On the Formalization of Foundations of Tarski’s System of Geometry

Pierre Boutry

University of Strasbourg - ICCube - CNRS

Computations and Proofs - Specfun - March 2016
Motivations

Geometry has played a central role in the history of mathematical proof:
- Axiomatic approach;
- Foundational crisis of mathematics;
- Metamathematics;
- Education.

Pierre Boutry
Formal Proofs in Tarski’s System of Geometry
Motivations

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Euclid
(325 B.C. - 265 B.C.)
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Alfred Tarski (1901 - 1983)
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Motivations

Tarski's system of geometry
Parallel postulates
Arithmetization of geometry
Perspectives

State of the art

Pierre Boutry
Formal Proofs in Tarski's System of Geometry
Motivations

- The **missing** concept in *Euclid’s Elements*
Motivations

- The **missing** concept in *Euclid’s Elements*: the betweenness.

Moritz Pasch
(1843 - 1930)
Motivations

- The **missing** concept in *Euclid’s Elements*: the betweenness.
- More than two millennia of **false proofs** of the parallel postulate.
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Posidonius
(135 B.C. - 51 B.C.)
Motivations

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Proclus (412 - 485)
Motivations

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Omar Khayyam
(1048 - 1131)
Motivations

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John Wallis
(1616 - 1703)
Motivations

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Jean-Henri Lambert  
(1728 - 1777)
Motivations

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- More than two millennia of **false proofs** of the parallel postulate.
- We can still make **mistakes**.
Motivations

- The **missing** concept in *Euclid’s Elements*: the betweenness.
- More than two millennia of **false proofs** of the parallel postulate.
- We can still make **mistakes**.

*It soon became clear that the only real long-term solution to the problems that I encountered is to start using computers in the verification of mathematical reasoning.*

(Vladimir Voevodsky, talk in March 2014 at the Institute for Advanced Studies at Princeton)
Motivations
Il n’en est pas moins certain que le théorème sur la somme des trois angles du triangle doit être regardé comme l’une de ces vérités fondamentales qu’il est impossible de contester, et qui sont un exemple toujours subsistant de la certitude mathématique qu’on recherche sans cesse et qu’on n’obtient que bien difficilement dans les autres branches des connaissances humaines.

(Adrien-Marie Legendre, Réflexions sur quelques manières de démontrer la théorie des parallèles ou le théorème sur la somme des trois angles du triangle)
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Formalizations of foundations of geometry
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- Synthetic approach
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid

Euclid
(325 av. J.-C. - 265 av. J.-C.)
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
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Euclid.
*Les éléments.*
Traduit par Bernard Vitrac.
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert

David Hilbert
(1862 - 1943)
Introduction

Tarski’s system of geometry

Parallel postulates

Arithmetic of geometry

Perspectives

Motivations

State of the art

Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert

David Hilbert.

*Foundations of Geometry (Grundlagen der Geometrie).*

Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
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Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski.

*Metamathematische Methoden in der Geometrie.*

Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski

- Analytic approach
Formalizations of foundations of geometry

- **Synthetic approach**: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski

- **Analytic approach**: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$. 
Introduction

Tarski’s system of geometry
Parallel postulates
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- Mixed analytic/synthetic approach
Formalizations of foundations of geometry

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- Analytic approach: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.

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Formalizations of foundations of geometry

- **Synthetic approach**: geometric objects and axioms about them.
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- **Analytic approach**: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.

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Formalizations of foundations of geometry

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- Analytic approach: a field $F$ is assumed and the space is defined as $F^n$.

- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
  - Birkhoff

George David Birkhoff.

A set of postulates for plane geometry (based on scale and protractors).

Formalizations of foundations of geometry

- **Synthetic approach**: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski

- **Analytic approach**: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.

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  - Birkhoff

- **Erlangen program**

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Felix Klein
(1849 - 1925)
Formalizations of foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert
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- Analytic approach: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.

- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
  - Birkhoff

- Erlangen program: a geometry is defined as a space of objects and a group of transformations acting on it.

Felix C. Klein.
A comparative review of recent researches in geometry, 1872.
Introduction

Tarski’s system of geometry

Parallel postulates

Arithmetization of geometry

Perspectives

Outline

1 Introduction

2 Tarski’s system of geometry
   - The axioms
   - Overview of the formalization

3 Parallel postulates

4 Arithmetization of geometry

5 Perspectives
Outline

1. Introduction

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The axioms

Introduction
Tarski's system of geometry
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The axioms
Overview of the formalization

The axioms

A single primitive type: point.
Two primitive predicates:
1 congruence
AB CD
2 betweenness
A B C

11 axioms.
A parameter controls the dimension.

Alfred Tarski
(1901 - 1983)
The axioms

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The axioms

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The axioms

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The axioms

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
  2. betweenness $A - B - C$.
- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoretical properties.

