Algorithms for solving fixed point equations of order 1

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Joint work with:
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Motivation: A non linear equation coming from combinatorics...

$K$ field of characteristic 0. \quad K = \mathbb{Q}, \mathbb{C}, \ldots $

**Starting point:** $F$, solution in $\mathbb{K}[u][[t]]$ of the fixed point equation (FPE) of order 1

$$F(t, u) = 1 + tu(uF(t, u)^2 + F(t, u) + \Delta F(t, u)),$$

where $\Delta$ is the divided difference operator $\Delta F(t, u) := \frac{F(t, u) - F(t, 1)}{u - 1}$. 


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where \( \Delta \) is the divided difference operator \( \Delta F(t, u) := \frac{F(t, u) - F(t, 1)}{u - 1} \).

Interest: Nature of \( F(t, 1) \).
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**Classical:** \( F \) and \( F(t, 1) \) are algebraic.
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**Interest:** Nature of \( F(t, 1) \).

**Classical:** \( F \) and \( F(t, 1) \) are algebraic.

**Goals:**

- Compute a polynomial \( R \in \mathbb{K}[t, z] \setminus \{0\} \) such that \( R(t, F(t, 1)) = 0. \)
- Estimate the size of \( R \) for any (FPE).
- Complexity estimates (ops. in \( \mathbb{K} \)) for the computation of \( R \).
...associated to planar maps enumeration

\[
\begin{align*}
\text{Count} \\
c_n &:= \# \{\text{planar maps with } n \text{ edges}\} \\
\downarrow \text{refinement} \\
c_{n,d} &:= \# \{\text{planar maps with } n \text{ edges,} \ \ d \text{ of them on the external face}\}
\end{align*}
\]
...associated to planar maps enumeration

Count

\[ c_n := \# \{ \text{planar maps with } n \text{ edges} \} \]
\[ \downarrow \text{refinement} \]
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Solution in \( \mathbb{K}[u][[t]] \)

\[ G(t) := \sum_{n=0}^{\infty} c_n t^n \quad \text{generating function} \]
\[ \downarrow \text{refinement} \]
\[ F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^{n} c_{n,d} u^d t^n \quad \text{complete generating function} \]
...associated to planar maps enumeration

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Solution in \( K[u][[t]] \)

\[ G(t) := \sum_{n=0}^{\infty} c_n t^n \] generating function
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(FPE) of order 1 [Tutte '68]

\[ F(t, u) = 1 + tu^2 F(t, u)^2 \]
\[ + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \]
Deletion-contraction of edges

(FPE) of order 1 [Tutte '68]

\[ F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \]
Theorem [Bousquet-Mélou, Jehanne ’06]

Let $f \in K[u]$ and $Q \in K[x, y, t, u]$. Let $F(t, u)$ be the unique solution in $K[u][[t]]$ of

$$F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u),$$

where $\Delta$ is the divided difference operator $\Delta F := \frac{F(t,u)-F(t,1)}{u-1}$.

Then $F$ is algebraic over $K(t, u)$. 
Theorem [Bousquet-Mélou, Jehanne ’06]
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Then $F$ is algebraic over $\mathbb{K}(t, u)$.

Systematic methods

[Tutte, Brown 60’s], [Zeilberger ’92]:
Guess-and-prove

[Brown ’65]:
Quadratic method

[Popescu ’86]:
Algebraicity result

[Bousquet-Mélou, Jehanne ’06]:
Polynomial elimination
**State of the art**

**Theorem [Bousquet-Mélo, Jehanne ’06]**
Let \( f \in \mathbb{K}[u] \) and \( Q \in \mathbb{K}[x, y, t, u] \). Let \( F(t, u) \) be the unique solution in \( \mathbb{K}[u][[t]] \) of
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F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u),
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where \( \Delta \) is the divided difference operator \( \Delta F := \frac{F(t, u) - F(t, 1)}{u-1} \).

Then \( F \) is **algebraic** over \( \mathbb{K}(t, u) \).

