Narayana polynomials and random walks in space

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Joint work with J. Borwein, A. Straub, V.H. Moll, T. Amdeberhan
Introduction

- some recent results by Lassalle
- their extension using a probabilistic approach
- about random walks
Narayana numbers and polynomials

Definitions:
- the Narayana polynomials

\[ N_r(z) = \sum_{k=1}^{r} N(r, k) z^{k-1} \]

or

\[ N_r(z) = \sum_{m \geq 0} \binom{r-1}{2m} C_m z^m (z + 1)^{r-2m-1}, \]

with

\[ C_m = \frac{1}{m+1} \binom{2m}{m}, \text{ Catalan number} \]
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- the Narayana numbers

\[ N (r, k) = \frac{1}{r} \binom{r}{k-1} \binom{r}{k} , \quad r \neq 0 \]
The Narayana Triangle

1

1 1
→ 2

1 3 1
→ 5

1 6 6 1
→ 14

1 10 20 10 1
→ 42
The Narayana Triangle

Each row sum is a Catalan number

\[ \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} = \frac{1}{n+2} \binom{2n+2}{n+1}. \]
Lassalle’s results


The numbers $A_n$ defined by

$$(z + 1) \mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n \mathcal{N}_{r-2n+1}(z)$$

satisfy the recurrence

$$(-1)^{n-1} A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}.$$
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The first values are

$$A_1 = 1, \ A_2 = 1, \ A_3 = 5, \ A_4 = 56, \ A_5 = 1092, \ A_6 = 32670$$
Lassalle’s results

Defined by the recurrence formula in Theorem 1, page 2 of Lasalle.

Table of n, a(n) for n=1..17.


FORMULA

\[ a(n) = (-1)^{(n-1)} \cdot (C(n) + \sum_{j=1}^{n-1} (-1)^j \cdot \binomial(2n-2j, l) \cdot a(j) \cdot C(n-j)), \]

where \( C() = \text{A000108}() \). - R. J. Mathar, Apr 17 2011, corrected by Vaclav Kotesovec, Feb 28 2014

E.g.f.: \( \sum_{k=0}^{\infty} a(k) x^k/(2k+1)! = \text{log}(x/\text{BesselJ}(1,2x)) \). - Sergei N. Gladkovskii, Dec 28 2011

\[ a(n) \approx (n!)^{-1/2} \cdot (\sqrt{Pi} \cdot n^{3/2} \cdot r^n), \]

where \( r = \text{BesselJZero}[1, 1]^{-2/16} = 0.91762316513274332857623611053391686855599186384686... \) - Vaclav Kotesovec, added Feb 28 2014, updated Mar 01 2014

MAPLE

\[ \text{A000108} := \text{proc}(n) \text{binomial}(2n, n)/(1+n) \text{; end proc;} \]

\[ \text{A180874} := \text{proc}(n) \text{option remember; if n = 1 then 1 else} \]

\[ \text{A000108}(n)+\text{add}((-1)^j \cdot \text{binomial}(2n-2j, l) \cdot \text{procname}(j) \cdot \text{A000108}(n-j), j=1..n-1) \text{; end if; end proc:} \]

- R. J. Mathar, Apr 16 2011

MATHEMATICA

\( nmax=20; \ a = \text{ConstantArray}[0, nmax]; \ a[[1]]=1; \text{Do}[a[[n]] = (-1)^{(n-1)} \cdot (\text{Binomial}[2n, n]/(n+1) + \text{Sum}((-1)^j \cdot \text{Binomial}[2n-2j, 2j-1] \cdot a[[j]]) \cdot \text{Binomial}[2*(n-j), n-j]/(n-j+1), \{j, 1, n-1\}], \{n, 2, nmax\}]; \ a (* \text{Vaclav Kotesovec, Feb 28 2014 *}) \)

CROSSREFS


KEYWORD

nonn,easy

AUTHOR

Jonathan Vos Post, Sep 22 2010

STATUS

{}
Lassalle’s results

Lassalle shows that

$$\{A_n\}_{n \in \mathbb{N}}$$

is an increasing sequence of positive integers.
Lassalle’s results

Lassalle shows that\[ \{A_n\}_{n \in \mathbb{N}} \]
is an increasing sequence of positive integers. D. Zeilberger suggested to study the sequence\[ a_n = \frac{2A_n}{C_n} \]
with first values\[ a_1 = 2, a_2 = 1, a_3 = 2, a_4 = 8, a_5 = 52, a_6 = 495, a_7 = 6470 \]
and that satisfies\[ (-1)^{n-1} a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1} \]
Lassalle’s results