Alfred Tarski (1901 - 1983)
Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

\[ AB \parallel CD \rightarrow AB \parallel EF, \quad CD \parallel EF \]

Axiom (Pseudo-reflexivity for congruence)

\[ AB \parallel BA \]

Axiom (Identity for congruence)

\[ AB \parallel CC \rightarrow A = B \]
Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

\[ AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF \]
Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

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### Axioms about congruence

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<td>[ AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF ]</td>
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<td>[ AB \equiv CC \Rightarrow A = B ]</td>
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Axiom about betweenness
Axiom (Identity for betweenness)

\[ A \sim B \sim A \Rightarrow A = B \]
Five-Segment Axiom
Five-Segment Axiom

Axiom (Five-Segment)

\[ AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land A\!-\!B\!-\!C \land A'\!-\!B'\!-\!C' \land A \neq B \Rightarrow CD \equiv C'D' \]
Five-Segment Axiom

Axiom (Five-Segment)

\[
AB \equiv A'B' \land BC \equiv B'C' \land \\
AD \equiv A'D' \land BD \equiv B'D' \land \\
A - B - C \land A' - B' - C' \land A \neq B \Rightarrow CD \equiv C'D'
\]
Axiom of Segment Construction
Axiom of Segment Construction

Axiom (Segment Construction)

$$\exists E, A-B-E \land BE \equiv CD$$
Axiom of Segment Construction

\[ \exists E, A - B - E \land BE \equiv CD \]
Pasch axiom
Pasch axiom

Axiom (Pasch)

\[ A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A \]
Pasch axiom

Axiom (Pasch)

\[ A - P - C \land B - Q - C \Rightarrow \exists X, \, P - X - B \land Q - X - A \]
Pasch axiom

Axiom (Pasch)

\[ A - P - C \wedge B - Q - C \Rightarrow \exists X, P - X - B \wedge Q - X - A \]
Pasch axiom

Axiom (Pasch)

\[ A - P - C \land B - Q - C \Rightarrow \exists X, P - X - B \land Q - X - A \]
2-Dimensional Axiom
2-Dimensional Axiom

Axiom (Lower 2-Dimensional)

$$\exists ABC, \neg A-B-C \land \neg B-C-A \land \neg C-A-B$$
2-Dimensional Axiom

Axiom (Lower 2-Dimensional)

\[ \exists ABC, \neg A-B-C \land \neg B-C-A \land \neg C-A-B \]

Axiom (Upper 2-Dimensional)

\[ AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow \]
\[ A-B-C \lor B-C-A \lor C-A-B \]
Euclid’s axiom

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The axioms
Overview of the formalization

Axiom (Euclid)

\[ AD^B DT^A \neq D ) \]

\[ 9 \ XY; \ A \ B X^A C Y^X T Y \]

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Formal Proofs in Tarski’s System of Geometry
Euclid’s axiom

Axiom (Euclid)

\[ A \neq D \land B \neq D \land C \neq D \Rightarrow \exists X, Y, A \neq B \land X \neq Y \land X \neq T \land Y \]
Euclid’s axiom

Axiom (Euclid)

\[ A - D - T \land B - D - C \land A \neq D \Rightarrow \exists XY, A - B - X \land A - C - Y \land X - T - Y \]
Euclid’s axiom

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\[ A \neq D \land B \neq D \land C \neq A \land A \neq D \Rightarrow \exists XY, A \neq B \land A \neq C \land X \neq Y \land X \neq T \land Y \]
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![Diagram of Euclid's axiom](image)
The axioms (summary)

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<th>$A - B - A \Rightarrow A = B$</th>
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<td>$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow A - B - C \lor B - C - A \lor C - A - B$</td>
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<td>$A - D - T \land B - D - C \land A \neq D \Rightarrow \exists XY, A - B - X \land A - C - Y \land X - T - Y$</td>
</tr>
<tr>
<td>Continuity</td>
<td>$\forall \Xi, (\exists A, (\forall XY, X \in \Xi \land Y \in \Upsilon \Rightarrow A - X - Y)) \Rightarrow \exists B, (\forall XY, X \in \Xi \land Y \in \Upsilon \Rightarrow X - B - Y)$</td>
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### The axioms (summary)

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<td>$\Rightarrow A - B - C \lor B - C - A \lor C - A - B$</td>
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<td>$\exists B, (\forall XY, X \in \Xi \land Y \in \gamma \Rightarrow X - B - Y)$</td>
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</tbody>
</table>
Overview of the formalization

W. Schwabhäuser
W. Szmielew A. Tarski

Metamathematische Methoden in der Geometrie
Mit 187 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

geocoq.github.io/GeoCoq/
### Overview of the formalization

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Neutral $\geq$ 2D</th>
<th>$= 2D$</th>
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<tbody>
<tr>
<td>Ch 2: Properties about betweenness</td>
<td>✓</td>
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Outline

1. Introduction

2. Tarski’s system of geometry

3. Parallel postulates
   - A syntaxic proof of the independence
   - Decidability of the predicates of the development
   - Equivalent statements

4. Arithmetization of geometry

5. Perspectives
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5. Perspectives
Euclid’s axiom

Axiom (Euclid)

\[ A \neq D \land B \neq C \land A \neq D \Rightarrow \exists XY, A \neq B \land A \neq C \land X \neq Y \]
Types of independence proofs
Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.
Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.

Hyperbolic geometry
Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.

