Algorithms for solving fixed point equations of order 1

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Motivation: A non-linear equation coming from combinatorics...

\[ \mathbb{K} \text{ effective field of characteristic 0.} \quad \mathbb{K} = \mathbb{Q}, \mathbb{Q}(y), \ldots \]

**Starting point:** \( F \), solution in \( \mathbb{K}[u][[t]] \) of the fixed point equation (FPE) of order 1

\[
F(t, u) = 1 + tu\left(uF(t, u)^2 + F(t, u) + \Delta F(t, u)\right),
\]

where \( \Delta \) is the divided difference operator \( \Delta F(t, u) := \frac{F(t, u) - F(t, 1)}{u - 1} \).
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**Interest:** Nature of \( F(t, 1) \).

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**Goals:**
- Compute a polynomial \( R \in \mathbb{K}[t, z] \setminus \{0\} \) such that \( R(t, F(t, 1)) = 0 \).
- Estimate the size of \( R \) for any (FPE).
- Complexity estimates (ops. in \( \mathbb{K} \)) for the computation of \( R \).
...associated to planar maps enumeration

**Count**

\[ c_n := \# \{ \text{planar maps with } n \text{ edges} \} \]

↓ refinement

\[ c_{n,d} := \# \{ \text{planar maps with } n \text{ edges, } d \text{ of them on the external face} \} \]
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Solution in \( \mathbb{K}[u][[t]] \)

\[ G(t) := \sum_{n=0}^{\infty} c_n t^n \quad \text{generating function} \]

\[ \downarrow \text{refinement} \]

\[ F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^{n} c_{n,d} u^d t^n \quad \text{complete generating function} \]
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(FPE) of order 1 [Tutte '68]

\[ F(t, u) = 1 + t u^2 F(t, u)^2 \]
\[ + t u \frac{u F(t, u) - F(t, 1)}{u - 1} \]
Deletion-contraction of edges

(FPE) of order 1 [Tutte ’68]

\[ F(t, u) = 1 + tu^2 F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \]
Theorem [Bousquet-Mélou, Jehanne ’06] see also [Popescu ’86]
Let \( f \in \mathbb{K}[u] \) and \( Q \in \mathbb{K}[x, y, t, u] \). Let \( F(t, u) \) be the unique solution in \( \mathbb{K}[u][[t]] \) of
\[
F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u),
\]
where \( \Delta \) is the divided difference operator \( \Delta F := \frac{F(t,u)-F(t,1)}{u-1} \).

Then \( F \) is algebraic over \( \mathbb{K}(t, u) \).
State of the art

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Then \( F \) is **algebraic** over \( \mathbb{K}(t, u) \).

**References**

[Tutte, Brown 60’s], [Zeilberger ’92]: Guess-and-prove

[Gessel, Zeilberger ’14]: Guess-and-prove

[Brown ’65]: Quadratic method

[Knuth ’68], [Banderier, Flajolet ’02]: Kernel method (linear case)

[Bousquet-Mélou, Jehanne ’06]: Polynomial elimination
Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne ’06]:

Fixed Point Equation (FPE)

\[ P(F(t, u), F(t, 1), t, u) = 0 \]

\[ \partial_u P(F(t, u), F(t, 1), t, u) = 0 \]
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Example: planar maps

\[ F(t, u) = 1 + tu^2F(t, u)^2 + tu \frac{uF(t, u) - F(t, 1)}{u - 1} \]

\[ 0 = (1 - F(t, u))(u - 1) + tu^2(u - 1)F(t, u)^2 \]

\[ + tu(uF(t, u) - F(t, 1)) \]

\[ 0 = \partial_u F(t, u) \cdot (1 - u + 2tu^2(u - 1)F(t, u)^2 + tu^2) \]

\[ + (1 - F(t, u) + tu(3u - 2)F(t, u)^2 + 2tu^2F(t, u) - tF(t, 1)) \]
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\[ \partial_u F(t, u) \cdot \partial_x P(F(t, u), F(t, 1), t, u) + \partial_u P(F(t, u), F(t, 1), t, u) = 0 \]

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solution \( u = U(t) \in \mathbb{K}[[t]] \) of

