sqrt{2}−1)x+1−\sqrt{2})\tilde{y}' + 4(x+\sqrt{2}−2)\tilde{y} = 0.\]

  Now, \(n_1 = \max\{2 − 1, 1 − 0, 0 − 0\} = 1\) and

  \[l_0(\alpha) = 2\sqrt{2}(\sqrt{2} − 1)\alpha(\alpha − 1) + 2\sqrt{2}(1 − \sqrt{2})\alpha = 2\sqrt{2}(\sqrt{2} − 1)\alpha(\alpha − 2)\]

  (this is \(l_{\sqrt{2}}\) of the original equation). The possible valuations are 0 and 2: the original equation has no pole at \(\sqrt{2}\). (Same with \(-\sqrt{2}\).)
Part II

Introduction
The three goals of this lecture

Indefinite summation: sum as a function of its upper bound

\[ \sum_{j=0}^{k} (-1)^j \binom{2020}{j} = \frac{(-1)^k k}{2020} \binom{2020}{k} \]

Definite summation: sum as a function of parameters of its summand

\[ \sum_{k \in \mathbb{Z}} (-1)^k \frac{(a + b)(a + c)(b + c)}{(a + k)(c + k)(b + k)} = \frac{(a + b + c)!}{a! b! c!} \]

Solving a recurrence

The linear recurrence

\[ (n + 2)u_{n+3} - 12(2n + 3)u_{n+2} + 192(n + 1)u_{n+1} - 256(2n + 1)u_n = 0 \]

admits solutions of the form

\[ u_n = \lambda \times 8^n + \mu \times n8^n + \nu \times n2^n \binom{2n}{n}. \]
The three goals of this lecture

Allows automatic proof of identities:


$$\sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^3 = \frac{n}{2} 8^n + 8^n - \frac{3n}{4} 2^n \binom{2n}{n}$$
Part III

Indefinite hypergeometric summation: Gosper’s algorithm
Hypergeometric sequences

Definition (a special case of polynomially recurrent sequence)

A sequence \((u_n)_{n \in \mathbb{N}}\) is hypergeometric (over some fixed field \(\mathbb{K}\)) if there exist two nonzero polynomials \(P\) and \(Q\) from \(\mathbb{K}[X]\) such that

\[
\forall n \in \mathbb{N}, \quad Q(n)u_{n+1} = P(n)u_n.
\]

Remarks (trivial but important)

- If \(Q(n) \neq 0\), then \(u_{n+1}\) only depends on \(u_n\).
- If \(\gcd(P, Q) =: G \neq 1\) and \(G(n_0) = 0\), then \(u_{n_0+1}\) and \(u_{n_0}\) are independent.
- If \(P(n_0) = 0 \neq Q(n)\) for all \(n \geq n_0\), then \(u_n = 0\) for all \(n > n_0\).

Solutions, initial conditions

- Provided \(0 \not\in Q(\mathbb{N})\), 1-dimensional vector space of solutions:

\[
u_n = u_0 \left( \frac{\text{l}(P)}{\text{l}(Q)} \right)^n \frac{\prod_{\alpha = 0} P(\alpha) (-\alpha)(-\alpha + 1) \cdots (-\alpha + n - 1)}{\prod_{\beta = 0} Q(\beta) (-\beta)(-\beta + 1) \cdots (-\beta + n - 1)}.
\]

- General case: higher-dimensional vector space of solutions, with initial conditions at indices 0 and some \(m\) for which \(Q(m - 1) = 0\).
Hypergeometric series in the theory of special functions

Definition of (generalized) hypergeometric series

Given \( p + q \) parameters \( a_1, \ldots, a_p, b_1, \ldots, b_q, \)

\[
pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},
\]

where

\[
(c)_n := c(c + 1) \cdots (c + n - 1) = \frac{\Gamma(c + n)}{\Gamma(c)}.
\]

Remark: Gauss studied \((p, q) = (2, 1)\).