**Systematic methods**

- [Tutte, Brown 60’s], [Zeilberger ’92]: Guess-and-prove
- [Brown ’65]: Quadratic method
- [Popescu ’86]: Algebraicity result
- [Bousquet-Mélo, Jehanne ’06]: Polynomial elimination

**Algorithms**

- [Knuth ’68], [Banderier, Flajolet ’02]: Kernel method (linear case)
- [Bousquet-Mélo, Jehanne ’06]: Polynomial elimination
- [Gessel, Zeilberger ’14]: Guess-and-prove
[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE)

\[
\begin{align*}
\text{(FPE) of order 1} & \quad \rightarrow \quad \text{solution } u = U(t) \in \mathbb{K}[[t]] \text{ of } \\
\partial_x P(F(t, u), F(t, 1), t, u) &= 0 & \rightarrow \quad \begin{cases} 
P(x, z, t, u) = 0, \\
\partial_x P(x, z, t, u) = 0, \\
\partial_u P(x, z, t, u) = 0.
\end{cases} \\
\end{align*}
\]

\( \text{(FPE)} \)
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take \( P \in \mathbb{K}[x, z, t, u] \) the “numerator” of (FPE) of order 1 → solution \( u = U(t) \in \mathbb{K}[[t]] \) of

\[
\frac{\partial x}{\partial} P(F(t, u), F(t, 1), t, u) = 0
\]

\[
\begin{align*}
P(x, z, t, u) &= 0, \\
\frac{\partial x}{\partial} P(x, z, t, u) &= 0, \\
\frac{\partial u}{\partial} P(x, z, t, u) &= 0. \\
\end{align*}
\]

(\( F(t, U(t)), F(t, 1), U(t) \)) zero in \( \mathbb{K}[[t]]^3 \)

---

Fixed Point Equation (FPE)

\[
\downarrow \text{numer}
\]

\[
P(F(t, u), F(t, 1), t, u) = 0
\]

\[
\downarrow \frac{\partial u}{\partial}
\]

\[
\frac{\partial u}{\partial} F(t, u) \cdot \frac{\partial x}{\partial} P(F(t, u), F(t, 1), t, u) + \frac{\partial u}{\partial} P(F(t, u), F(t, 1), t, u) = 0
\]
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE) of order 1 → solution $u = U(t) \in \mathbb{K}[[t]]$ of
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\begin{align*}
    \partial_x P(F(t, u), F(t, 1), t, u) &= 0 \\
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(F(t, U(t)), F(t, 1), U(t)) zero in $\mathbb{K}[[t]]^3$

Planar maps
\[
F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1}
\]
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne ’06]: Take $P \in \mathbb{K}[x, z, t, u]$ the “numerator” of (FPE)

\[(FPE) \text{ of order 1} \rightarrow \text{solution } u = U(t) \in \mathbb{K}[[t]] \text{ of}
\]
\[
\frac{\partial_x P(F(t, u), F(t, 1), t, u)}{\partial_y P(F(t, u), F(t, 1), t, u)} = 0
\]

\[
\left\{
\begin{array}{l}
P(x, z, t, u) = 0, \\
\partial_x P(x, z, t, u) = 0, \\
\partial_u P(x, z, t, u) = 0.
\end{array}
\right.
\]

\[(F(t, U(t)), F(t, 1), U(t)) \text{ zero in } \mathbb{K}[[t]]^3\]

Fixed Point Equation (FPE)

\[
\downarrow \text{numer}
\]

\[
P(F(t, u), F(t, 1), t, u) = 0
\]

\[
\downarrow \partial_u
\]

\[
\partial_u F(t, u) \cdot \partial_x P(F(t, u), F(t, 1), t, u) + \partial_u P(F(t, u), F(t, 1), t, u) = 0
\]

Planar maps

\[
F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1}
\]

\[
0 = (1 - F(t, u))(u - 1) + tu^2(u - 1)F(t, u)^2 + tu(uF(t, u) - F(t, 1))
\]
[Bousquet-Mélou, Jehanne '06]: Take \( P \in \mathbb{K}[x, z, t, u] \) the "numerator" of (FPE)

\[ P(x, z, t, u) = 0, \]
\[ \partial_x P(x, z, t, u) = 0, \]
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\((F(t, U(t)), F(t, 1), U(t))\) zero in \( \mathbb{K}[[t]]^3 \)