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\[ + (1 - F(t, u) + tu (3u - 2) F(t, u)^2 + 2tu^2 F(t, u) - t F(t, 1)) \]

solution \( u = U(t) \in \mathbb{K}[[t]] \) of

\( \partial_x P(F(t, u), F(t, 1), t, u) = 0 \)

\[ \begin{cases} P(x, z, t, u) = 0, \\ \partial_x P(x, z, t, u) = 0, \\ \partial_u P(x, z, t, u) = 0. \end{cases} \]

\( (F(t, U(t)), F(t, 1), U(t)) \) zero in \( \mathbb{K}[[t]]^3 \)
Our contributions

Inspired by [Bousquet-Mélou, Jehanne '06]

1. Geometric refinements of a method based on discriminants,
2. A new guess-and-prove method based on geometry,
3. A complexity result on the resolution of (FPE) of order 1.

Attention is paid to

• assumptions,
• degree bounds on the output,
• complexity estimates,
• potential for generalization.

Input: \( P := \text{numerator}(\text{FPE}) \),
Goal: \( \langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z] \).
Theorem [Bousquet-Mélou, Jehanne ’06]

Suppose \( \deg_x(P) \geq 2 \) and \( u = U(t) \in \mathbb{K}[[t]] \) is a root of

\[
\partial_x P(F(t, u), F(t, 1), t, u).
\]

Then \( u = U(t) \) is a double root of \( \text{disc}_x(P)(F(t, 1), t, u) \).

Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).
Algebraic elimination via iterated discriminants

\[ \text{disc}_x(P) = \text{Res}_x(P, \partial_x P) \text{ the discriminant of } P \text{ in } x. \]

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Hence, \( F(t, 1) \) is a root of \( \text{disc}_u(\text{disc}_x(P)) \).

\[
P := (1 - x)(u - 1) + tu^2(u - 1)x^2 \\
+ tu(ux - z)
\]

gives \( \text{disc}_u(\text{disc}_x(P)) \) equal to

\[
-256t^4 \cdot (27t^2z^2 - 18tz + 16t + z - 1) \\
\cdot (tz - 1)^2
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Algebraic elimination via iterated discriminants

\[ \text{disc}_x(P) = \text{Res}_x(P, \partial_x P) \] the discriminant of \( P \) in \( x \).

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\]

\[
P := 97t^3 u^2 + (-73u^4 - 56u^2x^2 + 87u^2x - 62x^2 + 124xz - 62z^2)t - xu^2 + u^2
\]
gives \( \text{disc}_x P \) equal to

\[
-16352t^2u^6 + \left( 21728t^4 - 10535t^2 + 50t + 1 \right) u^4 \\
+ 248t \left( 97t^3 - 56tz^2 + 87tz - z + 1 \right) u^2
\]

which has a double root at \( u = 0 \).
Contribution 1: ensuring non-nullity of double discriminant

\textbf{Theorem [Bostan, Chyzak, N., Safey El Din ’22]}

Suppose

\begin{itemize}
  \item (H0) $\deg_x(P) \geq 2$,
  \item (H1) $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
  \item (R) the zero set $V(P) \subset \overline{\mathbb{K}}^4$ is smooth outside $V(u - 1) \subset \overline{\mathbb{K}}^4$.
\end{itemize}

Set $D_0 := \text{disc}_x P$, $D_1 := \text{SqFreePart}(D_0)$ and $D_2 := \text{disc}_u D_1$.

Then

\begin{itemize}
  \item $R := \text{SqFreePart}(D_2)$ is non-zero in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.
\end{itemize}
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Suppose

- \( (\text{H0}) \) \( \deg_x(P) \geq 2 \),
- \( (\text{H1}) \) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- \( (\text{R}) \) the zero set \( V(P) \subset \overline{\mathbb{K}}^4 \) is smooth outside \( V(u - 1) \subset \overline{\mathbb{K}}^4 \).

Set \( D_0 := \text{disc}_x P, \ D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1 \).

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
- \( R \) has total size \( 16\delta^8 \) with degree in each variable at most \( 4\delta^4 \),
- \( R \) can be computed in \( O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \).
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Theorem [Bostan, Chyzak, N., Safey El Din ’22]
Suppose

- \((H0)\) \(\deg_x(P) \geq 2\),
- \((H1)\) \(\deg_u(\partial_x P(x, z, 0, u)) \geq 1\) and \(\partial_x P(F(t, c), F(t, 1), t, c) \neq 0\) for all \(c \in \mathbb{K}\),
- \((R)\) the zero set \(V(P) \subset \overline{\mathbb{K}}^4\) is smooth outside \(V(u - 1) \subset \overline{\mathbb{K}}^4\).