Examples: exponential, logarithm, Bessel, Airy, etc.

\[
\exp(x) = _0F_0\left(- \middle| x \right), \quad \log(1 + x) = x_1F_1\left(1, 1 \middle| -x \right),
\]

\[
J_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu + 1)} _0F_1\left(- \middle| \frac{x}{2} \right),
\]

\[
Ai(x) = -\frac{x^{\frac{6}{3}}}{2\pi} _0F_1\left(- \middle| \frac{x^{3}}{9} \right) + \frac{3}{2\pi} _0F_1\left(- \middle| \frac{x^{3}}{9} \right).
\]
Idealizing sequences by difference algebra

**Definition**

A *difference ring* is a commutative ring $A$ together with a ring endomorphism $\sigma$. A *difference field* is a difference ring that is also a field.

Elements of difference rings idealize sequences.

<table>
<thead>
<tr>
<th>field</th>
<th>endomorphism</th>
<th>term</th>
<th>abuse of notation</th>
<th>sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}$</td>
<td>$\sigma(1) = 1$</td>
<td>1</td>
<td>$\sigma(1) = 1$</td>
<td>$(1)_{n \in \mathbb{N}}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(X)$</td>
<td>$\sigma(X) = X + 1$</td>
<td>$n$</td>
<td>$\sigma(n) = n + 1$</td>
<td>$(n)_{n \in \mathbb{N}}$</td>
</tr>
<tr>
<td>$\mathbb{Q}(X, Y)$</td>
<td>$\sigma(Y) = (X + 1)Y$</td>
<td>$n!$</td>
<td>$\sigma(n!) = (n + 1)n!$</td>
<td>$(n!)_{n \in \mathbb{N}}$</td>
</tr>
</tbody>
</table>

**Definition**

Given a difference ring $A$ containing $\mathbb{K}(n)$, a nonzero element $t_n$ of $A$ is a *hypergeometric term* if there exists $R \in \mathbb{K}(n)$ such that

$$\frac{t_{n+1}}{t_n} = R(n).$$

Remark: no consideration of a definition/validity domain!
Hypergeometric indefinite summation: structural results

The problem

Given a hypergeometric sequence \((u_n)_{n \in \mathbb{N}}\), determine whether there exists another hypergeometric sequence \((U_n)_{n \in \mathbb{N}}\) satisfying \(U_{n+1} - U_n = u_n\) for all \(n \in \mathbb{N}\), and if so compute it.

Definition

A linear recurrence operator (with rational function coefficients) is a formal sum \(\ell_d(n) \sigma^d + \cdots + \ell_0(n)\) for some order \(d\) and some \(\ell_i(n) \in \mathbb{K}(n)\).

Lemma

If \(U_n\) is a hypergeometric term and \(L\) is a linear recurrence operator, then there exists a rational function \(r(n) \in \mathbb{K}(n)\) such that

\[
(L \cdot U)_n = \ell_d(n) U_{n+d} + \cdots + \ell_0(n) U_n = r(n) U_n.
\]

Proof: There exists \(\rho(n) \in \mathbb{K}(n)\) such that \(U_{n+1} = \rho(n) U_n\). So for all \(k \in \mathbb{N}\), \(U_{n+k} = \rho(n+k-1) \cdots \rho(n) U_n\). Multiply by the \(\ell_k(n)\) and add up terms.

Necessarily, \(U_n = R(n) u_n\) for some \(R \in \mathbb{K}(n)\).
### Hypergeometric indefinite summation: an algorithm

**Termiology: “decision algorithm”**

Returns a Yes/No answer to an existence problem (decision) in finite time for any input of its input class (completeness) by providing a description of all solutions (constructiveness).

<table>
<thead>
<tr>
<th>Input</th>
<th>a hypergeometric term $u_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>a hypergeometric term $U_n$ such that $U_{n+1} - U_n = u_n$, or the symbol $\nexists$ to mean “provably, no such $U_n$ exists”</td>
</tr>
</tbody>
</table>

1. Let $\rho(n)$ be the rational function $u_{n+1}/u_n$.
2. Solve the recurrence relation $\rho(n)R(n+1) - R(n) = 1$ for its rational solutions (decision algorithm by Abramov).
3. If solving yields a solution, return $R(n)u_n$, otherwise return $\nexists$.