Equivalently,

\[ (-1)^{n-1} a_n = 2 + \frac{1}{2} \sum_{j=1}^{n-1} (-1)^j \sigma_{n,j} a_j \]

with

\[ \sigma_{n,r} = \frac{2}{n} \binom{n}{r-1} \binom{n+1}{r+1} \]

which appears as OEIS A108838.
Lassalle’s results

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Remark that

\[ \sigma_{n,r} = \binom{n-1}{r-1} \binom{n+1}{r} - \binom{n-1}{r-2} \binom{n+1}{r+1} \].
Lassalle’s paper

A08838
Triangle of Dyck paths counted by number of long interior inclines.

Triangle

<table>
<thead>
<tr>
<th>A108838</th>
<th>Triangle of Dyck paths counted by number of long interior inclines.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3, 2, 4, 8, 2, 5, 20, 15, 2, 6, 24, 70, 175, 140, 35, 2, 8, 112, 420, 560, 280, 48, 2, 9, 168, 892, 1754, 1470, 504, 63, 2, 10, 420, 1680, 4704, 5890, 3360, 840, 80, 2, 11, 330, 2970, 11088, 19404, 16632, 6930, 1320, 99, 2</td>
<td></td>
</tr>
</tbody>
</table>

Comments

T(n,k) is the number of Dyck n-paths (A006318) containing k long interior inclines. An incline is an ascent or a descent where an ascent (resp. descent) is a maximal sequence of contiguous upsteps (resp. downsteps). An incline is long if it consists of at least 2 steps and is interior if it does not start or end the path.

T(n,k) is the number of Dyck (n+1)-paths whose last descent has length 2 and which contain n-k peaks. For example T(3,0)=3 counts UUDUDUD, UDUUDUDD, UEDUDDUD. - David Callan, Jul 05 2006

T(n,k) is the number of parallelogram polyominoes of semiperimeter n+1 having k corners. [Eric Deutsch, Oct 05 2006]

T(n,k) is the number of rooted ordered trees with n non-root nodes and k leaves; see example. - Joerg Arndt, Aug 18 2014

Links

Table of n, a(n) for n=2..56.
David Callan, Some Identities for the Catalan and Fine Numbers

Formula

G.F. T(n,k) = 2*binom[n+1, k+2]*binom[n-2, k]/(n+1). GF G(z, t) := Sum[T(n, k)*z^nt^k, {n>=1, k>=0}] satisfies z - (1-z)^2 - (2*t-t^2)*z^2 )G + (t^2-z)*G^2 = 0.

G.F.=1+z(1+z)^2, where r=x(t,z) is the Narayana function defined by (1+r) (1-tz)z=r, x(t,0)=0. - Eric Deutsch, Jul 23 2006

For n>=0, the row polynomials sum {k = 0..n} T(n+2,k)*x^k = 2/(n+1)*((1-x)^n*P(n,2,1,(1+x)/(1-x)), where P(n,a,b,x) denotes the Jacobi polynomial.
Lassalle’s results

Lassalle’s main result:

**Theorem**

The numbers \( \{A_n, \ n \geq 2\} \) are positive, increasing integers and given by

\[
A_{n+1} = \sum_{r=1}^{n} \frac{n - r + 1}{n + 2} \left( \begin{array}{c} 2n + 1 \\ 2r - 1 \end{array} \right) A_r A_{n+1-r}.
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The numbers \( \{a_n, \ n \geq 2\} \) are positive, increasing integers and given by

\[
a_{n+1} = \frac{1}{2} \sum_{r=1}^{n} \frac{1}{n + 1} \binom{n + 1}{r + 1} \binom{n + 1}{r - 1} a_r a_{n+1-r}.
\]

"Both sequences seem to be new"
A Bessel function approach

The Bessel function of the first kind

\[ I_\alpha (z) = \sum_{j \geq 0} \frac{1}{j! (j + \alpha)!} \left( \frac{z}{2} \right)^{2j+\alpha}. \]
A Bessel function approach

The Bessel function of the first kind

\[ I_\alpha (z) = \sum_{j \geq 0} \frac{1}{j! (j + \alpha)!} \left( \frac{z}{2} \right)^{2j+\alpha}. \]