Hyperbolic geometry

Elliptic geometry
Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.
  - Hyperbolic geometry
  - Elliptic geometry

- Syntaxic proofs: prove there does not exist a derivation of the axiom from the others.
### Syntaxic proof

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<td>$\exists ABC, \neg A - B - C \land \neg B - C - A \land \neg C - A - B$</td>
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<td>$A - D - T \land B - D - C \land A \neq D \Rightarrow \exists XY, A - B - X \land A - C - Y \land X - T - Y$</td>
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<td>Continuity</td>
<td>$\forall \Xi \forall \Upsilon, (\exists A, (\forall XY, X \in \Xi \land Y \in \Upsilon \Rightarrow A - X - Y)) \Rightarrow \exists B, (\forall XY, X \in \Xi \land Y \in \Upsilon \Rightarrow X - B - Y)$</td>
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AB ≡ CD ∧ AB ≡ EF ⇒ CD ≡ EF

AB ≡ BA

AB ≡ CC ⇒ A = B

∃ E, A–B–E ∧ BE ≡ CD

A–P–C ∧ B–Q–C ⇒ ∃ X, P–X–B ∧ Q–X–A

AB ≡ A′B′ ∧ BC ≡ B′C′∧

AD ≡ A′D′ ∧ BD ≡ B′D′∧

A–B–C ∧ A–B′–C′ ∧ A ≠ B ⇒ CD ≡ C′D′

∃ ABC, ¬A–B–C ∧ ¬B–C–A ∧ ¬C–A–B

AP ≡ AQ ∧ BP ≡ BQ ∧ CP ≡ CQ ∧ P ≠ Q ⇒

A–B–C ∨ B–C–A ∨ C–A–B

A–D–T ∧ B–D–C ∧ A ≠ D ⇒

∃ XY, A–B–X ∧ A–C–Y ∧ X–T–Y
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AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF
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A \rightarrow P \rightarrow C \land B \rightarrow Q \rightarrow C \Rightarrow \exists X, P \rightarrow X \rightarrow B \land Q \rightarrow X \rightarrow A
\]

\[
AB \equiv A' \rightarrow B' \land BC \equiv B' \rightarrow C'\land
\]

\[
AD \equiv A' \rightarrow D' \land BD \equiv B' \rightarrow D'\land
\]

\[
A \rightarrow B \rightarrow C \land A' \rightarrow B' \rightarrow C' \land A \neq B \Rightarrow CD \equiv C' \rightarrow D'
\]

\[
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\]

\[
AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow
\]

\[
A \rightarrow B \rightarrow C \lor B \rightarrow C \rightarrow A \lor C \rightarrow A \rightarrow B
\]

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<td>Five-Segment</td>
<td>( AB \equiv A' B' \land BC \equiv B' C' \land AD \equiv A' D' \land BD \equiv B' D' \land A \text{--} B \text{--} C \land A' \text{--} B' \text{--} C' \land A \neq B \Rightarrow CD \equiv C' D' )</td>
</tr>
<tr>
<td>Lower 2-Dimensional</td>
<td>( \exists ABC, \neg A \text{--} B \text{--} C \land \neg B \text{--} C \text{--} A \land \neg C \text{--} A \text{--} B )</td>
</tr>
<tr>
<td>Upper 2-Dimensional</td>
<td>( AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow A \text{--} B \text{--} C \lor B \text{--} C \text{--} A \lor C \text{--} A \text{--} B )</td>
</tr>
<tr>
<td>Euclid</td>
<td>( A \text{--} D \text{--} T \land B \text{--} D \text{--} C \land A \neq D \Rightarrow \exists XY, A \text{--} B \text{--} X \land A \text{--} C \text{--} Y \land X \text{--} T \text{--} Y )</td>
</tr>
<tr>
<td>\textbf{Continuity}</td>
<td>( \forall \exists \gamma, (\exists A, (\forall XY, X \in \exists \land Y \in \gamma \Rightarrow A \text{--} X \text{--} Y)) \Rightarrow \exists B, (\forall XY, X \in \exists \land Y \in \gamma \Rightarrow X \text{--} B \text{--} Y) )</td>
</tr>
</tbody>
</table>
Introduction

Tarski’s system of geometry

Parallel postulates

Arithmetization of geometry

Perspectives

Outline

1 Introduction

2 Tarski’s system of geometry

3 Parallel postulates
   - A syntaxic proof of the independence
   - Decidability of the predicates of the development
   - Equivalent statements

4 Arithmetization of geometry

5 Perspectives
One remark
One remark

Axiom (Playfair)

In a plane, there is at most one line parallel to another given line and passing by a given point.
One remark

Axiom (Playfair)

*In a plane, there is at most one line parallel to another given line and passing by a given point.*
Axiom (Playfair)

In a plane, there is at most one line parallel to another given line and passing by a given point.
Intuitionistic Logic

L. E. J. Brouwer (1881 - 1966)

Pierre Boutry

Formal Proofs in Tarski's System of Geometry
Intuitionistic Logic

Axiom (Excluded middle (not admitted))

\[ \forall A, A \lor \neg A \]
Intuitionistic Logic

Axiom (Excluded middle (not admitted))

\( \forall A, A \lor \neg A \)
Intuitionistic Logic

Axiom (Excluded middle (not admitted))

\[ \forall A, A \lor \neg A \]