**Remarks:**

- this is a variant of Gosper’s original algorithm (1978);
- Abramov’s decision algorithms for solving for rational solutions are used as a blackbox and have been described in the previous lecture.
Compute $S_n := \sum_{k=1}^{n-1} \frac{2^k(k - 1)}{k(k + 1)}$ by summing the term $u_n = \frac{2^n(n - 1)}{n(n + 1)}$:

$$U_n := \sum 2^n(n - 1) \frac{\delta n}{n(n + 1)} = ???$$

From $\frac{u_{n+1}}{u_n} = \frac{2n^2}{(n-1)(n+2)}$, we get to solve:

$$2n^2R(n + 1) - (n - 1)(n + 2)R(n) = (n - 1)(n + 2).$$

The unique rational solution is $R(n) = \frac{n + 1}{n - 1}$, so $U_n = \frac{2^n}{n}$.

Finally, for $n \geq 2$,

$$S_n = \sum_{k=1}^{n-1} \frac{2^k(k - 1)}{k(k + 1)} = \frac{2^n}{n} - 2.$$
Hypergeometric indefinite summation: some details

Again, we need to determine all rational solutions \( R(n) \) of:
\[
2n^2 R(n + 1) - (n - 1)(n + 2) R(n) = (n - 1)(n + 2).
\]

Let \( h \in \mathbb{N} \) be maximal such that there exist \( \alpha \) such that \( R(n) \) has poles at \( \alpha \) and \( \alpha + h \). Let \( e_i \) denote the pole order at \( \alpha + i \), so \( e_0 > 0 \) and \( e_h > 0 \):
\[
R(n + 1) = \frac{S(n + 1)}{(n - (\alpha - 1))^{e_0} \cdots (n - (\alpha + h - 1))^{e_h}}, \quad R(n) = \frac{S(n)}{(n - \alpha)^{e_0} \cdots (n - (\alpha + h))^{e_h}}.
\]

So, \( (n - (\alpha - 1))^{e_0} \mid 2n^2 \) and \( (n - (\alpha + h))^{e_h} \mid (n - 1)(n + 2) \), implying \( \alpha = 1, h = 0, e_0 = e_h = 1 \), and thus, \( R(n) = S(n) / (n - 1) \):
\[
2nS(n + 1) - (n + 2)S(n) = (n - 1)(n + 2).
\]

Same reasoning: \( S(n) \) has no more poles, so is a polynomial, but now \( (n + 2) \mid S(n + 1) \), and thus \( S(n) = (n + 1)P(n) \):
\[
2nP(n + 1) - (n + 1)P(n) = n - 1.
\]

Degree considerations imply \( P(n) \) is a constant, then \( P(n) = 1 \), hence
\[
R(n) = \frac{n + 1}{n - 1}.
\]
From terms \( u(X) \) back to sequences \( (u_n)_{n \in \mathbb{N}} \)

\[
(u_n)_{n \in \mathbb{N}} \quad \longrightarrow \quad u(X)
\]

\[
\forall n \in \mathbb{N}, \; Q(n) \neq 0 \Rightarrow u_{n+1} = \rho(n)u_n
\]

\[
\rho = \frac{P}{Q} = \frac{u(X + 1)}{u(X)}
\]

\[
\forall n \in \mathbb{N}, \; Q(n)V(n)V(n + 1) \neq 0
\]

\[
\Rightarrow \rho(n)R(n + 1) - R(n) = 1
\]

\[
\rho(X)R(X + 1) - R(X) = 1
\]

\[
\rho = \frac{P}{Q} = \frac{u(X + 1)}{u(X)}
\]

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\frac{P}{Q} = \frac{u(X + 1)}{u(X)}
\]

\[
S := \{n \in \mathbb{N} \mid Q(n)V(n)V(n + 1) = 0\}, \quad \tilde{U}_n := (V(n) = 0) ? 0 : R(n)u_n,
\]

\[
\forall n \in \mathbb{N} \setminus S, \; \tilde{U}_{n+1} - \tilde{U}_n = R(n + 1)u_{n+1} - R(n)u_n = \left(\rho(n)R(n + 1) - R(n)\right)u_n = u_n.
\]

