Transcendence of solutions of Mahler equations

Thomas Dreyfus
Joint work with Charlotte Hardouin and Julien Roques

1University Lyon 1, France
2University Toulouse 3, France
3University Grenoble 1, France

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Abstract

- Generating functions of automatic sequences are solutions of Mahler equations

\[ \phi_p^n y + a_{n-1} \phi_p^{n-1} y + \cdots + a_0 y = 0, \]

where \( p \geq 2, \phi_p y(z) := y(z^p) \ a_i \in \mathbb{C}(z), 0 \neq a_0. \)

- Many authors are interested about the differential-algebraic properties of such generating functions.

- In this talk we use parametrized differential Galois theory to study this question in a systematic way.
Case $n = 1$

Proposition (D., Hardouin, Roques)

Let $f \neq 0$ such that $\phi_p(f) = a_0 f$. The following statements are equivalent:

1. $f$ is hyperalgebraic over $\mathbb{C}(z)^1$;
2. there exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $u \in \mathbb{C}(z)^\times$ such that $a_0 = cz^m \frac{\phi_p(u)}{u}$.

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We say that $f$ is hyperalgebraic over $\mathbb{C}(z)$ if there is an algebraic relation over $\mathbb{C}(z)$ between $f$ and its derivatives.
Case \( n = 2 \)

**Theorem (D., Hardouin, Roques)**

Let \( f(z) \in \mathbb{C}((z)) \) be a nonzero solution of

\[
\phi^2_p y + a_1 \phi_p y + a_0 y = 0. \tag{1}
\]

Assume that (1) can not be reduced into an order one equation\(^2\). Then, \( f \) is hypertranscendental over \( \mathbb{C}(z) \).

\(^2\)More formally, we assume that the difference Galois group contains \( \text{SL}_2(\mathbb{C}) \).
The Baum-Sweet sequence

Example

The generating function of the Baum-Sweet sequence satisfies

\[ \phi_2^2 y + z\phi_2 y - y = 0. \]

It is hypertranscendental.
The Rudin-Shapiro sequence

Example

The generating function of the Rudin-Shapiro sequence satisfies

\[ \phi_2^2 y + \frac{1}{2z} \phi_2 y - \frac{1}{2z} y = 0. \]

It is hypertranscendental.
Difference Galois theory

Parametrized difference Galois theory

Hypertranscendence of solutions of Mahler equations
Consider the field
\[ K := \bigcup_{j \geq 1} \mathbb{C} \left( z^{1/j} \right), \]
we equip with the automorphism \( \phi_p \). Let
\[ \phi_p Y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} Y = AY, \quad (2) \]
which is equivalent to
\[ \phi_p^2 y + a \phi_p y + by = 0, \]
with \( a, b \in \mathbb{C}( z ) \), \( b \neq 0 \).
A Picard-Vessiot ring for (2) over $K$ is a difference ring extension $R|K$ such that

1) there exists $U \in \text{GL}_2(R)$ such that $\phi_p(U) = AU$;

2) $R$ is generated, as a $K$-algebra, by the entries of $U$ and $\det(U)^{-1}$;

3) the only $\phi_p$-ideals of $R$ are $\{0\}$ and $R$. 
Difference Galois group

Let $R \mid K$ be a Picard-Vessiot ring for (2). The difference Galois group $\text{Gal}(R/K)$ of $R$ over $K$ is the group of $K$-automorphisms of $R$ commuting with $\phi_p$:

$$\text{Gal}(R/K) := \{ \sigma \in \text{Aut}(R/K) \mid \phi_p \circ \sigma = \sigma \circ \phi_p \}.$$ 

The image

$$\text{Gal}(R/K) \rightarrow \text{GL}_2(\mathbb{C})$$

$$\sigma \mapsto U^{-1} \sigma(U)$$

is an algebraic subgroup of $\text{GL}_2(\mathbb{C})$. 
Proposition

*The algebraic dimension of $R|K$ equals to the dimension of* $\text{Gal}(R/K) \subset \text{GL}_2(\mathbb{C})$. 

Theorem (Roques)

One of the three following cases occurs.

1. $\text{Gal}(R/K)$ is conjugated to a group on upper triangular matrices. This happens if and only if there exists a solution $u \in K$ of the Riccati equation $(\phi_p(u) + a)u = -b$.

2. The first case does not occur and $\text{Gal}(R/K)$ is conjugated to a subgroup of 
\[
\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \bigg| \alpha, \beta \in \mathbb{C}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & \gamma \\ \epsilon & 0 \end{pmatrix} \bigg| \gamma, \epsilon \in \mathbb{C}^\times \right\}. 
\] This happens if and only if the first case does not occur and there exists a solution $u \in K$ of the Riccati equation
\[
\left( \phi_p^2(u) + \left( \phi_p^2 \left( \frac{b}{a} \right) - \phi_p(a) + \frac{\phi_p(b)}{a} \right) \right)u = -\frac{\phi_p(b)b}{a^2}. 
\]

3. $\text{Gal}(R/K)$ contains $\text{SL}_2(\mathbb{C})$. 