A particular instance of the excluded middle

\[ \forall ABCD, (\exists I, \text{Col } AB I \land \text{Col } CD I) \lor \\
(\exists I, \text{Col } AB I \land \text{Col } CD I) \Rightarrow \\
(\exists I, \text{Col } AB I \land \text{Col } CD I) \}

L. E. J. Brouwer (1881 - 1966)
Intuitionistic Logic

Axiom (Excluded middle (not admitted))

\[ \forall A, A \lor \neg A \]

A particular instance of the excluded middle

\[ \forall ABCD, (\exists I, \text{Col } AB I \land \text{Col } CD I) \lor \neg (\exists I, \text{Col } AB I \land \text{Col } CD I) \]

The most frequent instance of the excluded middle

\[ \forall AB : \text{Point}, A = B \lor A \neq B \]
A first equivalence

We proved that the following formulas are equivalent in Tarski's system of geometry in intuitionistic logic:

- $\forall AB : Point; A = B \_ A \neq B$;
- $\forall ABC; AB \angle AC$;
- $\forall ABCD; AB \angle CD$. 

Pierre Boutry

Formal Proofs in Tarski's System of Geometry
A first equivalence

We proved that the following formulas are equivalent in Tarski’s system of geometry in intuitionistic logic:
A first equivalence

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\[ \forall AB : \text{Point}, A = B \lor A \neq B; \]
A first equivalence

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- $\forall AB : \text{Point}, A = B \lor A \neq B$;
- $\forall ABC, A \dashv B \dashv C \lor \neg A \dashv B \dashv C$;
A first equivalence

We proved that the following formulas are equivalent in Tarski’s system of geometry in intuitionistic logic:

- $\forall AB : Point, A = B \lor A \neq B$;
- $\forall ABC, A \underline{B} C \lor \neg A \underline{B} C$;
- $\forall ABCD, AB \equiv CD \lor \neg AB \equiv CD$. 
### Results

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<thead>
<tr>
<th>Identity for betweenness</th>
<th>$A \overline{B} A \Rightarrow A = B$</th>
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<tbody>
<tr>
<td>Transitivity for congruence</td>
<td>$AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF$</td>
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<td>$AB \equiv CC \Rightarrow A = B$</td>
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<td>Segment Construction</td>
<td>$\exists E, \overline{A-B-E} \land BE \equiv CD$</td>
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<tr>
<td>Pasch</td>
<td>$\overline{A-P-C} \land \overline{B-Q-C} \Rightarrow \exists X, \overline{P-X-B} \land \overline{Q-X-A}$</td>
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<tr>
<td>Five-Segment</td>
<td>$AB \equiv A'B' \land BC \equiv B'C' \land AD \equiv A'D' \land BD \equiv B'D' \land A-B-C \land A'-B'-C' \land A \neq B \Rightarrow CD \equiv C'D'$</td>
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<td>Lower 2-Dimensional</td>
<td>$\exists ABC, \neg A-B-C \land \neg B-C-A \land \neg C-A-B$</td>
</tr>
<tr>
<td>Upper 2-Dimensional</td>
<td>$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow A-B-C \lor B-C-A \lor C-A-B$</td>
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<tr>
<td>Euclid</td>
<td>$\overline{A-D-T} \land \overline{B-D-C} \land A \neq D \Rightarrow \exists XY, \overline{A-B-X} \land \overline{A-C-Y} \land \overline{X-T-Y}$</td>
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<td>$A \rightarrow D \rightarrow T \land B \rightarrow D \rightarrow C \land A \neq D \Rightarrow$</td>
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<tr>
<td></td>
<td>$\exists XY, A \rightarrow B \rightarrow X \land A \rightarrow C \rightarrow Y \land X \rightarrow T \rightarrow Y$</td>
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<tr>
<td>Decidability of equality</td>
<td>$A = B \lor A \neq B$</td>
</tr>
</tbody>
</table>
Results

We proved the decidability of:

- **Bet** \( \forall ABC, A \rightarrow B \rightarrow C \vee \neg A \rightarrow B \rightarrow C; \)
- **Cong** \( \forall ABCD, AB \equiv CD \vee \neg AB \equiv CD; \)
- **Col** \( \forall ABC, \text{Col } A B C \vee \neg \text{Col } A B C; \)
- **Out** \( \forall ABC, A \leftarrow B \leftarrow C \vee \neg A \leftarrow B \leftarrow C; \)
- **Per** \( \forall ABC, \triangle A B C \vee \neg \triangle A B C; \)
- **Perp at** \( \forall ABCDP, AB \perp \downarrow CD \vee \neg AB \perp \downarrow CD; \)
- **TS** \( \forall ABCD, \overrightarrow{D} \overrightarrow{C} \overrightarrow{B} \vee \neg \overrightarrow{A} \overrightarrow{C} \overrightarrow{B}; \)
- **OS** \( \forall ABCD, \overrightarrow{C} \overrightarrow{D} \overrightarrow{B} \vee \neg \overrightarrow{A} \overrightarrow{C} \overrightarrow{D} \overrightarrow{B}; \)
- **CongA** \( \forall ABCDEF, AB \equiv D \equiv E \equiv F \vee \neg AB \equiv D \equiv E \equiv F; \)
- **Reflect** . . .
Results