From the recurrence

\[ (-1)^{n-1} a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1}, \]

we deduce

**Theorem**

[V.H.M, T.A., C.V.] The numbers \( \{a_n\} \) satisfy

\[ \sum_{j=1}^{+\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = \frac{2}{\sqrt{x}} \frac{l_2 (2\sqrt{x})}{l_1 (2\sqrt{x})}. \]
A Bessel function approach

Using classical contiguity properties of Bessel $I$ functions, we recover Lassale’s

**Theorem**

The numbers $\{a_n\}$ satisfy the recurrence, with $a_1 = 1$,

$$2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \quad n \geq 2.$$ 

As a corollary, the numbers $\{a_n\}$ are positive.
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As a corollary, the numbers \( \{a_n\} \) are positive.

Using the recurrence, we show moreover, by induction on \( n \),

**Theorem**

*The numbers \( \{a_n\} \) are integers, and \( a_n \) is even if \( n \) is odd.*
A Bessel function approach

Theorem

For \( n \geq 3 \), the sequence \( \{a_n\} \) is increasing.

Proof.

Start from

\[
2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \quad n \geq 2
\]

so that

\[
a_n \geq \frac{1}{2n} \left[ \binom{n}{0} \binom{n}{2} a_1 a_{n-1} + \binom{n}{n-2} \binom{n}{2} a_{n-1} a_1 \right] = (n-1) a_{n-1}
\]

hence for \( n \geq 3 \),

\[
a_n - a_{n-1} \geq (n-2) a_{n-1} > 0.
\]
A probabilistic approach

The symmetric beta distribution

\[ f_\mu(x) = \begin{cases} 
\frac{1}{B(\mu + \frac{1}{2}, \frac{1}{2})} (1 - x^2)^{\mu - \frac{1}{2}} & x \in [-1, 1] \\
0 & \text{else}
\end{cases} \]
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The odd moments equal zero and

\[ \mathbb{E} X_{\mu}^{2m} = \frac{B(\mu + \frac{1}{2}, 2m + \frac{1}{2})}{B(\mu + \frac{1}{2}, \frac{1}{2})} = \frac{\Gamma(\mu + 1) (2m)!}{\Gamma(\mu + m + 1) 2^{2m} m!}. \]
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Remark: for \( \mu = 1 \),

\[ \mathbb{E} (2X_1)^{2m} = C_m \]

and for \( \mu = 0 \),

\[ \mathbb{E} (2X_0)^{2m} = \binom{2m}{m}. \]
A probabilistic approach

The moment generating function is

$$\varphi_\mu (z) = \mathbb{E} e^{zX_\mu} = \Gamma (\mu + 1) 2^\mu \frac{l_\mu (z)}{z^\mu}.$$
A probabilistic approach

The moment generating function is

\[ \varphi_\mu (z) = \mathbb{E}e^{zX_\mu} = \Gamma (\mu + 1) 2^\mu \frac{I_\mu (z)}{z^\mu}. \]

It admits the Weierstrass factorization

\[ \varphi_\mu (z) = \prod_{k \geq 1} \left( 1 + \frac{z^2}{j_{\mu,k}^2} \right) \]

where \( \{j_{\mu,k}\} \) are the zeros of the Bessel function \( J_\mu \).
A probabilistic approach

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The cumulants \( \kappa_{\mu} (n) \) are defined as

$$
\log \varphi_{\mu} (z) = \sum_{n \geq 1} \kappa_{\mu} (n) \frac{z^n}{n!}.
$$
A probabilistic approach

Define the **Bessel zeta function** - or Rayleigh function

\[ \zeta_\mu (s) = \sum_{k \geq 1} \frac{1}{j_{\mu,k}^s}. \]
A probabilistic approach

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The cumulants of the symmetric beta distribution are

\[ \kappa_{\mu}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 \left(-1\right)^{n/2+1} (n-1)! \zeta_{\mu}(n) & \text{if } n \text{ is even}. \end{cases} \]
A probabilistic approach