Differentially closed field

Definition

Let $\mathcal{C}$ be a field equipped with a derivation $\delta$. We say that $(\mathcal{C}, \delta)$ is differentially closed if, for every (finite) set of $\delta$-polynomials $\mathcal{F}$ in coefficients in $\mathcal{C}$, if the system of differential equations $\mathcal{F} = 0$ has a solution with entries in some $\delta$-field extension $\mathcal{L}$, then it has a solution with entries in $\mathcal{C}$. Any $\delta$-field $\mathcal{C}$ has a differential closure $\tilde{\mathcal{C}}$. 
Consider the derivation

\[ \delta := z \log(z) \partial_z, \text{ such that } \delta \circ \phi_p = \phi_p \circ \delta. \]

Let \((\tilde{C}, \delta)\) be a differential closure of \((C, \delta)\). Let

\[ L := \text{Frac} \left( \tilde{C} \otimes_C K(\log) \right). \]
Parametrized Picard-Vessiot extension

A parametrized Picard-Vessiot ring for (2) over $\mathbf{L}$ is a differential-difference ring extension $S|\mathbf{L}$ such that
1) there exists $U \in \text{GL}_2(S)$ such that $\phi_p(U) = AU$;
2) $S$ is generated, as a $\delta$-$\mathbf{L}$-algebra, by the entries of $U$, and $\det(U)^{-1}$;
3) the only $(\delta, \phi_p)$-ideals of $S$ are $\{0\}$ and $S$. 
Let $S|L$ be a parametrized Picard-Vessiot ring for (2). The parametrized difference Galois group $\text{PGal}(S/L)$ of $S$ over $L$ is the group of $L$-automorphisms of $S$ commuting with $\phi_p$ and $\delta$:

$$\text{PGal}(S/L) := \{ \sigma \in \text{Aut}(S/L) \mid \phi_p \circ \sigma = \sigma \circ \phi_p, \delta \circ \sigma = \sigma \circ \delta \}.$$
Linear differential algebraic group

Definition
We say that a subgroup $G$ of $\text{GL}_2(\tilde{\mathbb{C}})$ is a differential algebraic group if there exist $P_1, \ldots, P_k$, $\delta$-polynomials in 4 variables and in coefficients in $\tilde{\mathbb{C}}$ such that for $A = (a_{i,j}) \in \text{GL}_2(\tilde{\mathbb{C}})$,

$$A \in G \iff P_1(a_{i,j}) = \cdots = P_k(a_{i,j}) = 0.$$ 

The image

$$\text{PGal}(S/L) \rightarrow \text{GL}_2(\tilde{\mathbb{C}})$$

$$\sigma \mapsto U^{-1} \sigma(U)$$

is a differential algebraic subgroup of $\text{GL}_2(\tilde{\mathbb{C}})$. 
Proposition (Hardouin-Singer)

The differential dimension of $S|L$ equals to the dimension of $\text{PGal}(S/L) \subset \text{GL}_2 \left( \widehat{\mathbb{C}} \right)$. 
Case \( n = 1 \)

**Proposition (D., Hardouin, Roques)**

Let \( f \neq 0 \) such that \( \phi_p(f) = af \) with \( a \neq 0 \). We have the following alternative:

1. \( f \) is hypertranscendental over \( \mathbb{C}(z) \). In this case \( \text{PGal}(S/L) = \tilde{\mathbb{C}}^\times \); 
2. \( f \) is hyperalgebraic over \( \mathbb{C}(z) \). In this case \( \text{PGal}(S/L) \) is conjugated to a subgroup of \( \mathbb{C}^\times \).

Furthermore, the last case occurs if and only if there exist \( c \in \mathbb{C}^\times \), \( m \in \mathbb{Z} \) and \( u \in \mathbb{C}(z)^\times \) such that \( a = cz^m \phi_p(u) \).
From now, we consider

\[ \phi_p Y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} Y = AY, \]  

(2)

and assume that \( \text{Gal}(R/K) \) contains \( \text{SL}_2(\mathbb{C}) \). This implies that \( \text{PGal}(S/L) \) contains \( \text{SL}_2(\mathbb{C}) \).

Let \( U \in \text{GL}_2(S) \) be a fundamental solution. \( \det(U) \) is solution of

\[ \phi_p \det(U) = \det(A) \det(U) = b \det(U). \]
det(\(U\)) is hypertranscendental

Assume that \(\det(U)\) is hypertranscendental over \(\mathbb{C}(z)\). We have the following alternative:

1. \(\text{PGal}(S/L)\) is conjugated to \(\tilde{\mathbb{C}} \times \text{SL}_2(\mathbb{C})\);
2. \(\text{PGal}(S/L)\) is equal to a \(\text{GL}_2(\tilde{\mathbb{C}})\).

Moreover, the first case holds if and only if there exists \(B \in K^{2 \times 2}\) such that

\[
\rho_{\phi p}(B) = ABA^{-1} + z\partial_z(A)A^{-1} - \frac{1}{2} z\partial_z(b)b^{-1}l_2.
\]
det($U$) is hypertranscendental

**Theorem (D., Hardouin, Roques)**

Assume that det($U$) is hypertranscendental over $\mathbb{C}(z)$. Assume that $\phi^2_y + a\phi_y + by = 0$ admits a nonzero solution $f \in \mathbb{C}((z))$. Then, $f$ is hypertranscendental over $\mathbb{C}(z)$. 
det$(U)$ is hyperalgebraic

Theorem (D., Hardouin, Roques)

Assume that det$(U)$ is hyperalgebraic over $\mathbb{C}(z)$. Then, the parametrized difference Galois group $\text{PGal}(S/L)$ is a subgroup of $\mathbb{C}^\times \text{SL}_2(\tilde{\mathbb{C}})$ containing $\text{SL}_2(\tilde{\mathbb{C}})$. Furthermore, if $\phi_p^2y + a\phi_py + by = 0$ admits a nonzero solution $f \in \mathbb{C}((z))$, then $f$ is hypertranscendental over $\mathbb{C}(z)$. 