Algorithms for solving fixed point equations of order 1

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Joint work with:
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Motivation: A non linear equation coming from combinatorics...

$K$ field of characteristic 0. $K = \mathbb{Q}, \mathbb{C}, \ldots$

**Starting point:** $F$, solution in $K[u][[t]]$ of the fixed point equation (FPE) of order 1

$$F(t, u) = 1 + tu(uF(t, u)^2 + F(t, u) + \Delta F(t, u)),$$

where $\Delta$ is the divided difference operator $\Delta F(t, u) := \frac{F(t, u) - F(t, 1)}{u - 1}$.
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Goals:

- Compute a polynomial \( R \in \mathbb{K}[t, z] \setminus \{0\} \) such that \( R(t, F(t, 1)) = 0 \).
- Estimate the size of \( R \) for any (FPE).
- Complexity estimates (ops. in \( \mathbb{K} \)) for the computation of \( R \).
...associated to planar maps enumeration

Count

\[ c_n := \# \{ \text{planar maps with } n \text{ edges} \} \]

\[ \downarrow \text{refinement} \]

\[ c_{n,d} := \# \{ \text{planar maps with } n \text{ edges,} \]
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Solution in \( \mathbb{K}[u][[t]] \)

\[ G(t) := \sum_{n=0}^{\infty} c_n t^n \]

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complete generating function
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(FPE) of order 1 [Tutte '68]

\[ F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \]
Theorem [Bousquet-Mélou, Jehanne '06]
Let $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y, t, u]$. Let $F(t, u)$ be the unique solution in $\mathbb{K}[u][[t]]$ of

$$F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u),$$

where $\Delta$ is the divided difference operator $\Delta F := \frac{F(t,u)-F(t,1)}{u-1}$.

Then $F$ is algebraic over $\mathbb{K}(t, u)$. 
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Systematic methods

[Tutte, Brown 60’s], [Zeilberger ’92]:
Guess-and-prove

[Brown ’65]: Quadratic method

[Popescu ’86]: Algebraicity result

[Bousquet-Mélou, Jehanne ’06]: Polynomial elimination
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Polynomial elimination

Algorithms

[Knuth ’68], [Banderier, Flajolet ’02]:
Kernel method (linear case)

[Bousquet-Mélou, Jehanne ’06]:
Polynomial elimination

[Gessel, Zeilberger ’14]:
Guess-and-prove
[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE) of order 1

\[ P(x, z, t, u) = 0, \]
\[ \partial_x P(F(t, u), F(t, 1), t, u) = 0, \]
\[ \partial_u P(x, z, t, u) = 0. \]

(F(t, U(t)), F(t, 1), U(t)) zero in $\mathbb{K}[[t]]^3$
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE)

(FPE) of order 1 $\rightarrow$ solution $u = U(t) \in \mathbb{K}[[t]]$ of $
\partial_x P(F(t, u), F(t, 1), t, u) = 0$

Fixed Point Equation (FPE)

\[ P(F(t, u), F(t, 1), t, u) = 0 \]

\[ \partial_u F(t, u) \cdot \partial_x P(F(t, u), F(t, 1), t, u) + \partial_u P(F(t, u), F(t, 1), t, u) = 0 \]

\[ \begin{aligned}
P(x, z, t, u) &= 0, \\
\partial_x P(x, z, t, u) &= 0, \\
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(FPE)

Planar maps

$F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1}$
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE)

\[ (F\text{PE}) \text{ of order } 1 \rightarrow \text{solution } u = U(t) \in \mathbb{K}[[t]] \text{ of} \]

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\[ \begin{align*}
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P(x, z, t, u) = 0, \\
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\[ (F(t, U(t)), F(t, 1), U(t)) \text{ zero in } \mathbb{K}[[t]]^3 \]

---

**Fixed Point Equation (FPE)**

\[ P(F(t, u), F(t, 1), t, u) = 0 \]

\[ \downarrow \text{numer} \]

\[ \partial_u F(t, u) \cdot \partial_x P(F(t, u), F(t, 1), t, u) + \partial_u P(F(t, u), F(t, 1), t, u) = 0 \]