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Neutral</th>
<th>2D</th>
<th>= 2D</th>
<th>Euclid</th>
<th>Decidability of equality</th>
<th>Excluded middle</th>
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2. Tarski’s system of geometry

3. Parallel postulates
   - A syntaxic proof of the independence
   - Decidability of the predicates of the development
   - Equivalent statements

4. Arithmetization of geometry

5. Perspectives
Equivalent statements
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<tr>
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Imply the decidability of intersection of lines

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Formal Proofs in Tarski’s System of Geometry
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Imply the decidability of intersection of lines

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Imply the decidability of intersection of lines

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Imply the decidability of intersection of lines

X

Pierre Boutry  Formal Proofs in Tarski’s System of Geometry
The role of continuity
The role of continuity

Here, we consider two postulates:
The role of continuity

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- Triangle postulate;

\[ \begin{align*}
\text{Tarski's system of geometry} \\
\text{Parallel postulates} \\
\text{Arithmetization of geometry} \\
\text{Perspectives}
\end{align*} \]

A syntaxic proof of the independence
Decidability of the predicates of the development
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An extra axiom is needed to prove their equivalence. Indeed:
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The role of continuity

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- Triangle postulate;
- Playfair’s postulate.

An extra axiom is needed to prove their equivalence. Indeed:

- Playfair’s postulate $\Rightarrow$ Triangle postulate;
- \[ \text{Triangle postulate} \nRightarrow \text{Playfair's postulate} \]
  (Max Dehn);
The role of continuity

Here, we consider two postulates:

- Triangle postulate;
- Playfair’s postulate.

An extra axiom is needed to prove their equivalence. Indeed:

- Playfair’s postulate $\Rightarrow$ Triangle postulate;
- Triangle postulate $\not\Rightarrow$ Playfair’s postulate (Max Dehn);
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Equivalent statements

<table>
<thead>
<tr>
<th>Triangle postulate</th>
<th>Equivalent to Playfair's postulate without continuity axiom</th>
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</tbody>
</table>

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Equivalent statements

1. ... 12. ...
8. ... 18. ...
10. ... 20. ...

Equivalent to Playfair’s postulate without continuity axiom

21. ...
22. ...
23. Tarski’s version of the parallel postulate
24. ...

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Equivalent statements

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Equivalent to Playfair’s postulate without continuity axiom
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Decidability of the predicates of the development

Equivalent statements

Equivalent statements

Triangle postulate

Playfair’s postulate

Tarski’s version of the parallel postulate

Equivalent to Playfair’s postulate without continuity axiom
Equivalent statements

1 ... 12 ...
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X ✓ ✓
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5. Perspectives
Several ways to define the foundations of geometry
Several ways to define the foundations of geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski

- Analytic approach: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.

- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
  - Birkhoff

- Erlangen program: a geometry is defined as a space of objects and a group of transformations acting on it.
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Arithmetization of geometry

These approaches seem very different. In 1637, Descartes proved that the analytic approach can be derived from the synthetic approach. This is called arithmetization and coordination of geometry.

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Arithmetization of geometry

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A page from *La Géométrie* of Descartes
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Arithmetization of geometry

As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality, and thenceforth marched on at a rapid pace toward perfection.

(Joseph-Louis Lagrange, Lecons élémentaires sur les mathématiques; quoted by Morris Kline, Mathematical Thought from Ancient to Modern Times, p. 322)

First presented by Descartes, the arithmetization of geometry is the culminating result of both Hilbert’s and Tarski’s developments.

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Alfred Tarski (1901 - 1983)
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2. Tarski’s system of geometry
3. Parallel postulates
4. Arithmetization of geometry
   - Construction of an ordered field
   - Automated proofs of algebraic characterization
5. Perspectives
Outline

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Arithmetic operations

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Arithmetic operations

To define the operations we need three points: 1 point defines the neutral element of the addition; 1 point defines the neutral element of the multiplication; these 2 points define the line on which we will define the operations; 1 to define a line needed for the ruler and compass constructions.

Definition
Ar2 O E E' A B C :=

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To define the operations we need three points:
Arithmetic operations

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These properties are summarized as:
Arithmetic operations

To define the operations we need three points

- 1 point defines the neutral element of the addition
- 1 point defines the neutral element of the multiplication
- These 2 points define the line on which we will define the operations
- 1 to define a line needed for the ruler and compass constructions

These properties are summarized as:

\[
\text{Definition } \text{Ar2 } O E E' A B C := \\
\sim \text{Col } O E E' \land \text{Col } O E A \land \text{Col } O E B \land \text{Col } O E C.
\]
Addition (a first approach)
Addition (a first approach)

Let us prolong $OB$ but the length of $OA$. But: this does not work for negative points. Therefore, we need to be able to handle the negative points.
Let us prolong $\overline{OB}$ but the length of $\overline{OA}$.
Addition (a first approach)

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Addition

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Addition

Definition: Sum O E E' A B C :=
Ar2 O E E' A B C /
exists A', exists C',
Pj E E' A A' /
Col O E' A' /
Pj O E A' C' /
Pj O E' B C' /
Pj E' E C' C.

Properties of parallelograms to prove properties about Sum.
Addition

Originally from Descartes.
Originally from Descartes.