Define three sequences by

\[
\phi_n := (n \in S) ? u_n - (\tilde{U}_{n+1} - \tilde{U}_n) : 0, \quad \Phi_n := \sum_{i=0}^{n-1} \phi_i, \quad U_n := \tilde{U}_n + \Phi_n.
\]

**Theorem**

If Gosper’s algorithm succeeds in finding an indefinite sum in the form of a hypergeometric term \( R(X)u(X) \), then:

- as a sequence, \((U_n)_{n \in \mathbb{N}}\) is an indefinite sum: \( \forall n \in \mathbb{N}, \; U_{n+1} - U_n = u_n \);
- \((\Phi_n)_{n \in \mathbb{N}}\) is piecewise constant, with finitely many computable jumps.
The partial sum of $e = \sum_{k \geq 0} 1/k!$ is not hypergeometric

If

$$S(n) := \sum_{k=1}^{n-1} \frac{1}{k!}$$

were hypergeometric, there would be a rational $R \in \mathbb{Q}(n)$ satisfying

$$S(n) = \frac{R(n)}{n!}$$

and

$$\frac{R(n+1)}{(n+1)!} - \frac{R(n)}{n!} = S(n+1) - S(n) = \frac{1}{n!},$$

and therefore

$$R(n + 1) - (n + 1)R(n) = n + 1.$$ 

No possible pole, only possible degree is 0, no possible constant $R \neq 0$, so no possible $R$. 
Two nonzero hypergeometric terms $u(n)$ and $v(n)$ are similar if their quotient $u(n)/v(n)$ is a rational function of $n$.

The problem

Given pairwise similar hypergeometric terms $u_1(n), \ldots, u_s(n)$, determine whether there exist, and if so compute, another hypergeometric term $U(n)$ and constants $\lambda_1, \ldots, \lambda_s$ satisfying

$$U(n+1) - U(n) = \lambda_1 u_1(n) + \cdots + \lambda_s u_s(n).$$

Approach: After writing

$$U(n+1) - U(n) = \left[ \lambda_1 + \lambda_2 \frac{u_2(n)}{u_1(n)} + \cdots + \lambda_s \frac{u_s(n)}{u_1(n)} \right] u_1(n),$$

the bracket is a rational term in $n$. Gosper's and Abramov’s algorithms generalize by identifying linear constraints on the $\lambda_i$ that make the search for $R(n) = U(n)/u_1(n)$ successful.
Part IV

Definite hypergeometric summation: 
Zeilberger’s algorithm
Bivariate hypergeometric sequences

**Definition**

A sequence \((u_{n,k})_{(n,k)\in\mathbb{N}^2}\) is *hypergeometric with respect to n and k* (over some fixed field \(\mathbb{K}\)) if there exist a polynomial \(Z \in \mathbb{K}[X, Y]\) and rational functions \(\rho\) and \(\sigma\) from \(\mathbb{K}(X, Y)\) such that

\[
\forall (n, k) \in \mathbb{N}^2, \quad Z(n, k) \neq 0 \Rightarrow \frac{u_{n+1,k}}{u_{n,k}} = \rho(n, k) \quad \text{and} \quad \frac{u_{n,k+1}}{u_{n,k}} = \sigma(n, k).
\]

**Example: binomial coefficients**

\[
\begin{align*}
\binom{n}{k} = \frac{n!}{k! \,(n-k)!}, & \quad \rho(n, k) = \frac{n+1}{n+1-k}, & \quad \sigma(n, k) = \frac{n-k}{k+1}.
\end{align*}
\]

**Property** [would generalize to a noncommutative Gröbner-basis theory in a longer course]

\[
\rho(X, Y+1)\sigma(X, Y) = \rho(X, Y)\sigma(X+1, Y)
\]
Definite hypergeometric summation