**Planar maps**

\[ F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \quad \text{(FPE)} \]

\[ 0 = (1 - F(t, u))(u - 1) + tu^2(u - 1)F(t, u)^2 + tu(uF(t, u) - F(t, 1)) \]
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE)

(FPE) of order 1 $\Rightarrow$ solution $u = U(t) \in \mathbb{K}[[t]]$ of
$$\frac{\partial}{\partial x} P(F(t, u), F(t, 1), t, u) = 0$$

$P(x, z, t, u) = 0,$
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(F(t, U(t)), F(t, 1), U(t)) zero in $\mathbb{K}[[t]]^3$

Planar maps

$$F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \quad (FPE)$$

$$0 = (1 - F(t, u))(u - 1) + tu^2 (u - 1) F(t, u)^2 + tu(uF(t, u) - F(t, 1))$$

$$0 = \frac{\partial}{\partial u} F(t, u) \cdot (1 - u + 2tu^2 (u - 1) F(t, u)^2 + tu^2) + (1 - F(t, u) + tu(3u - 2) F(t, u)^2 + 2tu^2 F(t, u) - tF(t, 1))$$
Our contributions

Work based on [Bousquet-Mélou, Jehanne ’06]

1. Geometric **refinements** of a method based on discriminants,
2. A new guess-and-prove method based on **geometry**,
3. A **complexity result** on the resolution of (FPE) of order 1.

Attention is paid to

- assumptions,
- degree bounds on the output,
- complexity estimates,
- potential for generalization.

**Input:** $P := \text{numerator}(\text{FPE})$,  
**Goal:** $\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z]$. 
disc \( x(P) = \text{Res}_x(P, \partial_x P) \) the discriminant of \( P \) in \( x \).

**Theorem** [Bousquet-Mélou, Jehanne ’06]

Suppose \( \deg_x(P) \geq 2 \) and \( u = U(t) \in \mathbb{K}[[t]] \) is a root of

\[
\partial_x P(F(t, u), F(t, 1), t, u).
\]

Then \( u = U(t) \) is a **double root** of \( \text{disc}_x(P)(F(t, 1), t, u) \).

Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).
Algebraic elimination via iterated discriminants

disc_x(P) = \text{Res}_x(P, \partial_x P) \text{ the discriminant of } P \text{ in } x.

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Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).

\[
P := (1 - x)(u - 1) + tu^2(u - 1)x^2
+ tu(ux - z)
\]

gives \( \text{disc}_u(\text{disc}_x P) \) equal to

\[
-256t^4 \cdot (27t^2z^2 - 18tz + 16t + z - 1)
\cdot (tz - 1)^2
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\[
P := 97t^3u^2 + (-73u^4 - 56u^2x^2 + 87u^2x - 62x^2 + 124xz - 62z^2)t - xu^2 + u^2
\]
gives \( \text{disc}_xP \) equal to

\[
-16352t^2u^6 \\
+ \left(21728t^4 - 10535t^2 + 50t + 1\right)u^4 \\
+ 248t \left(97t^3 - 56tz^2 + 87tz - z + 1\right)u^2
\]
which has a double root at \( u = 0 \).
Theorem [Bostan, Chyzak, N., Safey El Din ’22]

Suppose

1. \((H0)\) \(\deg_x(P) \geq 2\),
2. \((H1)\) \(\deg_u(\partial_x P(x, z, 0, u)) \geq 1\) and \(\partial_x P(F(t, c), F(t, 1), t, c) \neq 0\) for all \(c \in \mathbb{K}\),
3. \((R)\) the zero set \(V(P) \subset \overline{\mathbb{K}}^4\) is smooth outside \(V(u - 1) \subset \overline{\mathbb{K}}^4\).

Set \(D_0 := \text{disc}_x P\), \(D_1 := \text{SqFreePart}(D_0)\) and \(D_2 := \text{disc}_u D_1\).

Then

1. \(R := \text{SqFreePart}(D_2)\) is non-zero in \(\mathbb{K}[z, t]\) and satisfies \(R(F(t, 1), t) = 0\).
**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

Suppose

- (H0) \( \deg_x(P) \geq 2 \),
- (H1) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- (R) the zero set \( V(P) \subset \mathbb{K}^4 \) is smooth outside \( V(u - 1) \subset \mathbb{K}^4 \).

