Random Walks in the Quarter-Plane: Explicit Criterions for the Finiteness of the Associated Group in the Genus 1 Case

Guy Fayolle * Roudolf Iasnogorodski†

*INRIA - Domaine de Voluceau, Rocquencourt - BP 105 - 78153 Le Chesnay - France
†Saint-Petersbourg - Russia.

Introduction

- Piecewise homogeneous random walk with sample paths in $\mathbb{Z}_+^2$, the lattice in the positive quarter plane. In the strict interior of $\mathbb{Z}_+^2$, the size of the jumps is 1, and $\{p_{ij}, |i|, |j| \leq 1\}$ will denote the generator of the process for this region. Thus a transition $(m, n) \rightarrow (m + i, n + j), m, n > 0,$ can take place with probability $p_{ij}$, and

$$\sum_{|i|,|j|\leq1} p_{ij} = 1.$$ 

- No strong assumption about the boundedness of the upward jumps on the axes, neither at $(0, 0)$. In addition, the downward jumps on the $x$ [resp. $y$] axis are bounded by $L$ [resp. $M$], where $L$ and $M$ are arbitrary finite integers.

- Original question : Find an explicit form for the invariant measure of such process.
The basic functional equation

The invariant measure \( \{ \pi_{i,j}, i, j \geq 0 \} \) satisfies the fundamental bivariate functional equation

\[
Q(x, y) \pi(x, y) = q(x, y) \pi(x) + \tilde{q}(x, y) \tilde{\pi}(y) + \pi_0(x, y), \tag{1}
\]

where in (1) the unknown functions \( \pi(x, y), \pi(x), \tilde{\pi}(y) \) are sought to be analytic in the region \( \{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1 \} \), and continuous on their respective boundaries.
$$\pi(x, y) = \sum_{i,j \geq 1} \pi_{ij} x^{i-1} y^{j-1},$$
$$\pi(x) = \sum_{i \geq L} \pi_{i0} x^{i-L}, \quad \tilde{\pi}(y) = \sum_{j \geq M} \pi_{0j} y^{j-M},$$
$$Q(x, y) = xy \left[ 1 - \sum_{i, j \in S} p_{ij} x^i y^j \right], \quad \sum_{i, j \in S} p_{ij} = 1,$$
$$q(x, y) = x^L \left[ \sum_{i \geq -L, j \geq 0} p'_{ij} x^i y^j - 1 \right] \equiv x^L (P_{L0}(x, y) - 1),$$
$$\tilde{q}(x, y) = y^M \left[ \sum_{i \geq 0, j \geq -M} p''_{ij} x^i y^j - 1 \right] \equiv y^M (P_{0M}(x, y) - 1),$$
$$\pi_0(x, y) = \sum_{i=1}^{L-1} \pi_{i0} x^i [P_{i0}(x, y) - 1] + \sum_{j=1}^{M-1} \pi_{0j} y^j [P_{0j}(x, y) - 1] + \pi_{00} (P_{00}(xy) - 1).$$

$S$ is the set of allowed jumps, and $q, \tilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).
Group and Genus

The function $Q(x, y)$, often referred to as the kernel of (1), can be rewritten in the two following equivalent forms

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \quad (2)$$

$$a(x) = p_{1,1}x^2 + p_{0,1}x + p_{-1,1} \quad \tilde{a}(y) = p_{1,1}y^2 + p_{1,0}y + p_{1,-1},$$
$$b(x) = p_{1,0}x^2 + (p_{0,0} - 1)x + p_{-1,0} \quad \tilde{b}(y) = p_{0,1}y^2 + (p_{0,0} - 1)y + p_{0,-1},$$
$$c(x) = p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1} \quad \tilde{c}(y) = p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}.$$  

We shall also need the discriminants

$$D(x) \overset{\text{def}}{=} b^2(x) - 4a(x)c(x), \quad \tilde{D}(y) \overset{\text{def}}{=} \tilde{b}^2(y) - 4\tilde{a}(y)\tilde{c}(y). \quad (3)$$

The polynomials $D$ and $\tilde{D}$ are of degree 4, respectively in $x$ and $y$. 

Let $\mathbb{C}(x), \mathbb{C}(y)$ and $\mathbb{C}(x, y)$ denote the respective fields of rational functions of $x, y$ and $(x, y)$ over $\mathbb{C}$. Since in general $Q$ is assumed to be irreducible, the quotient field $\mathbb{C}(x, y)$ with respect to $Q$ will be denoted by $\mathbb{C}_Q(x, y)$.

**Definition 1.** The group of the random walk is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x, y)$ generated by $\xi$ and $\eta$ given by

$$
\xi(x, y) = \left( x, \frac{c(x)}{ya(x)} \right), \quad \eta(x, y) = \left( \frac{\tilde{c}(y)}{xa(\tilde{y})}, y \right).
$$

Here $\xi$ and $\eta$ are involutions satisfying $\xi^2 = \eta^2 = I$.

**Lemma 2.** Let

$$
\delta \overset{\text{def}}{=} \eta \xi.
$$

Then $\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{ \delta^n, n \in \mathbb{Z} \}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. $\blacksquare$

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• The group $\mathcal{H}$ is finite of order $2n$ if, and only if,

$$\delta^n = I.$$  \hfill (5)

• The product $\delta = \eta \xi$ is non-commutative, except for $\delta^2 = I$, in which case the group is of order 4.