P(F(t, u), F(t, 1), t, u) = 0

\( \partial_x P(F(t, u), F(t, 1), t, u) = 0 \)

\( \partial_u P(F(t, u), F(t, 1), t, u) = 0 \)

Planar maps

\( F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \) (FPE)

\[ 0 = (1 - F(t, u))(u - 1) + tu^2 (u - 1)F(t, u)^2 \]
\[ + tu(uF(t, u) - F(t, 1)) \]

\[ 0 = \partial_u F(t, u) \cdot (1 - u + 2tu^2(u - 1)F(t, u)^2 + tu^2) \]
\[ + (1 - F(t, u) + tu(3u - 2)F(t, u)^2 + 2tu^2 F(t, u) - tF(t, 1)) \]
Our contributions

Work based on [Bousquet-Mélou, Jehanne ’06]

1. Geometric refinements of a method based on discriminants,
2. A new guess-and-prove method based on geometry,
3. A complexity result on the resolution of (FPE) of order 1.

Attention is paid to

• assumptions,
• degree bounds on the output,
• complexity estimates,
• potential for generalization.

Input: $P := \text{numerator}(\text{FPE}),$
Goal: $\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z].$
Algebraic elimination via iterated discriminants

\[ \text{disc}_x(P) = \text{Res}_x(P, \partial_x P) \] the discriminant of \( P \) in \( x \).

**Theorem [Bousquet-Mélou, Jehanne ’06]**

Suppose \( \deg_x(P) \geq 2 \) and \( u = U(t) \in \mathbb{K}[[t]] \) is a root of

\[ \partial_x P(F(t, u), F(t, 1), t, u). \]

Then \( u = U(t) \) is a **double root** of \( \text{disc}_x(P)(F(t, 1), t, u) \).

Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).
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Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).

\[
P := (1 - x)(u - 1) + tu^2(u - 1)x^2 \\
+ tu(ux - z)
\]

gives \( \text{disc}_u(\text{disc}_x(P)) \) equal to

\[
-256t^4 \cdot (27t^2z^2 - 18tz + 16t + z - 1) \\
\cdot (tz - 1)^2
\]
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\]

\[
P := 97t^3u^2 + (-73u^4 - 56u^2x^2 + 87u^2x - 62x^2 + 124xz - 62z^2)t - xu^2 + u^2
\]

gives \( \text{disc}_x P \) equal to
\[
-16352t^2u^6 \\
+ \left( 21728t^4 - 10535t^2 + 50t + 1 \right) u^4 \\
+ 248t \left( 97t^3 - 56tz^2 + 87tz - z + 1 \right) u^2
\]

which has a double root at \( u = 0 \).
Contribution 1: ensuring non-nullity of double discriminant

Theorem [Bostan, Chyzak, N., Safey El Din ’22]

Suppose

- \( (H_0) \) \( \deg_x(P) \geq 2 \),
- \( (H_1) \) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- \( (R) \) the zero set \( V(P) \subset \mathbb{K}^4 \) is smooth outside \( V(u - 1) \subset \mathbb{K}^4 \).

Set \( D_0 := \text{disc}_x P, D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1 \).

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
Contribution 1: ensuring non-nullity of double discriminant

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Suppose

- (H0) \( \deg_x(P) \geq 2 \),
- (H1) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in K \),
- (R) the zero set \( V(P) \subset \overline{K}^4 \) is smooth outside \( V(u - 1) \subset \overline{K}^4 \).

Set \( D_0 := \text{disc}_x P \), \( D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1 \).

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
- \( R \) has total size \( 16\delta^8 \) with degree in each variable at most \( 4\delta^4 \),
- \( R \) can be computed in \( O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \).
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- \( \text{(H0)} \) \( \deg_x(P) \geq 2 \),
- \( \text{(H1)} \) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- \( \text{(R)} \) the zero set \( V(P) \subset \overline{\mathbb{K}}^4 \) is smooth outside \( V(u - 1) \subset \overline{\mathbb{K}}^4 \).