Define the random variable $Y_\mu$ as

$$Y_\mu = \sum_{k \geq 1} \frac{L_k}{j_{\mu, k}}$$

where $\{L_k\}$ are i.i.d. Laplace random variables. Then

$$\mathbb{E} e^{izY_\mu} = \prod_{k \geq 1} \left( 1 + \frac{z^2}{j_{\mu, k}^2} \right)^{-1} = \frac{1}{\mathbb{E} e^{izX_\mu}}$$

so that, with $X_\mu$ and $Y_\mu$ independent,

$$f(X_\mu + IzY_\mu + z) = f(z).$$
A probabilistic approach

The case $\mu = 1$ gives Lassale’s sequence with

$$f_1 (x) = \begin{cases} 
\frac{2}{\pi} \sqrt{1 - x^2}, & x \in [-1, 1] \\
0 & \text{else}
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and

$$\mathbb{E} (2X_1)^{2m} = C_m.$$
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The Narayana polynomials read

$$\mathcal{N}_r(z) = \mathbb{E} \left( 1 + z + 2\sqrt{z}X_1 \right)^{r-1}, \ r \geq 1.$$

which we rewrite as

$$\mathcal{N}_r(z) = (2\sqrt{z})^{r-1} \mathbb{E} \left( \frac{1 + z}{2\sqrt{z}} + X_1 \right)^{r-1}, \ r \geq 1.$$
We need the following result.

**Theorem**

If

\[ P_n(z) = \mathbb{E} (z + X)^n, \]

then \( P_n \) satisfies the recurrence

\[ P_{n+1}(z) - zP_n(z) = \sum_{m \geq 1} \left( \begin{array}{c} n \\ 2m - 1 \end{array} \right) \kappa_1(2m) P_{n-2m+1}(z). \]
A probabilistic approach

Apply to Narayana polynomials

\[ \mathcal{N}_{r+1}(z) - (1 + z) \mathcal{N}_r(z) = \sum_{m \geq 1} \binom{r - 1}{2m - 1} \kappa_1(2m) 2^m z^m \mathcal{N}_{r+1-2m}(z) \]
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and compare to

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so that

\[ A_n = (-1)^{n+1} \kappa(2n) 2^{2n} = 2^{2n+1} (2n - 1)! \zeta_1(2n) \]
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and

\[ a_n = \frac{A_n}{C_n} = 2^{2n+1} (n+1)! (n-1)! \zeta_1(2n). \]
A generalization

The symmetric beta distribution

\[ f_\mu(x) = \begin{cases} 
\frac{1}{B(\mu+\frac{1}{2}, \frac{1}{2})} (1 - x^2)^{\mu-\frac{1}{2}} & \text{if } x \in [-1,1] \\
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The cumulants are

\[ \kappa_{\mu}(2n) = (-1)^{n+1} 2^{2n+1} (2n - 1)! \zeta_{\mu}(2n) \]
A generalization

\[ a_n = \frac{2 \, (-1)^{n+1} \, \kappa_1(2n)}{\mathbb{E} (2X_1)^{2n}} \]
A generalization

Define

\[ a_n = \frac{2 (-1)^{n+1} \kappa_1 (2n)}{\mathbb{E} (2X_1)^{2n}} \]

\[ a_n^{(\mu)} = \frac{2 (-1)^{n+1} \kappa_\mu (2n)}{\mathbb{E} (2X_\mu)^{2n}}. \]
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The recurrence

\[ (n + \mu) \zeta_\mu(2n) = \sum_{r=1}^{n-1} \zeta_\mu(2r) \zeta_\mu(2n-2r), \]

translates, for \( \mu = 1 \), into

\[ 2na_n = \sum_{k=1}^{n-1} \binom{n}{k - 1} \binom{n}{k + 1} a_k a_{n-k}, \quad n \geq 2. \]
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\[2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \quad n \geq 2.\]

\[ a_n^{(\mu)} = \frac{1}{2 \binom{n+\mu-1}{n-1}} \sum_{k=1}^{n-1} \binom{n+\mu-1}{n-k-1} \binom{n+\mu-1}{k-1} a_k^{(\mu)} a_{n-k}^{(\mu)}.\]
We deduce

**Theorem (T.A, V.H.M., C.V.)**

*The coefficients $a_n^{(\mu)}$ are positive and increasing for $n \geq \left\lfloor \frac{\mu + 3}{2} \right\rfloor$.***
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**Theorem (T.A, V.H.M., C.V.)**

The coefficients $a_n^{(\mu)}$ are **positive and increasing** for $n \geq \left\lfloor \frac{\mu+3}{2} \right\rfloor$.