Goal: develop semi-automatic proofs of parametrized sum identities like

\[ \sum_{k \in \mathbb{Z}} \binom{n}{k} = 2^n, \quad \sum_{k \in \mathbb{Z}} \binom{n}{k}^2 = \binom{2n}{n}, \quad \sum_{k \in \mathbb{Z}} (-1)^k \binom{2n}{n+k}^3 = \frac{(3n)!}{(n!)^3}, \quad \text{etc.} \]

Sums with “natural bounds”, a simpler situation

Because of the equivalence \( \binom{n}{k} = 0 \iff k \notin \{0, \ldots, n\} \) for any given \( n \in \mathbb{N} \):

\[ \sum_{k \in \mathbb{Z}} \rightarrow \sum_{k=0}^{n}, \quad \sum_{k \in \mathbb{Z}} \rightarrow \sum_{k=0}^{n}, \quad \sum_{k \in \mathbb{Z}} \rightarrow \sum_{k=-n}^{n}, \quad \text{respectively.} \]

Principle of summation by creative telescoping

bivariate hypergeometric term \( \rightarrow \) structured recurrence on summand \( \rightarrow \) recurrence on sum \( \rightarrow \) closed form (existence?)

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Operator notation for linear recurrence equations

(Forward) Shift operator
- on sequences: $S_n : (t_{n,k})_{(n,k) \in \mathbb{N}^2} \mapsto (t_{n+1,k})_{(n,k) \in \mathbb{N}^2}$
- or on terms: $(S_n \cdot t)(n, k) = t(n + 1, k)$
Similar notation for $S_k$.

General recurrence operator

Given a term $t(n, k)$, a rational function $c(n, k)$, nonnegative integers $i, j$,

$\left( c(n, k)S^i_n S^j_k \cdot t \right)(n, k) = c(n, k)t(n + i, k + j)$.

Representing an equation

$$\sum_{(i,j)} c_{i,j}(n, k) t_{n+i, k+j} = 0 \quad \rightarrow \quad \sum_{(i,j)} c_{i,j}(n, k) S^i_n S^j_k$$
The method of creative telescoping: a simple example

Let us prove: \( U(n) := \sum_{k \in \mathbb{Z}} \binom{n}{k} = 2^n. \)

Pascal’s triangle rule,
\[
\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k},
\]
is represented in operator notation by
\[
S_n S_k - S_k - 1 = (S_n - 2) + (S_k - 1)(S_n - 1).
\]
This reflects the explicit relation
\[
\left( \left( \binom{n+1}{k} - 2 \binom{n}{k} \right) + \left( \binom{n+1}{k+1} - \binom{n}{k+1} \right) - \left( \binom{n+1}{k} - \binom{n}{k} \right) \right) = 0.
\]
Sum over \( k \in \mathbb{Z} \) and discard telescoping sums. There remains:
\[
\left( (S_n - 2) \cdot U \right)(n) = U(n+1) - 2U(n) = 0.
\]
Verifying the initial condition \( U(0) = 1 \) ends the proof.
Creative telescoping for (general) sequences

Given a sequence \((u_{n,k})_{(n,k)\in\mathbb{N}^2}\), suppose one can find two polynomials

\[
P(n, S_n) = \sum_{0 \leq i \leq r} p_i(n)S_n^i \quad \text{and} \quad Q(n, k, S_n, S_k) = \sum_{0 \leq i+j \leq r} q_{i,j}(n, k)S_n^i S_k^j
\]

satisfying

\[
\forall (n, k) \in \mathbb{N}^2, (L \cdot u)(n, k) = 0,
\]

for \(L(n, k, S_n, S_k) = P(n, S_n) - (S_k - 1)Q(n, k, S_n, S_k)\).