Set \( D_0 := \text{disc}_x P, \ D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1. \)

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
- \( R \) has total size \( 16\delta^8 \) with degree in each variable at most \( 4\delta^4 \),
- \( R \) can be computed in \( O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \).
Contribution 1: ensuring non-nullity of double discriminant

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

Suppose

- (H0) $\deg_x(P) \geq 2$,
- (H1) $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- (R) the zero set $V(P) \subset \overline{\mathbb{K}}^4$ is smooth outside $V(u - 1) \subset \overline{\mathbb{K}}^4$.

Set $D_0 := \text{disc}_x P$, $D_1 := \text{SqFreePart}(D_0)$ and $D_2 := \text{disc}_u D_1$.

Then

- $R := \text{SqFreePart}(D_2)$ is non-zero in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.
- $R$ has total size 16$\delta^8$ with degree in each variable at most 4$\delta^4$,
- $R$ can be computed in $O_{\log(\delta^{10})}$ ops. in $\mathbb{K}$.

$D_1 := \text{SqFreePart} \left( \text{disc}_x(P) \right)$ satisfies

\[
\partial_u D_1(U(t), F(t, 1), t) = 0.
\]

\[
\begin{cases}
(\partial_u D_1 \; \partial_z D_1 \; \partial_t D_1) \cdot (u \; z \; t)^T = 0, \\
(\partial_z D_1 \; \partial_t D_1) \cdot (z \; t)^T = 0
\end{cases}
\]
Contribution 1 (cont’d): using geometry arguments to refine the complexity

$P \in \mathbb{K}[x, z, t, u]$ and $\delta := \deg(P)$.

Theorem [Bostan, Chyzak, N., Safey El Din ’22]

Suppose
- (H1) $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- $\langle P, \partial_x P, \partial_u P \rangle : (u - 1)^\infty \subset \mathbb{K}(t)[x, z, u]$ is radical and 0-dimensional over $\mathbb{K}(t)$.

Then one can compute $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$
- with degree in each variable at most $\delta^3$ and total size $\delta^6$,
- in $O_{\log}(L\delta^6 + \delta^{7.89}) \subset O_{\log}(\delta^{10})$ ops. in $\mathbb{K}$, where $L = \text{cost of evaluating } P \text{ at } (x, z, t, u) \in \mathbb{K}^4$. 
Contribution 2: Guess-and-prove based on geometry

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \text{deg}(P) \).  
**Output:** \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1) \), i.e. \( R(t, F_1) = 0 \).
**Contribution 2: Guess-and-prove based on geometry**

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P) \).

**Output:** \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1) \), i.e. \( R(t, F_1) = 0 \).

---

**geometry**

1. **Functional equation**

2. **Polynomial system**

3. **Bounds**
   - \( \deg_t(R) \leq b_t \)
   - \( \deg_z(R) \leq b_z \)

---

(expand on tools: Newton iteration, Algebraic approximants, "seriestoalgeq", Multiplicity lemma)
Input: \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P) \).
Output: \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1) \), i.e. \( R(t, F_1) = 0 \).

**guess-and-prove**

- (4) Expand \( F_1 \)
- (5) Compute \( R \in \mathbb{K}[t, z] \) s.t. \( R(t, F_1) = O(t^{\sim b_t b_z}) \)
- (6) Certify that \( R(t, F_1) = 0 \)

**geometry**

- (1) Functional equation
- (2) Polynomial system
- (3) Bounds
  - \( \deg_t(R) \leq b_t \)
  - \( \deg_z(R) \leq b_z \)
Contribution 2: Guess-and-prove based on geometry

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P) \).

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<tr>
<td>(1) Functional equation</td>
<td>(4) Expand ( F_1 )</td>
<td>● Newton iteration</td>
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<tr>
<td>(2) Polynomial system</td>
<td>(5) Compute ( R \in \mathbb{K}[t, z] ) s.t. ( R(t, F_1) = O(t^{b_t b_z}) )</td>
<td>● Algebraic approximants “seriestoalgeq”</td>
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</table>
| (3) Bounds  
  - \( \deg_t(R) \leq b_t \),  
  - \( \deg_z(R) \leq b_z \) | (6) Certify that \( R(t, F_1) = 0 \) | ● Multiplicity lemma: \( R(t, F_1) = O(t^{2b_t b_z}) \) implies \( R(t, F_1) = 0 \) |
Define $A_u := (F(t, u), F(t, 1), u)$ and assume that
- there exists $u = U(t) \in \mathbb{K}[[t]] \setminus \{1\}$ solution of $\partial_x P(F(t, u), F(t, 1), t, u) = 0$,
- the Jacobian of $(P, \partial_x P, \partial_u P)$ w.r.t $\{x, z, u\}$ is invertible at $A_{U(t)} \in \mathbb{K}[[t]]^3$.