Set \( D_0 := \text{disc}_x P, D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1 \).

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
- \( R \) has total size \( 16\delta^8 \) with degree in each variable at most \( 4\delta^4 \),
- \( R \) can be computed in \( O_{\text{log}}(\delta^{10}) \) ops. in \( \mathbb{K} \).

\[ \delta = \deg(P) \]

\[ D_1 := \text{SqFreePart}(\text{disc}_x(P)) \text{ satisfies} \]

\[ \partial_u D_1(U(t), F(t, 1), t) = 0. \]

\[ \left\{ \begin{align*}
(\partial_u D_1 \ & \partial_z D_1 \ & \partial_t D_1) \cdot (u \ z \ t)^T = 0, \\
(\partial_z D_1 \ & \partial_t D_1) \cdot (z \ t)^T &= 0
\end{align*} \right. \]
Contribution 1 (cont’d): using geometry arguments to refine the complexity

\( P \in \mathbb{K}[x, z, t, u] \) and \( \delta := \text{deg}(P) \).

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

Suppose

- (H1) \( \text{deg}_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- \( \langle P, \partial_x P, \partial_u P \rangle : (u - 1)\infty \subset \mathbb{K}(t)[x, z, u] \) is radical and 0-dimensional over \( \mathbb{K}(t) \).

Then one can compute \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F(t, 1) \)

- with degree in each variable at most \( \delta^3 \) and total size \( \delta^6 \),
- in \( O_{\log}(L\delta^6 + \delta^{7.89}) \subset O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \),

where \( L = \text{cost of evaluating } P \) at \( (x, z, t, u) \in \mathbb{K}^4 \).
**Contribution 2: Guess-and-prove based on geometry**

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P) \).

**Output:** \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1) \), i.e. \( R(t, F_1) = 0 \).
**Contribution 2: Guess-and-prove based on geometry**

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P). \)

**Output:** \( R \in K[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1), \) i.e. \( R(t, F_1) = 0. \)

**geometry**

(1) Functional equation

\[ \downarrow \]

(2) Polynomial system

\[ \downarrow \]

(3) Bounds
- \( \deg_t(R) \leq \beta_t, \)
- \( \deg_z(R) \leq \beta_z. \)
**Contribution 2: Guess-and-prove based on geometry**

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \text{deg}(P) \).

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**geometry**

(1) Functional equation

(2) Polynomial system

(3) Bounds
   - \( \text{deg}_t(R) \leq b_t \)
   - \( \text{deg}_z(R) \leq b_z \)

**guess-and-prove**

(4) Expand \( F_1 \)

(5) Compute \( R \in \mathbb{K}[t, z] \) s.t. \( R(t, F_1) = O(t^{b_t b_z}) \)

(6) Certify that \( R(t, F_1) = 0 \)
Contribution 2: Guess-and-prove based on geometry

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \text{deg}(P). \)

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---

**guess-and-prove**

(4) Expand \( F_1 \)

(5) Compute \( R \in K[t, z] \) s.t.
\[ R(t, F_1) = O(t^{b_t b_z}) \]

(6) Certify that \( R(t, F_1) = 0 \)

---

**tools**

- Newton iteration
- Algebraic approximants “seriestoalgeq”
- Multiplicity lemma: \( R(t, F_1) = O(t^{2 b_t b_z}) \) implies \( R(t, F_1) = 0 \)
Theorem [Bostan, Chyzak, N., Safey El Din ’22]

Define $A_u := (F(t, u), F(t, 1), u)$ and assume that

- there exists $u = U(t) \in \mathbb{K}[t] \setminus \{1\}$ solution of $\partial_x P(F(t, u), F(t, 1), t, u) = 0$,
- the Jacobian of $(P, \partial_x P, \partial_u P)$ w.r.t. $\{x, z, u\}$ is invertible at $A_{U(t)} \in \mathbb{K}[t]^3$.