Set \(D_0 := \text{disc}_x P\), \(D_1 := \text{SqFreePart}(D_0)\) and \(D_2 := \text{disc}_u D_1\).

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- \(R\) has total size \(16\delta^8\) with degree in each variable at most \(4\delta^4\),
- \(R\) can be computed in \(O_{\log}(\delta^{10})\) ops. in \(\mathbb{K}\).

\(D_1 := \text{SqFreePart}(\text{disc}_x(P))\) satisfies
\[
\partial_u D_1(U(t), F(t, 1), t) = 0.
\]
\[
\begin{cases} 
(\partial_u D_1 \quad \partial_z D_1 \quad \partial_t D_1) \cdot (u \ z \ t)^T = 0, \\
(\partial_z D_1 \quad \partial_t D_1) \cdot (z \ t)^T = 0
\end{cases}
\]
Contribution 1 (cont’d): using geometry arguments to refine the complexity

\( P \in \mathbb{K}[x, z, t, u] \) and \( \delta := \deg(P) \).

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

Suppose

- (H1) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- \( \langle P, \partial_x P, \partial_u P \rangle : (u - 1)^\infty \subset \mathbb{K}(t)[x, z, u] \) is radical and 0-dimensional over \( \mathbb{K}(t) \).

Then one can compute \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F(t, 1) \)

- with degree in each variable at most \( \delta^3 \) and total size \( \delta^6 \),
- in \( O_{\log}(L\delta^6 + \delta^{7.89}) \subset O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \),
  where \( L = \text{cost of evaluating } P \text{ at } (x, z, t, u) \in \mathbb{K}^4 \).

- Geometric resolution: [Giusti, Lecerf, Salvy ’01]

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9/14
Contribution 2: Guess-and-prove based on geometry

Input: \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \text{deg}(P). \)

Output: \( R \in \mathbb{K}[t, z] \setminus \{0\} \) annihilating \( F_1 = F(t, 1) \), i.e. \( R(t, F_1) = 0. \)
Contribution 2: Guess-and-prove based on geometry

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P). \)

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**geometry**

1. **Functional equation**

2. **Polynomial system**

3. **Bounds**
   - \( \deg_t(R) \leq b_t, \)
   - \( \deg_z(R) \leq b_z. \)
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**guess-and-prove**

1. Expand \( F_1 \)
2. Compute \( R \in \mathbb{K}[t, z] \) s.t. \( R(t, F_1) = O(t^{b_t b_z}) \)
3. Certify that \( R(t, F_1) = 0 \)

**geometry**

1. Functional equation
2. Polynomial system
3. Bounds
   - \( \deg_t(R) \leq b_t \)
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Contribution 2: Guess-and-prove based on geometry

**Input:** \( P(F(t, u), F(t, 1), t, u) = 0, \delta := \deg(P). \)

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**geometry**

1. Functional equation
   - (1) Functional equation

2. Polynomial system
   - (2) Polynomial system

3. Bounds
   - (3) Bounds
     - \( \deg_t(R) \leq b_t, \)
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**guess-and-prove**

4. Expand \( F_1 \)
   - (4) Expand \( F_1 \)

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     - \( R(t, F_1) = O(t^{\sim b_t b_z}) \)

6. Certify that \( R(t, F_1) = 0 \)
   - (6) Certify that \( R(t, F_1) = 0 \)

**tools**

- Newton iteration
- Algebraic approximants “seriestoalgeq”
- Multiplicity lemma:
  - \( R(t, F_1) = O(t^{\sim 2b_t b_z}) \)
  - implies \( R(t, F_1) = 0 \)
\( \theta \in [2, 3] \) a feasible exponent of matrix multiplication

**Theorem [Bostan, Chyzak, N., Safey El Din '22]**

Define \( A_u := (F(t, u), F(t, 1), u) \) and assume that

- there exists \( u = U(t) \in \mathbb{K}[[t]] \setminus \{1\} \) solution of \( \partial_x P(F(t, u), F(t, 1), t, u) = 0 \),
- the Jacobian of \( (P, \partial_x P, \partial_u P) \) w.r.t \( \{x, z, u\} \) is invertible at \( A_{U(t)} \in \mathbb{K}[[t]]^3 \).