Define the generalized Narayana polynomials as

$$N_r^{(\mu)}(z) = \mathbb{E} \left( 1 + z + 2\sqrt{z}X_{\mu} \right)^{r-1}, \quad r \geq 1.$$
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**Theorem (T.A, V.H.M., C.V.)**

_The coefficients $a_n^{(\mu)}$ are positive and increasing for $n \geq \left\lfloor \frac{\mu+3}{2} \right\rfloor$._

Define the generalized Narayana polynomials as

$$\mathcal{N}^{(\mu)}_r(z) = \mathbb{E} \left( 1 + z + 2\sqrt{z}X_\mu \right)^{r-1}, \quad r \geq 1.$$  

They satisfy the recurrence

$$\mathcal{N}^{(\mu)}_{r+1}(z) - (1 + z) \mathcal{N}^{(\mu)}_r(z) = - \sum_{m \geq 1} \binom{r-1}{2m-1} \kappa_\mu(2m) 2^{2m} z^m \mathcal{N}^{(\mu)}_{r+1-2m}(z).$$
The Gegenbauer polynomials $C_n^{(\mu)}(z)$ are defined by the horizontal generating function

$$
\sum_{n \geq 0} C_n^{(\mu)}(z) t^n = (1 - 2xt + t^2)^{-\mu}
$$
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and are given by

\[
C_n^{(\mu)}(z) = \frac{(2\mu)_n}{n!} \mathbb{E} \left( z + \sqrt{z^2 - 1} X_{\mu - \frac{1}{2}} \right)^n
\]
Link with classical polynomials

The Gegenbauer polynomials $C_n^{(\mu)}(z)$ are defined by the horizontal generating function

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and are given by

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C_n^{(\mu)}(z) = \frac{(2\mu)_n}{n!} E \left( z + \sqrt{z^2 - 1} \right)^n
$$

so that

$$
\mathcal{N}_{n+1}^{(\mu)}(z) = \frac{n!}{(2\mu + 1)_n} (1 - z)^n C_n^{(\mu + \frac{1}{2})} \left( \frac{1 + z}{1 - z} \right).
$$
The usual Narayana polynomials ($\mu = 1$) are given by

\[
N_{n+1}(z) = \frac{2}{(n+1)(n+2)} (1-z)^n C_n^{(\frac{3}{2})} \left( \frac{1+z}{1-z} \right)
\]

\[
= (1-z)^n \ _2F_1 \left( \frac{-n}{2}, n+3; \frac{z}{z-1} \right)
\]

\[
= \frac{(2n+2)!}{(n+2)! (n+1)!} z^n \ _2F_1 \left( \frac{-n}{-2n-2}, -n-1; \frac{z-1}{z} \right)
\]

\[
= \ _2F_1 \left( \frac{-n}{2}, -n-1; z \right).
\]
The case $\mu = \frac{1}{2}$

The density $f_{\frac{1}{2}}$ is the uniform density on $[-1, 1]$ with

$$\mathbb{E}(2X_1)^{2n} = \frac{2^{2n}}{2n + 1}$$

and

$$\kappa_{\frac{1}{2}}(2n) = 2^{2n} \frac{B_{2n}}{2n}$$
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Theorem

The sequence

$$a_n^{\left(\frac{1}{2}\right)} = 2^{2n} \frac{2n + 1}{n} \left| B_{2n} \right|$$

with first terms

$$a_1^{\left(\frac{1}{2}\right)} = 2, \quad a_2^{\left(\frac{1}{2}\right)} = \frac{4}{3}, \quad a_3^{\left(\frac{1}{2}\right)} = \frac{32}{9}, \quad a_4^{\left(\frac{1}{2}\right)} = \frac{96}{5}, \quad a_5^{\left(\frac{1}{2}\right)} = \frac{512}{3}$$

is positive and increasing.
The case $\mu = \frac{1}{2}$

The convolution identity

$$(n + \mu) \zeta_\mu (2n) = \sum_{r=1}^{n-1} \zeta_\mu (2r) \zeta_\mu (2n - 2r)$$

with $\mu = \frac{1}{2}$ gives the well-known

$$\sum_{k=1}^{n-2} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n + 1) B_{2n}, \quad n \geq 1$$
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The moments-cumulants identity

$$\kappa_{\frac{1}{2}} (n) = \mathbb{E} X_{\frac{1}{2}}^n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_{\frac{1}{2}} (j) \mathbb{E} X_{\frac{1}{2}}^{n-j}$$

gives

$$\sum_{j=1}^{n} \binom{2n + 1}{2j} 2^{2j} B_{2j} = 2n, \quad n \geq 1.$$
The limit case $\mu = -\frac{1}{2}$