Then the geometry-driven guess-and-prove computes $R \in \mathbb{K}[t, z] \setminus \{0\}$

- such that $R(t, F(t, 1)) = 0$,
- having its partial degrees bounded by $\delta^3$ and total size $\delta^6$,
- in $O_{\log}(\delta^{10.14})$ arithmetic operations in $\mathbb{K}$.
Complexity result / degree bounds for geometry-driven guess-and-prove

\[ \theta \in [2, 3] \text{ a feasible exponent of matrix multiplication} \]

**Theorem [Bostan, Chyzak, N., Safey El Din '22]**

Define \( A_u := (F(t, u), F(t, 1), u) \) and assume that
- there exists \( u = U(t) \in \mathbb{K}[[t]] \setminus \{1\} \) solution of \( \partial_x P(F(t, u), F(t, 1), t, u) = 0 \),
- the Jacobian of \( (P, \partial_x P, \partial_u P) \) w.r.t \( \{x, z, u\} \) is invertible at \( A_{U(t)} \in \mathbb{K}[[t]]^3 \).

Then the geometry-driven guess-and-prove computes \( R \in \mathbb{K}[t, z] \setminus \{0\} \)
- such that \( R(t, F(t, 1)) = 0 \),
- having its partial degrees bounded by \( \delta^3 \) and total size \( \delta^6 \),
- in \( O_{\log}(\delta^{10.14}) \) arithmetic operations in \( \mathbb{K} \).
- \( O_{\log}(L\delta^6 + \delta^{3\theta+3}) \) ops. in \( \mathbb{K} \), where \( L = \text{cost for evaluating } P \) at \( (x, z, t, u) \in \mathbb{K}^4 \).
Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log(\delta^{14})}$ arithmetic operations in $\mathbb{K}$. 
Contribution 3: a polynomial time complexity for solving a \((\text{FPE})\) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists \(R \in \mathbb{K}[t, z] \setminus \{0\}\) annihilating \(F(t, 1)\) of total arithmetic size \(\delta^6\). Moreover, one can compute \(R\) in \(O_{\log(\delta^{14})}\) arithmetic operations in \(\mathbb{K}\).

Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]

\[
P, \delta, \quad \langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z] \quad \text{ideal of } \mathbb{K}(t)[x, z, u]
\]

\[
\rightarrow \quad P_\epsilon, \delta_\epsilon = O(\delta), \quad \langle P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon \rangle \cap \mathbb{K}[t, \epsilon, z] \quad \text{radical, 0-dimensional}
\]
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F(t, 1) \) of total arithmetic size \( \delta^6 \). Moreover, one can compute \( R \) in \( O_{\log} (\delta^{14}) \) arithmetic operations in \( \mathbb{K} \).

Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]

\[
\begin{align*}
P, \delta, \\
\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z] \\
\mathcal{J} \text{ ideal of } \mathbb{K}(t)[x, z, u]
\end{align*}
\]

\[
\begin{align*}
P_{\epsilon}, \delta_{\epsilon} = O(\delta), \\
\langle P_{\epsilon}, \partial_x P_{\epsilon}, \partial_u P_{\epsilon} \rangle \cap \mathbb{K}[t, \epsilon, z] \\
\mathcal{J}_{\epsilon} \text{ ideal of } \mathbb{K}(t, \epsilon)[x, z, u] \\
\text{radical, 0-dimensional}
\end{align*}
\]

\[
\begin{align*}
(FPE) \\
\rightarrow \\
(FPE) + \epsilon \sqrt{t} \Delta F
\end{align*}
\]
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log}(\delta^{14})$ arithmetic operations in $\mathbb{K}$.

Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]
  \[ \mathcal{J}_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u] \text{ radical, 0-dimensional} \]
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

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Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]
  
  $\rightarrow J_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u]$ radical, 0-dimensional

- “Stickelberger’s theorem” [Stickelberger 1897], [Cox ’20]
  
  $\rightarrow$ take $R$ char. pol. of a linear map $m_z$ defined over $\mathbb{K}(t, \epsilon)[x, z, u]/J_\epsilon$

- Parametric geometric resolution [Schost ’03]
  
  $O_{\log}(L_\epsilon \delta^9_\epsilon)$ ops. in $\mathbb{K}$, with $L_\epsilon = O(\delta L) \rightarrow z = \frac{V(t, \epsilon, \lambda)}{\partial_\lambda W(t, \epsilon, \lambda)}$, $W(t, \epsilon, \lambda) = 0$.