Definition $\text{Sum } O \ E \ E' \ A \ B \ C :=$

$\text{Ar2 } O \ E \ E' \ A \ B \ C /\$

exists $A'$, exists $C'$,

$\text{Pj } E \ E' \ A \ A' /\ \text{Col } O \ E' \ A' /\$

$\text{Pj } O \ E \ A' \ C' /\ \text{Pj } O \ E' \ B \ C' /\$

$\text{Pj } E' \ E \ C' \ C$. 
Addition

Originally from Descartes.

Definition \text{Sum} O E E' A B C :=
\text{Ar}_2 O E E' A B C \land
\text{exists } A', \text{ exists } C',
\text{Pj} E E' A A' \land \text{Col} O E' A' \land
\text{Pj} O E A' C' \land \text{Pj} O E' B C' \land
\text{Pj} E' E C' C.
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Pj O E A’ C’ /
Pj O E’ B C’ /
Pj E’ E C’ C.
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Addition

Originally from Descartes.

Definition $\text{Sum } O \ E \ E' \ A \ B \ C :=$
\[
\text{Ar2 } O \ E \ E' \ A \ B \ C /\
\exists A', \exists C', \
\text{Pj } E \ E' \ A \ A' /\ \text{Col } O \ E' \ A' /\ 
\text{Pj } O \ E \ A' \ C' /\ \text{Pj } O \ E' \ B \ C' /\ 
\text{Pj } E' \ E \ C' \ C.
\]
Addition

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Definition Sum $O E E' A B C :=$

\[
\text{Ar2 } O E E' A B C \land \\
\text{exists } A', \text{exists } C', \\
Pj E E' A A' \land \text{Col } O E' A' \land \\
Pj O E A' C' \land Pj O E' B C' \land \\
Pj E' E C' C.
\]
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Definition Sum O E E' A B C :=
   Ar2 O E E' A B C /
   exists A', exists C',
   Pj E E' A A' /
   Col O E' A' /
   Pj O E A' C' /
   Pj O E' B C' /
   Pj E' E C' C.
Addition

Originally from Descartes.

Definition \[ \text{Sum } O \ E \ E' \ A \ B \ C := \]
\[ \text{Ar2 } O \ E \ E' \ A \ B \ C /\]
\[ \text{exists A', exists C',}
\[ \text{Pj } E \ E' \ A \ A'/\text{Col } O \ E' \ A'/\]
\[ \text{Pj } O \ E \ A' \ C'/\text{Pj } O \ E' \ B \ C'/\]
\[ \text{Pj } E' \ E \ C' \ C. \]
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$\text{Pj } O E A' C' \land \text{Pj } O E' B C' \land$

$\text{Pj } E' E C' C.$
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Definition: \( \text{Sum} \ O \ E \ E' \ A \ B \ C := \)
\[
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\exists A', \exists C',
\text{Pj} \ E \ E' \ A \ A' \ /
\text{Col} \ O \ E' \ A' \ /
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Originally from Descartes.

Definition \( \text{Sum} \ O \ E \ E' \ A \ B \ C := \)
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exists \( A' \), exists \( C' \),
\[ \text{Pj} \ E \ E' \ A \ A' \ \land \ \text{Col} \ O \ E' \ A' \ \land \]
\[ \text{Pj} \ O \ E \ A' \ C' \ \land \ \text{Pj} \ O \ E' \ B \ C' \ \land \]
\[ \text{Pj} \ E' \ E \ C' \ C. \]

Properties of parallelograms to prove properties about \( \text{Sum} \).
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Multiplication

Originally from Descartes.

Definition Prod O E E' A B C :=
Ar2 O E E' A B C /
exists B',
Pj E E' B B' /
Col O E' B' /

Using Pappus theorem, we proved the
commutativity of Prod
and, using
Desargues theorem, its associativity.

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Originally from Descartes.
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Definition Prod O E E’ A B C :=
Ar2 O E E’ A B C /
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Pj E E’ B B’ /
Col O E’ B’ /
Pj E’ A B’ C.
Multiplication

Originally from Descartes.

Definition Prod $O E E' A B C :=$

$\text{Ar2 } O E E' A B C \land \exists B'$,

$\text{Pj } E E' B B' \land \text{Col } O E' B' \land$

$\text{Pj } E' A B' C.$
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Multiplication

Originally from Descartes.

Definition Prod O E E’ A B C :=
  Ar2 O E E’ A B C /\ exists B’,
  Pj E E’ B B’ /\ Col O E’ B’ /\
  Pj E’ A B’ C.
Multiplication

Originally from Descartes.

Definition Prod $0\ E\ E'\ A\ B\ C :=$

$$\text{Ar2}\ 0\ E\ E'\ A\ B\ C /\ \text{exists}\ B',$$
$$\text{Pj}\ E\ E'\ B\ B' /\ \text{Col}\ 0\ E'\ B' /\$$
$$\text{Pj}\ E'\ A\ B'\ C.$$
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Multiplication

Originally from Descartes.

Definition Prod 0 E E’ A B C :=
Ar2 0 E E’ A B C /\ exists B’,
Pj E E’ B B’ /\ Col 0 E’ B’ /\ Pj E’ A B’ C.

\[
\frac{OB}{OE} = \frac{OB'}{OE'}
\]

\[
\frac{OC}{OA} = \frac{OB'}{OE'}
\]
Multiplication

Originally from Descartes.