The density $f_{-\frac{1}{2}}$ is the discrete uniform density on $\{-1, 1\}$ with

$$\mathbb{E} \left( 2X_{-\frac{1}{2}} \right)^{2n} = 2^{2n}$$

and the cumulants are expressed in terms of Euler numbers

$$\kappa_{-\frac{1}{2}} (2n) = -2^{4n-1} E_{2n-1}$$
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The sequence

$$a_n\left(-\frac{1}{2}\right) = (-1)^n 2^{2n} E_{2n-1}$$

with first terms

$$a_{\frac{1}{2}} = 2, \quad a_{2\frac{1}{2}} = 4, \quad a_{1\frac{1}{2}} = 32, \quad a_{4\frac{1}{2}} = 544, \quad a_{5\frac{1}{2}} = 15872$$

is positive and increasing.
The limit case $\mu = -\frac{1}{2}$

The convolution identity

$$(n + \mu) \zeta_{\mu}(2n) = \sum_{r=1}^{n-1} \zeta_{\mu}(2r) \zeta_{\mu}(2n - 2r)$$

gives the well-known

$$\sum_{k=1}^{n-1} \binom{2n - 2}{2k - 1} E_{2k-1} E_{2n-2k-1} = 2E_{2n-1}, \ n \geq 1$$
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gives

$$\sum_{k=1}^{n} \binom{2n - 1}{2k - 1} 2^{2k-1} E_{2k-1} = 1, \quad n \geq 1.$$
Random Walks: densities

The probability density $p_n(\nu; x)$ of the distance to the origin in $d \geq 2$ dimensions after $n \geq 2$ steps is, for $x > 0$,

$$p_n(\nu; x) = \frac{1}{2^{\nu} \nu!} \int_0^\infty (tx)^{\nu+1} J_\nu(tx) j_\nu^n(t) \, dt$$

with

$$\nu = \frac{d}{2} - 1.$$
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with

$$\nu = \frac{d}{2} - 1.$$ 

For $x > 0$ and $n = 1, 2, \ldots$ the function

$$\psi_n(\nu; x) = \frac{\nu!}{2\pi^{\nu+1}} \frac{p_n(\nu; x)}{x^{2\nu+1}}$$

satisfies

$$\psi_n(\nu; x) = \frac{\nu! 2^{2\nu}}{(2\nu)! \pi} \int_{-1}^{+1} \psi_{n-1} \left( \nu; \sqrt{1 + 2\lambda x + x^2} \right) (1 - \lambda^2)^{\nu - \frac{1}{2}} \, d\lambda.$$
Random Walks: 3 steps
Random Walks: 4 steps
Random Walks: moments

The moments

\[ W_n (\nu; s) = \int_0^\infty x^s f_n (\nu; x) \, dx \]

satisfy

\[ W_n (\nu; 2k) = \frac{(k + \nu)! \nu!^{n-1}}{(k + \nu n)!} \sum_{k_1 + \cdots + k_n = k} \binom{k}{k_1, \ldots, k_n} \binom{k + n \nu}{k_1 + \nu, \ldots, k_n + \nu} \]
Random Walks: moments

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and the recursion

\[ W_{n_1 + n_2} (\nu; 2k) = \sum_{j=0}^k \binom{k}{j} \frac{(k + \nu)! \nu!}{(k - j + \nu)! (j + \nu)!} W_{n_1} (\nu; 2j) W_{n_2} (\nu; 2k - 2j) \]
Random Walks: moments

The moments

\[ W_n (\nu; s) = \int_0^\infty x^s f_n (\nu; x) \, dx \]

satisfy

\[ W_n (\nu; 2k) = \frac{(k + \nu)!\nu!^{n-1}}{(k + \nu n)!} \sum_{k_1+\ldots+k_n=k} \binom{k}{k_1, \ldots, k_n} \binom{k + \nu}{k_1 + \nu, \ldots, k_n + \nu} \]

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\[ W_{n_1+n_2} (\nu; 2k) = \sum_{j=0}^k \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)! (j + \nu)!} W_{n_1} (\nu; 2j) W_{n_2} (\nu; 2k - 2j) \]

and in particular

\[ W_n (\nu; 2k) = \sum_{j=0}^k \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)! (j + \nu)!} W_{n-1} (\nu; 2j) \]
Random Walks: 2 dimensions