Then:

- If \(U_n = \sum_{k\in\mathbb{Z}} u_{n,k}\) is over natural bounds, then
  \[
  \forall n \in \mathbb{N}, (P \cdot U)(n) = p_r(n)U_{n+r} + \cdots + p_0(n)U_n = 0.
  \]

- If \(U_n = \sum_{k=a}^{b} u_{n,k}\) for \((a, b)\) independent of \(n\), then
  \[
  \forall n \in \mathbb{N}, (P \cdot U)(n) = (Q(n, k, S_n, S_k) \cdot u)(n, b+1) - (Q(n, k, S_n, S_k) \cdot u)(n, a).
  \]
Structured recurrences for hypergeometric terms

Given a hypergeometric term \( u(n, k) \), suppose one can find two polynomials

\[
P(n, S_n) = \sum_{0 \leq i \leq r} p_i(n) S_n^i \quad \text{and} \quad Q(n, k, S_n, S_k) = \sum_{0 \leq i+j \leq r} q_{i,j}(n, k) S_n^i S_k^j
\]

satisfying

\[
P(n, S_n) \cdot u = (S_k - 1)Q(n, k, S_n, S_k) \cdot u.
\]

Then, there exists a rational function \( R(n, k) \) satisfying

\[
\boxed{P(n, S_n) \cdot u = (S_k - 1) \cdot (R(n, k) u)}.
\]

Problem when back to sequences: values of \( k \) that are poles of \( R \)?
Structured recurrences for terms: examples

For the three examples, respectively:

\[
\begin{align*}
\frac{u(n + 1, k) - 2u(n, k)}{S_n - 2} &= (S_k - 1) \cdot \left( \frac{k}{k - n - 1} u(n, k) \right) - \frac{k}{n+1} u(n+1, k) \\
\frac{(n + 1)u(n + 1, k) - 2(2n + 1)u(n, k)}{(n+1)S_n - 2(2n+1)} &= (S_k - 1) \cdot \left( \frac{k^2(2k - 3 - 3n)}{(k - n - 1)^2} u(n, k) \right) - \frac{k^2(2k - 3 - 3n)}{(n+1)^2} u(n+1, k) \\
\frac{(n + 1)^2u(n + 1, k) - 3(3n + 2)(3n + 1)u(n, k)}{(n+1)^2S_n - 3(3n+2)(3n+1)} &= (S_k - 1) \cdot \left( \frac{p(n, k)}{2(k - n - 1)^3} u(n, k) \right) - \frac{(k+n+1)^3p(n, k)}{16(n+1)^3(2n+1)^3} u(n+1, k)
\end{align*}
\]

where

\[
p(n, k) = 9k^4n - 30k^2n^3 + 37n^5 + 6k^4 - 18k^3n - 54k^2n^2 + 30kn^3 + 116n^4
\]

\[
- 12k^3 - 18k^2n + 54kn^2 + 136n^3 + 3k^2 + 27kn + 74n^2 + 3k + 19n + 2.
\]
Zeilberger’s fast algorithm: the idea

To solve $P(n, S_n) \cdot u = (S_k - 1) \cdot (R(n, k) u)$, if $P$ was known, Gosper’s algorithm could find $R$!

**Ideas**

- work with increasing target shift order $r$ for $P$ (degree in $S_n$)
- introduce undetermined coefficients for $P$
- employ parametrized Gosper’s algorithm
Zeilberger’s fast algorithm: the description

**Input** A hypergeometric term $u(n, k)$ and an integer $m \in \mathbb{N}$.

**Output** A family $(\lambda_i(n))_{0 \leq i \leq r}$ of rational functions, with $r \leq m$, and a rational function $R(n, k)$ such that

$$
\lambda_r(n)u_{n+r,k} + \cdots + \lambda_0(n)u_{n,k} = R(n, k + 1)u_{n,k+1} - R(n, k)u_{n,k},
$$

or the symbol $\not\exists$.

1. For $r$ from 0 to $m$:
   1. fix $n$ as a parameter and introduce the similar hypergeometric terms $u_i(k) = u(n + i, k)$, for $0 \leq i \leq r$;
   2. solve
      $$
      S(k + 1) - S(k) = \lambda_0 u_0(k) + \cdots + \lambda_r u_r(k)
      $$
      for a hypergeometric term $S(k) = R(k)u_0(k) = R(n, k)u(n, k)$ given by a rational function $R = R(n, k)$ and for rational functions $\lambda_i = \lambda_i(n)$ independent of $k$;
   3. if solving provides a solution, return the family $(\lambda_i)_{i=0}^r$ and $R$.