- Bivariate resultants [Villard ’18], [Hyun, Neiger, Schost ’19]
  
  $O_{\log}(\delta^{10.89}_\epsilon)$ ops. in $\mathbb{K} \rightarrow R = \text{Res}_\lambda (z - E(t, \epsilon, \lambda), W(t, \epsilon, \lambda))$. 
Conclusion

- Refinement of an existing method based on discriminants,
- Design of a new guess-and-prove algorithm based on geometric bounds,
- A general complexity result for solving (FPE) of order 1.

Future works

- Improve the previous complexity estimates,
- Implement and compare the algorithms,
- Study the case of higher order equations.
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Example where (H1) is not satisfied

Consider the functional equation

\[ F(t, u) = 1 + t((u - 1)F(t, u)^2 + F(t, u) - F(t, 1)) . \] (1)

Here \( P = 1 - x + t((u - 1)x^2 + x - z) \).

Therefore, \( \partial_x P(x, z, 0, u) = 1 \), hence assumption (H1) is not satisfied.

Algorithm DD of page 8:

1. \( \text{disc}_x P = 4t^2 uz - 4t^2 z + t^2 - 4tu + 2t + 1 \),
2. \( \text{disc}_u (\text{disc}_x (P)) = 1 \).

The output is \( R = 1 \), which is obviously wrong.

In fact, the unique solution \( F(t, u) \) of (1) in \( \mathbb{Q}[u][[t]] \) satisfies \( F(t, 1) = 1 \), and is a root of \( R := t(u - 1)x^2 + (t - 1)x + 1 - t \).
### Recap

#### Generic case

<table>
<thead>
<tr>
<th>Page</th>
<th>Contribution</th>
<th>Hypothesis</th>
<th>Total size</th>
<th>Complexity</th>
<th>Relative exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>$\delta^8$</td>
<td>$O(\log(\delta^{10}))$</td>
<td>$\frac{10}{8} = 1.25$</td>
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<td>9</td>
<td>Geom</td>
<td>(H1), radical, 0-dim</td>
<td>$\delta^6$</td>
<td>$O(\log(L\delta^6 + \delta^{7.89}))$</td>
<td>$\frac{10}{6} = 1.6$</td>
</tr>
<tr>
<td>11</td>
<td>G&amp;P</td>
<td>(H1), Jac $\neq 0$</td>
<td>$\delta^6$</td>
<td>$O(\log(L\delta^6 + \delta^{3\theta+3}))$</td>
<td>$\frac{10.14}{6} = 1.69$</td>
</tr>
<tr>
<td>13</td>
<td>General</td>
<td>None</td>
<td>$\delta^6$</td>
<td>$O(\log(\delta^{14}))$</td>
<td>$\frac{14}{6} \sim 2.33$</td>
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#### Sparse case

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<th>Page</th>
<th>Contribution</th>
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<tr>
<td>8</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>$\delta^8$</td>
<td>?</td>
<td>?</td>
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<tr>
<td>9</td>
<td>Geom</td>
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<td>$\delta^6$</td>
<td>$O(\log(\delta^{7.89}))$</td>
<td>$\frac{7.89}{6} = 1.315$</td>
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<td>$\delta^6$</td>
<td>$O(\log(\delta^{3\theta+3}))$</td>
<td>$\frac{\theta+1}{2} \sim 1.69 \rightarrow \frac{\theta}{2} \sim 1.19$</td>
</tr>
<tr>
<td>13</td>
<td>General</td>
<td>None</td>
<td>$\delta^6$</td>
<td>$O(\log(\delta^{10.89}))$</td>
<td>$\frac{10.89}{6} \sim 1.815$</td>
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