Then, the geometry-driven guess-and-prove computes \( R \in \mathbb{K}[t, z] \setminus \{0\} \)

- such that \( R(t, F(t, 1)) = 0 \),
- having its partial degrees bounded by \( \delta^3 \) and total size \( \delta^6 \),
- in \( O_{\log}(\delta^{10.14}) \) arithmetic operations in \( \mathbb{K} \).
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- such that $R(t, F(t, 1)) = 0$,
- having its partial degrees bounded by $\delta^3$ and total size $\delta^6$,
- in $O\log(\delta^{10.14})$ arithmetic operations in $\mathbb{K}$,
- $O\log(\delta^6 + \delta^{3\theta+3})$ ops. in $\mathbb{K}$, where $L = \text{cost for evaluating } P \text{ at } (x, z, t, u) \in \mathbb{K}^4$. 

$\theta \in [2, 3]$ a feasible exponent of matrix multiplication
Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log}(\delta^{14})$ arithmetic operations in $\mathbb{K}$. 
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

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Sketch of the proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]
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Sketch of the proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]

\[ P, \delta, \quad \langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z] \]
\[ J \text{ ideal of } \mathbb{K}(t)[x, z, u] \]
\[ (\text{FPE}) \]
\[ \rightarrow \]
\[ P_\epsilon, \delta_\epsilon = O(\delta), \quad \langle P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon \rangle \cap \mathbb{K}[t, \epsilon, z] \]
\[ J_\epsilon \text{ ideal of } \mathbb{K}(t, \epsilon)[x, z, u] \]
\[ \text{radical, 0-dimensional} \]
\[ \rightarrow \]
\[ (\text{FPE}) + \epsilon \sqrt{t} \Delta F \]
Theorem [Bostan, Chyzak, N., Safey El Din ’22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log}(\delta^{14})$ arithmetic operations in $\mathbb{K}$.

Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne ’06]
  \[\mathcal{J}_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u]\] radical, 0-dimensional
Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

There exists $R \in K[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size $\delta^6$. Moreover, one can compute $R$ in $O_{\log}(\delta^{14})$ arithmetic operations in $K$.

**Sketch of proof:**

- **Symbolic homotopy** [Bousquet-Mélou, Jehanne ’06]
  \[ J_\epsilon \subset K(t, \epsilon)[x, z, u] \text{ radical, 0-dimensional} \]

- **“Stickelberger’s theorem”** [Stickelberger 1897], [Cox ’20]
  \[ \text{take } R \text{ char. pol. of a linear map } m_z \text{ defined over } K(t, \epsilon)[x, z, u]/J_\epsilon \]

- **Parametric geometric resolution** [Schost ’03]
  \[ O_{\log}(L_\epsilon \delta^9_\epsilon) \text{ ops. in } K, \text{ with } L_\epsilon = O(\delta L) \rightarrow z = \frac{V(t, \epsilon, \lambda)}{\partial_\lambda W(t, \epsilon, \lambda)}, W(t, \epsilon, \lambda) = 0. \]

- **Bivariate resultants** [Villard ’18], [Hyun, Neiger, Schost ’19]
  \[ O_{\log}(\delta^{10.89}_\epsilon) \text{ ops. in } K \rightarrow R = \text{Res}_\lambda(z - E(t, \epsilon, \lambda), W(t, \epsilon, \lambda)). \]
Conclusion and future works

Conclusion

- Refinement of an existing method based on discriminants
- Design of a new guess-and-prove algorithm based on geometric bounds
- A general complexity result for solving (FPE) of order 1

Future works

- Improve the previous complexity estimates
- Implement and compare the algorithms
- Study the case of higher order equations
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Contribution 1: ensuring non-nullity of double discriminant

**Theorem [Bostan, Chyzak, N., Safey El Din ’22]**

Suppose

- (H0) \( \deg_x(P) \geq 2 \),
- (H1) \( \deg_u(\partial_x P(x, z, 0, u)) \geq 1 \) and \( \partial_x P(F(t, c), F(t, 1), t, c) \neq 0 \) for all \( c \in \mathbb{K} \),
- (R) the zero set \( V(P) \subset \overline{\mathbb{K}}^4 \) is smooth outside \( V(u - 1) \subset \overline{\mathbb{K}}^4 \).