Definition \( \text{Prod} \ O \ E \ E' \ A \ B \ C := \)
\[
\text{Ar2} \ O \ E \ E' \ A \ B \ C \ \land \ \exists B',
\text{Pj} \ E \ E' \ B \ B' \ \land \ \text{Col} \ O \ E' \ B' \ \land
\text{Pj} \ E' \ A \ B' \ C.
\]

\[
\frac{OB}{OE} = \frac{OB'}{OE'}
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Using Pappus' theorem, we proved the commutativity of \( \text{Prod} \) and, using Desargues' theorem, its associativity.
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Definition $\text{Prod} \ O \ E \ E' \ A \ B \ C :=$
\begin{align*}
\text{Ar2} & \ O \ E \ E' \ A \ B \ C /\ \exists \ B', \\
\text{Pj} & \ E \ E' \ B \ B' /\ \text{Col} \ O \ E' \ B' /\ \\
\text{Pj} & \ E' \ A \ B' \ C.
\end{align*}

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Using **Pappus' theorem**, we proved the commutativity of $\text{Prod}$ and, using **Desargues' theorem**, its associativity.
From predicates to function symbols
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Problems linked to the use of predicates:
From predicates to function symbols

Problems linked to the use of predicates:

- Statements become quickly unreadable;
From predicates to function symbols

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Lemma `sum_assoc`: for all `O E E' A B C AB BC ABC`,

```
Lemma sum_assoc : forall O E E' A B C AB BC ABC,
    Sum O E E' A B AB ->
    Sum O E E' B C BC ->
    (Sum O E E' A BC ABC <-> Sum O E E' AB C ABC).
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From predicates to function symbols

Problems linked to the use of predicates:

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Lemma sum_assoc : forall O E E' A B C AB BC ABC,
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- We cannot apply the standard Coq tactics ring and field.

Axiom constructive_definite_description :
forall (A : Type) (P : A->Prop),
(exists! x, P x) -> { x : A | P x }.

However, this axiom turns the intuitionistic logic of Coq into an almost classical logic.
Problems linked to the use of predicates:

- Statements become quickly unreadable;

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Lemma sum_assoc : forall O E E' A B C AB BC ABC,
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- We cannot apply the standard Coq tactics ring and field.

We used an axiom which turns a relation which has been proved to be functional into a proper Coq function.
From predicates to function symbols

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- Statements become quickly unreadable;

```
Lemma sum_assoc : forall O E E' A B C AB BC ABC,
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```

However, this axiom turns the intuitionistic logic of Coq into an almost classical logic.
Total functions

Definition: Sum of E E' A B C :=
Ar^2 O E E' A B C /\ exists A', exists C',
Pj E E' A A' /\ Col O E' A' /\ Pj O E A' C' /
Pj O E' B C' /\ Pj E' E C' C.

Nothing but total functions are allowed in Coq. Therefore we defined a dependent type to represent the points belonging to the ruler.

Definition: F : Type := {P: Tpoint | Col O E P}.

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Total functions

- The function are only defined for points which belong to our ruler.
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Definition Sum O E E’ A B C :=
Ar2 O E E’ A B C /
exists A’, exists C’,
Pj E E’ A A’ /
Col O E’ A’ /
Pj O E A’ C’ /
Pj O E’ B C’ /
Pj E’ E C’ C.
The function are only defined for points which belong to our ruler.

Definition Sum O E E' A B C :=
Ar2 O E E' A B C /
exists A', exists C',
Pj E E' A A' /
Col O E A' /
Pj O E' B C' /
Pj E' E C' C.

Nothing but total functions are allowed in Coq.
Total functions

- The function are only defined for points which belong to our ruler.

Definition Sum O E E’ A B C :=
Ar2 O E E’ A B C /
exists A’, exists C’,
Pj E E’ A A’ /
Col O E A’ /
Pj O E A’ C’ /
Pj O E’ B C’ /
Pj E’ E C’ C.

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- Therefore we defined a dependent type to represent the points belonging to the ruler.
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An ordered field