Then the geometry-driven guess-and-prove computes $R \in \mathbb{K}[t, z] \setminus \{0\}$
- such that $R(t, F(t, 1)) = 0$,
- having its partial degrees bounded by $\delta^3$ and total size $\delta^6$,
- in $O_{\log}(\delta^{10.14})$ arithmetic operations in $\mathbb{K}$.
\[ \theta \in [2, 3] \text{ a feasible exponent of matrix multiplication} \]

**Theorem [Bostan, Chyzak, N., Safey El Din '22]**

Define \( A_u := (F(t, u), F(t, 1), u) \) and assume that
- there exists \( u = U(t) \in \mathbb{K}[t] \setminus \{1\} \) solution of \( \partial_x P(F(t, u), F(t, 1), t, u) = 0 \),
- the Jacobian of \( (P, \partial_x P, \partial_u P) \) w.r.t \( \{x, z, u\} \) is invertible at \( A_{U(t)} \in \mathbb{K}[t]^3 \).

Then the geometry-driven guess-and-prove computes \( R \in \mathbb{K}[t, z] \setminus \{0\} \)
- such that \( R(t, F(t, 1)) = 0 \),
- having its partial degrees bounded by \( \delta^3 \) and total size \( \delta^6 \),
- in \( O_{\log}(\delta^{10.14}) \) arithmetic operations in \( \mathbb{K} \).
- \( O_{\log}(L\delta^6 + \delta^{3\theta+3}) \) ops. in \( \mathbb{K} \), where \( L = \text{cost for evaluating } P \) at \( (x, z, t, u) \in \mathbb{K}^4 \).
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log}(\delta^{14})$ arithmetic operations in $\mathbb{K}$. 
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

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Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]

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\begin{align*}
P, \delta, \\
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\text{(FPE)} & \\
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\[
\begin{align*}
P_{\epsilon}, \delta_{\epsilon} & = O(\delta), \\
\langle P_{\epsilon}, \partial_x P_{\epsilon}, \partial_u P_{\epsilon} \rangle \cap \mathbb{K}[t, \epsilon, z] & \\
\mathcal{J}_{\epsilon} \text{ ideal of } \mathbb{K}(t, \epsilon)[x, z, u] & \quad \text{radical, 0-dimensional} \\
\text{(FPE)} + \epsilon \sqrt{t} \Delta F & \quad \rightarrow \\
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- **Symbolic homotopy** [Bousquet-Mélou, Jehanne '06]
  \[ J_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u] \text{ radical, 0-dimensional} \]
- **“Stickelberger’s theorem”** [Stickelberger 1897], [Cox '20]
  \[ \text{take } R \text{ char. pol. of a linear map } m_z \text{ defined over } \mathbb{K}(t, \epsilon)[x, z, u]/J_\epsilon \]
- **Parametric geometric resolution** [Schost '03]
  \[ O_{\log}(L_\epsilon \delta_\epsilon^9) \text{ ops. in } \mathbb{K}, \text{ with } L_\epsilon = O(\delta L) \rightarrow z = \frac{V(t, \epsilon, \lambda)}{\frac{\partial}{\partial \lambda} W(t, \epsilon, \lambda)}, W(t, \epsilon, \lambda) = 0. \]
- **Bivariate resultants** [Villard '18], [Hyun, Neiger, Schost '19]
  \[ O_{\log}(\delta_\epsilon^{10.89}) \text{ ops. in } \mathbb{K} \rightarrow R = \text{Res}_\lambda(z - E(t, \epsilon, \lambda), W(t, \epsilon, \lambda)). \]
Conclusion and future works

Conclusion

- Refinement of an existing method based on discriminants,
- Design of a new guess-and-prove algorithm based on geometric bounds,
- A general complexity result for solving (FPE) of order 1.