Set \( D_0 := \text{disc}_x P, D_1 := \text{SqFreePart}(D_0) \) and \( D_2 := \text{disc}_u D_1 \).

Then

- \( R := \text{SqFreePart}(D_2) \) is non-zero in \( \mathbb{K}[z, t] \) and satisfies \( R(F(t, 1), t) = 0 \).
- \( R \) has total size \( 16\delta^8 \) with degree in each variable at most \( 4\delta^4 \),
- \( R \) can be computed in \( O_{\log}(\delta^{10}) \) ops. in \( \mathbb{K} \).

\[ D_1 := \text{SqFreePart}(\text{disc}_x(P)) \] satisfies

\[ \partial_u D_1(U(t), F(t, 1), t) = 0. \]

\[ \begin{cases} (\partial_u D_1 \ \partial_z D_1 \ \partial_t D_1) \cdot (u \ z \ t)^T = 0, \\ (\partial_z D_1 \ \partial_t D_1) \cdot (z \ t)^T = 0 \end{cases} \]
Example where (H1) is not satisfied

Example

Consider the functional equation

\[ F(t, u) = 1 + t((u - 1)F(t, u)^2 + F(t, u) - F(t, 1)). \]  \hspace{1cm} (1)

Here \( P = 1 - x + t((u - 1)x^2 + x - z) \).

Therefore, \( \partial_x P(x, z, 0, u) = 1 \), hence assumption \((H1)\) is not satisfied.

Algorithm DD of page 16:

1. \( \text{disc}_x P = 4t^2 uz - 4t^2 z + t^2 - 4tu + 2t + 1 \),
2. \( \text{disc}_u(\text{disc}_x(P)) = 1 \).

The output is \( R = 1 \), which is obviously wrong.

In fact, the unique solution \( F(t, u) \) of (1) in \( \mathbb{Q}[u][[t]] \) satisfies \( F(t, 1) = 1 \), and is a root of \( R := t(u - 1)x^2 + (t - 1)x + 1 - t \).
## Recap

### Generic case

<table>
<thead>
<tr>
<th>Page</th>
<th>Contribution</th>
<th>Hypothesis</th>
<th>Total size</th>
<th>Complexity</th>
<th>Relative exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>(\delta^8)</td>
<td>(O_{\log}(\delta^{10}))</td>
<td>(\frac{10}{8} = 1.25)</td>
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<tr>
<td>9</td>
<td>Geom</td>
<td>(H1), radical, 0-dim</td>
<td>(\delta^6)</td>
<td>(O_{\log}(L\delta^6 + \delta^{7.89}))</td>
<td>(\frac{10}{6} = 1.6)</td>
</tr>
<tr>
<td>11</td>
<td>G&amp;P</td>
<td>(H1), Jac(\neq 0)</td>
<td>(\delta^6)</td>
<td>(O_{\log}(L\delta^6 + \delta^{3\theta+3}))</td>
<td>(\frac{10.14}{6} = 1.69)</td>
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<tr>
<td>13</td>
<td>General</td>
<td>None</td>
<td>(\delta^6)</td>
<td>(O_{\log}(\delta^{14}))</td>
<td>(\frac{14}{6} \sim 2.33)</td>
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### Sparse case

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<th>Page</th>
<th>Contribution</th>
<th>Hypothesis</th>
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<th>Complexity</th>
<th>Relative exponent</th>
</tr>
</thead>
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<td>16</td>
<td>DD</td>
<td>(H0), (H1), (R)</td>
<td>(\delta^8)</td>
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<td>(\delta^6)</td>
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<td>(\delta^6)</td>
<td>(O_{\log}(\delta^{3\theta+3}))</td>
<td>(\frac{\theta+1}{2} \sim 1.69 \rightarrow \frac{\theta}{2} \sim 1.19)</td>
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<tr>
<td>13</td>
<td>General</td>
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<td>(\delta^6)</td>
<td>(O_{\log}(\delta^{10.89}))</td>
<td>(\frac{10.89}{6} \sim 1.815)</td>
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