2. Return $\not\exists$. 
Existence of a recurrence for the sum — Termination

Proper hypergeometric terms

\[ u(n, k) = P(n, k)\eta^n\xi^k \prod_{1 \leq \ell \leq L} (a_\ell n + b_\ell k + c_\ell)!^{e_\ell} \]

for \( P \in \mathbb{K}[n, k], \eta \in \mathbb{K}, \xi \in \mathbb{K}, \ L \in \mathbb{N}, \) and all of the \( a_\ell, b_\ell, e_\ell \) in \( \mathbb{Z} \).

Remarks: (i) General hyp. terms have a rational function in place of \( P \).
(ii) Some (rational) denominators may be absorbed into factorials.

Theorem

An unbounded variant of Zeilberger’s algorithm \((m = \infty)\) terminates on any proper hypergeometric input.

Proof:
- Find \( A(n, S_n, S_k) \neq 0 \) such that \( A \cdot u = 0 \) (lemma on next slide).
- Find \( \tilde{P} \neq 0 \) and suitable \( \tilde{Q} \) and \( p \) to write it
  \[ A = (S_k - 1)^p \left[(S_k - 1)\tilde{Q}(n, S_k, S_n) + \tilde{P}(n, S_n)\right]. \]
- Observe the existence of \( Q(n, k, S_n, S_k) \) for which
  \[ k(k - 1) \cdots (k - p + 1)A \cdot u(n, k) = (-1)^p p! \tilde{P} \cdot u(n, k) + (S_k - 1)Q \cdot u(n, k). \]
Elimination property of proper hypergeometric terms

**Lemma**

There exists a nonzero \( A(n, S_n, S_k) \) annihilating \( u(n, k) \) with \( \deg_{S_n} A \leq r \) and \( \deg_{S_k} A \leq s \),

\[
r = 2\beta, \quad s = 2(2\alpha - 1)\beta + \deg_k P + 1, \quad \alpha = \sum_{\ell=1}^{L} |e_\ell a_\ell|, \quad \beta = \sum_{\ell=1}^{L} |e_\ell b_\ell|.
\]

Explanation on the simplified case \( u = \binom{n}{k} \): for \( 0 \leq i \leq r \) and \( 0 \leq j \leq s \),

\[
\binom{n+i}{k+j} = \frac{(n+i)!}{(k+j)! (n-k+i-j)!} = \frac{n!}{(k+s)! (n-k+r)!} \times (n+1) \cdots (n+i) \\
\times (k+j+1) \cdots (k+s) \times (n-k+i-j+1) \cdots (n-k+r). \tag{deg_k=(s-j)+(r-i+j)\leq r+s=O(r+s)}
\]

As there are \( O((r+s)^2) \) choices for \((i,j)\), linear algebra over \( \mathbb{K}(n) \) to get a dependency in \( \mathbb{K}(n)[k] \leq r+s \) proves the existence of \( A \).
Part V

Solving for hypergeometric solutions: Petkovšek’s algorithm
Hypergeometric solutions of a recurrence: the problem

The problem

Given a linear recurrence operator \( L = a_r(n)S_r^n + \cdots + a_0(n) \), find all its hypergeometric-term solutions \( u(n) \).

\[
\begin{align*}
  u(n + 1) &= \frac{P(n)}{Q(n)} u(n) \quad \rightarrow \quad u(n + \ell) = \frac{P(n + \ell - 1)}{Q(n + \ell - 1)} \cdots \frac{P(n)}{Q(n)} u(n)
\end{align*}
\]

Necessarily,

\[
\begin{align*}
  a_r(n) P(n + r - 1)P(n + r - 2) \cdots P(n) + \\
  \underline{\text{only term without } Q(n+r-1)}
  a_{r-1}(n)Q(n + r - 1)P(n + r - 2) \cdots P(n) + \cdots \\
  a_1(n)Q(n + r - 1) \cdots Q(n - 1)P(n) + \\
  a_0(n) \underline{Q(n + r - 1) \cdots Q(n)} = 0,
\end{align*}
\]

so we need to consider all \( \gcd(P(n), Q(n + \ell)) \), which are unconstrained.
Lemma

Given a rational function $P/Q \in \mathbb{C}(n)$, there exists a unique tuple $(\zeta, A, B, C) \in \mathbb{C} \times \mathbb{C}[n]^3$ with monic $A$, $B$, and $C$ that satisfies

$$
\frac{P(n)}{Q(n)} = \zeta \frac{A(n)}{B(n)} \frac{C(n + 1)}{C(n)},
$$

with $\gcd(A(n), C(n)) = \gcd(B(n), C(n + 1)) = \gcd(A(n), B(n + i)) = 1$ for $i \in \mathbb{N}$.