In two dimensions:

\[ W_2(0; 2k) : 1; 2; 6; 20; 70; 252; 924; 3432; 12870; \]
\[ W_3(0; 2k) : 1; 3; 15; 93; 639; 4653; 35169; 272835; 2157759; \]
\[ W_4(0; 2k) : 1; 4; 28; 256; 2716; 31504; 387136; 4951552; \]
\[ W_5(0; 2k) : 1; 5; 45; 545; 7885; 127905; 2241225; 41467725; \]
\[ W_6(0; 2k) : 1; 6; 66; 996; 18306; 384156; 8848236; 218040696; \]
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In fact

\[ W_n(0; 2k) = \sum_{k_1 + \cdots + k_n = k} \binom{k}{k_1, \ldots, k_n}^2 \]

count abelian squares.
Random Walks: 4 dimensions

In four dimensions:

\( W_2(1; 2k) : 1; 2; 5; 14; 42; 132; 429; 1430; 4862; \) (Catalan)

\( W_3(1; 2k) : 1; 3; 12; 57; 303; 1743; 10629; 67791; \)

\( W_4(1; 2k) : 1; 4; 22; 148; 1144; 9784; 90346; \)

\( W_5(1; 2k) : 1; 5; 35; 305; 3105; 35505; 444225; \)

\( W_6(1; 2k) : 1; 6; 51; 546; 6906; 99156; 1573011 \)
Random Walks: Narayana polynomials again

The distance to the origin $R_n$ verifies

$$R_{n+1} \sim \sqrt{1 + 2\Lambda R_n + R_n^2}$$

with

$$\Lambda = \cos \theta \sim \frac{\nu!}{\sqrt{\pi} (\nu - \frac{1}{2})!} (1 - \lambda^2)^{\nu - \frac{1}{2}}, \quad -1 \leq \lambda \leq +1$$
Random Walks: Narayana polynomials again

The distance to the origin $R_n$ verifies

$$R_{n+1} \sim \sqrt{1 + 2\Lambda R_n + R_n^2}$$

with

$$\Lambda = \cos \theta \sim \frac{\nu!}{\sqrt{\pi} (\nu - \frac{1}{2})!} (1 - \lambda^2)^{\nu - \frac{1}{2}} , -1 \leq \lambda \leq +1$$

As a consequence

$$\mathbb{E} R_{n+1}^{2k} = \mathbb{E} \left( 1 + 2\Lambda R_n + R_n^2 \right)^k$$

and since

$$\mathcal{N}_{k}^{(\nu)} (z) = \mathbb{E} \left( 1 + 2\Lambda \sqrt{z} + z \right)^{k-1} ,$$

we deduce ($\mu$ is now $\nu$)

$$\mathbb{E} \left( R_{n+1}^{2k} \right) = \mathbb{E} \mathcal{N}_{k+1}^{(\nu)} (R_n^2) .$$
Random Walks: 4 steps

The recursion

\[ W_n (\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)! (j + \nu)!} W_{n-1} (\nu; 2j) \]
Random Walks: 4 steps

The recursion

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Denote

\[ A_{k,j} (\nu) = \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)! (j + \nu)!} \]
Random Walks: 4 steps

The recursion

\[ W_n(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)! (j + \nu)!} W_{n-1}(\nu; 2j) \]

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and build the Narayana triangle (or Catalan triangle A001263)

\[
A(1) = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & \\
1 & 3 & 1 & 0 & \\
1 & 6 & 6 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \\
\end{bmatrix}, \quad A^3(1) = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots \\
3 & 1 & 0 & 0 & \\
12 & 9 & 1 & 0 & \\
57 & 72 & 18 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \\
\end{bmatrix}
\]
Random Walks: 4 steps

The recursion

\[
W_n (\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k + \nu)! \nu!}{(k - j + \nu)! (j + \nu)!} W_{n-1} (\nu; 2j)
\]

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57 & 72 & 18 & 1 \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]

For example,

\[
W_3 (1; 2k) : 1; 3; 12; 57;
\]
References 1/2


References 2/2


- J. Borwein, A. Straub and C. Vignat, Densities of short uniform random walks in higher dimensions, arXiv:1508.04729