Future works

- Improve the previous complexity estimates,
- Implement and compare the algorithms,
- Study the case of higher order equations.
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An Empirical Method for Solving (Rigorously!) Algebraic-Functional Equations of the Form $F(P(x,t), P(x,1), x, t) = 0$.
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Implementations of efficient univariate polynomial matrix algorithms and application to bivariate resultants.
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É. Schost.
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W. T. Tutte.
On the enumeration of planar maps.

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In *ISSAC 2018, 43rd International Symposium on Symbolic and Algebraic Computation*, New York, USA, July 16-19, 2018,
New York, United States, July 2018.

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A proof of Julian West’s conjecture that the number of two-stack-sortable permutations of length $n$ is $2(3n)!/((n+1)(2n+1)!)$.
Example where (H1) is not satisfied

Consider the functional equation

\[ F(t, u) = 1 + t((u - 1)F(t, u)^2 + F(t, u) - F(t, 1)). \]  

(1)

Here \( P = 1 - x + t((u - 1)x^2 + x - z) \).
Therefore, \( \partial_x P(x, z, 0, u) = 1 \), hence assumption (H1) is not satisfied.

Algorithm DD of page 7:

1. \( \text{disc}_x P = 4t^2uz - 4t^2z + t^2 - 4tu + 2t + 1 \),
2. \( \text{disc}_u(\text{disc}_x(P)) = 1 \).

The output is \( R = 1 \), which is obviously wrong.

In fact, the unique solution \( F(t, u) \) of (1) in \( \mathbb{Q}[u][[t]] \) satisfies \( F(t, 1) = 1 \), and is a root of \( R := t(u - 1)x^2 + (t - 1)x + 1 - t \).
### Recap

#### Generic case

<table>
<thead>
<tr>
<th>Page</th>
<th>Contribution</th>
<th>Hypothesis</th>
<th>Total size</th>
<th>Complexity</th>
<th>Relative exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>$\delta^8$</td>
<td>$O_{\log}(\delta^{10})$</td>
<td>$\frac{10}{8} = 1.25$</td>
</tr>
<tr>
<td>8</td>
<td>Geom</td>
<td>(H1), radical, 0-dim</td>
<td>$\delta^6$</td>
<td>$O_{\log}(L\delta^6 + \delta^{7.89})$</td>
<td>$\frac{10}{6} = 1.6$</td>
</tr>
<tr>
<td>10</td>
<td>G&amp;P</td>
<td>(H1), Jac$\neq 0$</td>
<td>$\delta^6$</td>
<td>$O_{\log}(L\delta^6 + \delta^{3\theta+3})$</td>
<td>$\frac{10.14}{6} = 1.69$</td>
</tr>
<tr>
<td>12</td>
<td>General</td>
<td>None</td>
<td>$\delta^6$</td>
<td>$O_{\log}(\delta^{14})$</td>
<td>$\frac{14}{6} \sim 2.33$</td>
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</tbody>
</table>

#### Sparse case

<table>
<thead>
<tr>
<th>Page</th>
<th>Contribution</th>
<th>Hypothesis</th>
<th>Total size</th>
<th>Complexity</th>
<th>Relative exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>$\delta^8$</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>8</td>
<td>Geom</td>
<td>(H1), radical, 0-dim</td>
<td>$\delta^6$</td>
<td>$O_{\log}(\delta^{7.89})$</td>
<td>$\frac{7.89}{6} = 1.315$</td>
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<tr>
<td>10</td>
<td>G&amp;P</td>
<td>(H1), Jac$\neq 0$</td>
<td>$\delta^6$</td>
<td>$O_{\log}(\delta^{3\theta+3})$</td>
<td>$\frac{\theta+1}{2} \sim 1.69 \rightarrow \frac{\theta}{2} \sim 1.19$</td>
</tr>
<tr>
<td>12</td>
<td>General</td>
<td>None</td>
<td>$\delta^6$</td>
<td>$O_{\log}(\delta^{10.89})$</td>
<td>$\frac{10.89}{6} \sim 1.815$</td>
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