Proof: Choose $\zeta$ to reduce to coprime monic $P$ and $Q$. Set $(A_0, B_0, C_0) := (P, Q, 1)$ and iterate $G_k(n) := \gcd(A_k(n - k - 1), B_k(n))$ in

$$(A_{k+1}(n), B_{k+1}(n), C_{k+1}(n)) := \left(\frac{A_k(n)}{G_k(n + k + 1)}, \frac{B_k(n)}{G_k(n)}, C_k(n)G_k(n) \cdots G_k(n + k)\right),$$

with invariant

$$
\frac{P(n)}{Q(n)} = \frac{A_k(n)}{B_k(n)} \frac{C_k(n + 1)}{C_k(n)}.
$$

Take $(A, B, C) := \lim_{k \to \infty} (A_k, B_k, C_k)$. Observe that for all $0 \leq k < i, j$, $A_i(n) \mid A_{k+1}(n)$, $B_j(n) \mid B_{k+1}(n)$, so $\gcd(A_i(n), B_j(n + k + 1)) = 1$. Good choices for $(i, j, k)$ give the rest.
Hypergeometric solutions of a recurrence: the analysis

\[ u(n + 1) = \frac{P(n)}{Q(n)} u(n) = \zeta \frac{A(n)}{B(n)} \frac{C(n + 1)}{C(n)} u(n) \]

Combining

\[ \zeta^r a_r(n) A(n + r - 1) A(n + r - 2) \cdots A(n) C(n + r) + \]

only term without \( B(n+r-1) \)

\[ \zeta^{r-1} a_{r-1}(n) B(n + r - 1) A(n + r - 2) \cdots A(n) C(n + r - 1) + \cdots \]

\[ \zeta a_1(n) B(n + r - 1) \cdots B(n - 1) A(n) C(n + 1) + \]

\[ a_0(n) B(n + r - 1) \cdots B(n) C(n) = 0, \]

only term without \( A(n) \)

with \( \gcd(A(n), C(n)) = \gcd(B(n), C(n + 1)) = \gcd(A(n), B(n + i)) = 1 \) for \( i \in \mathbb{N} \)

yields

\[ A(n) \mid a_0(n), \quad B(n + r - 1) \mid a_r(n). \]
Hypergeometric solutions of a recurrence: the algorithm

Input  A recurrence equation $a_r(n)u(n + r) + \cdots + a_0(n)u(n) = 0$ with polynomial coefficients $a_i(n)$.

Output  The set of hypergeometric terms $t_n$ solving the recurrence.

1. Initialize $S$ to the empty set and factor the polynomials $a_0$ and $a_r$.
2. For each monic $A(n) \mid a_0(n)$ and for each monic $B(n) \mid a_r(n - r + 1)$:
   1. extract the leading coefficient $d(\zeta)$ of
      \[
      \sum_{i=0}^{r} \zeta^i a_i(n) B(n + r - 1) \cdots B(n + i + 1) A(n + i) \cdots A(n) C(n + i);
      \]
   2. factor $d(\zeta)$;
   3. for each monic irreducible factor $p(\zeta)$ of $d(\zeta)$:
      1. fix $\zeta$ satisfying $p(\zeta) = 0$ by extending the constant field if necessary,
      2. compute polynomials $C(n)$ that annihilate the recurrence above.
      3. for each obtained solution $C$, define a hypergeometric term $t_n$ by
         \[
         t_{n+1}/t_n = \frac{A(n)}{B(n)} \frac{C(n + 1)}{C(n)}
         \]
      and add to $S$;
3. Return $S$. 