We proved some lemmas asserting that the operations are morphisms relative to our defined equality. For example, the lemma asserting that if $A = A'$ and $B = B'$ implies $A + B = A' + B'$ is defined in Coq as:

```
Global Instance addF_morphism :
  Proper (EqF ==> EqF ==> EqF) AddF.
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Finally, we can prove we have a field:

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Lemma fieldF :
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Outline

1. Introduction
2. Tarski’s system of geometry
3. Parallel postulates
4. Arithmetization of geometry
   - Construction of an ordered field
   - Automated proofs of algebraic characterization
5. Perspectives
Characterization of the predicates of the theory

Formal Proofs in Tarski’s System of Geometry
Characterization of the predicates of the theory

We formalized the characterizations of the predicates of the theory.
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\text{Lemma characterization_of_congruence_F : forall A B C D,} \\
\text{Cong A B C D <->} \\
\text{let (Ac, HA) := coordinates_of_point_F A in let (Ax,Ay) := Ac in} \\
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\text{let (Dc, HD) := coordinates_of_point_F D in let (Dx,Dy) := Dc in} \\
\text{(Ax - Bx) * (Ax - Bx) + (Ay - By) * (Ay - By) -} \\
\text{((Cx - Dx) * (Cx - Dx) + (Cy - Dy) * (Cy - Dy)) =F= 0.}
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coordinates_of_point_F is a one-to-one correspondence between the pairs of points on the ruler representing the coordinates and the points of the plane.
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This was proved using the first synthetic and formal proofs of the \textit{intercept} and \textit{Pythagoras’} theorems.
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A bootstrapping approach
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• Wu’s approach: prove manually the characterizations **then**
  use these characterizations to obtain theorems automatically.
A bootstrapping approach

- Wu’s approach: prove manually the characterizations *then* use these characterizations to obtain theorems automatically.
- Our approach: prove manually only the first three characterizations and obtain *automatically* the others.
Automated proofs of characterizations

We used the Gröbner basis method to prove new characterizations from already proven ones. For example, to characterize the parallelism, we did not use its definition, namely:

\[
\text{Definition Par}\_\text{stric}t\ A\ B\ C\ D := \\
A\neq B \land C\neq D \land \text{Coplanar} A\ B\ C\ D /\neg \exists X, \text{Col}\ X\ A\ B \land \text{Col}\ X\ C\ D.
\]

\[
\text{Definition Par}\ A\ B\ C\ D := \\
\text{Par}\_\text{stric}t\ A\ B\ C\ D \lor (A\neq B \land C\neq D \land \text{Col}\ A\ C\ D \land \text{Col}\ B\ C\ D).
\]

But an equivalent statement:

\[
\text{Lemma characterization\_of\_parallelism\_F\_aux} : \\
\forall A\ B\ C\ D, \\
\text{Par}\ A\ B\ C\ D \iff A\neq B \land C\neq D \land \exists P, \text{Midpoint}\ C\ A\ P \land \exists Q, \text{Midpoint}\ Q\ B\ P \land \text{Col}\ C\ D\ Q.
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Pierre Boutry

Formal Proofs in Tarski's System of Geometry
Automated proofs of characterizations

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\text{Definition Par}_\text{strict} \ A \ B \ C \ D := \not(\text{Col} \ X \ A \ B) \lor \not(\text{Col} \ X \ C \ D).
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But an equivalent statement:

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\end{array}
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<td>$AB \equiv CD$</td>
<td>$(x_A - x_B)^2 + (y_A - y_B)^2 - (x_C - x_D)^2 + (y_C - y_D)^2 = 0$</td>
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| $A\parallel B\parallel C$ | $\exists t, 0 \leq t \leq 1 \land \begin{align*} t(x_C - x_A) &= x_B - x_A \\
 t(y_C - y_A) &= y_B - y_A \end{align*}$ |
| Col $ABCD$          | $(x_A - x_B)(y_B - y_C) - (y_A - y_B)(x_B - x_C) = 0$ |
| $A\parallel I\parallel B$ | $2x_I - (x_A + x_B) = 0 \land 2y_I - (y_A + y_B) = 0$ |
| $\triangle ABD$     | $(x_A - x_B)(x_B - x_C) + (y_A - y_B)(y_B - y_C) = 0$ |
| $AB \parallel CD$  | $(x_A - x_B)(x_C - x_D) + (y_A - y_B)(y_C - y_D) = 0 \land (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \land (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$ |
| $AB \perp CD$       | $(x_A - x_B)(y_C - y_D) - (y_A - y_B)(x_C - x_D) = 0 \land (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \land (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$ |
## Automated proofs of characterizations

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We first proved the characterization of the midpoint predicate manually and then automatically and the script of the proof by computation was eight times shorter than our original one.
An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

- The midpoints of each side of the triangle;
- The feet of each altitude;
- The midpoints of the line-segments from each vertex of the triangle to the orthocenter.
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Lemma nine_point_circle:
forall A B C A1 B1 C1 A2 B2 C2 A3 B3 C3 H O,
~ Col A B C ->
Col A B C2 -> Col B C A2 -> Col A C B2 ->
Perp A B C C2 -> Perp B C A A2 -> Perp A C B B2 ->
Perp A B C2 H -> Perp B C A A2 H -> Perp A C B2 H ->
Midpoint A3 A H -> Midpoint B3 B H -> Midpoint C3 C H ->
Midpoint C1 A B -> Midpoint A1 B C -> Midpoint B1 C A ->
Cong O A1 O B1 -> Cong O A1 O C1 ->
Cong 0 A2 0 A1 /\ Cong 0 B2 0 A1 /\ Cong 0 C2 0 A1 /\ Cong 0 A3 0 A1 /\ Cong 0 B3 0 A1 /\ Cong 0 C3 0 A1.

We did not prove a theorem about polynomials but a geometric statement. The nine-point circle theorem is true in any model of Tarski’s Euclidean geometry axioms (without continuity) and not only in a specific one.
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Midpoint C1 A B -> Midpoint A1 B C -> Midpoint B1 C A ->
Cong O A1 O B1 -> Cong O A1 O C1 ->
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Formal Proofs in Tarski's System of Geometry
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- Extend our formalization of geometry to **higher dimension** geometry.
Thank you!