• We shall write $f_\alpha = \alpha(f)$, for any automorphism $\alpha \in \mathcal{H}$ and any function $f \in \mathbb{C}_Q(x, y)$.

• The fundamental equation (1), together with $\xi, \eta, \delta$, can be lifted onto the universal covering $\mathbb{C}$ (the finite complex plane).
Let \( \{x_\ell\}_{1 \leq \ell \leq 4} \) be the 4 roots of the discriminant \( D(x) \) [see equation (3)], which are the branch points of the Riemann surface

\[
\mathcal{K} = \{ (x, y) \in \mathbb{C}^2 : Q(x, y) = 0 \}.
\]

They are always real, with \( |x_1| \leq |x_2| \leq |x_3| \leq |x_4| \).

Moreover \( x_1 \leq x_2, \ [x_1x_2] \subset [-1, +1] \) and \( 0 \leq x_2 \leq x_3 \).

Here \( \mathcal{K} \) is assumed to be of genus 1 (the torus), so that the algebraic curve \( Q(x, y) = 0 \) admits a uniformization given in terms of the Weierstrass \( \wp \) function with periods \( \omega_1, \omega_2 \) and its derivatives. Indeed, letting

\[
D(x) = b^2(x) - 4a(x)c(x) \overset{\text{def}}{=} d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0,
\]

\[
z \overset{\text{def}}{=} 2a(x)y + b(x),
\]

the following formulae hold (see the Yellow Book [FIM]).
1. If $d_4 \neq 0$ (4 finite branch points $x_1, \ldots, x_4$) then $D'(x_4) > 0$ and

$$\begin{align*}
  x(\omega) &= x_4 + \frac{D'(x_4)}{\varphi(\omega) - \frac{1}{6}D''(x_4)}, \\
  z(\omega) &= \frac{D'(x_4)\varphi'(\omega)}{2 \left( \varphi(\omega) - \frac{1}{6}D''(x_4) \right)^2}.
\end{align*}$$

(6)

2. If $d_4 = 0$ (3 finite branch points $x_1, x_2, x_3$ and $x_4 = \infty$) then

$$\begin{align*}
  x(\omega) &= \frac{\varphi(\omega) - \frac{d_2}{3}}{d_3}, \\
  z(\omega) &= -\frac{\varphi'(\omega)}{2d_3}.
\end{align*}$$
\[ \omega_1 = 2i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-D(x)}}, \quad \omega_2 = 2 \int_{x_2}^{x_3} \frac{dx}{\sqrt{D(x)}}, \quad \omega_3 = 2 \int_{x(y_1)}^{x_1} \frac{dx}{\sqrt{D(x)}}. \]

\( \omega_1 \) is purely imaginary, while \( 0 < \omega_3 < \omega_2 \).

- It was proved in [FIM] that the group \( \mathcal{H} \) is finite of order \( 2n \) if and only if

\[ n\omega_3 = 0 \mod (\omega_1, \omega_2), \]

or, since \( \omega_3 \) is real,

\[ n\omega_3 = 0 \mod (\omega_2), \]

(7)

where \( n \) stands for the minimal positive integer with this property.
On the *universal covering* \( \mathbb{C} \) (the finite complex plane), the automorphisms \( \xi, \eta, \delta \) become (see [FIM], Section 3.3)

\[
\xi^*(\omega) = -\omega + \omega_2,
\eta^*(\omega) = -\omega + \omega_2 + \omega_3,
\delta^*(\omega) = \eta^*\xi^* = \omega + \omega_3. \quad (8)
\]

Here \( \delta = \eta\xi \) corresponds to \( \delta^* = \eta^*\xi^* \). Thus, for any \( f(x, y) \in C_Q(x, y) \),

\[
\delta(f(x, y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \quad \omega \in \mathbb{C}.
\]

In particular,

\[
\begin{cases}
\delta(x) = x(\delta^*(\omega)) = x(\omega + \omega_3), \\
\eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3).
\end{cases} \quad (9)
\]
$\mathcal{H}$ is generated by the elements $\xi$ and $\eta$, and we can define the homomorphism

$$h(R(x, y)) \overset{\text{def}}{=} R(h(x), h(y)), \quad \forall h \in \mathcal{H}, \forall R \in \mathcal{C}_Q(x, y).$$

For any $R \in \mathcal{C}_Q(x, y)$, the following equivalences hold:

$${\begin{cases}  
\xi(R) = R & \iff R \in \mathcal{C}(x), \\
\eta(R) = R & \iff R \in \mathcal{C}(y),
\end{cases}}$$

so that $\mathcal{C}(x)$ (resp. $\mathcal{C}(y)$) is the set of elements of $\mathcal{C}_Q(x, y)$ invariant with respect to $\xi$ (resp. $\eta$). Indeed, $R$ has the general form

$$R(x, y) = A(x) + B(x)y \mod Q(x, y),$$

where $A(x)$ and $B(x)$ are elements of $\mathcal{C}(x)$. [Hint: $\xi(R) = R$ and $\xi(y) \neq y$, so that necessarily $B(x) \equiv 0$.]

Introduce the matrix
\[
\mathbb{P} = \begin{pmatrix}
p_{11} & p_{10} & p_{1,-1} \\
p_{01} & p_{00} - 1 & p_{0,-1} \\
p_{-1,1} & p_{-1,0} & p_{-1,-1}
\end{pmatrix},
\]
and let \( \vec{C}_1, \vec{C}_2, \vec{C}_3 \) (resp. \( \vec{D}_1, \vec{D}_2, \vec{D}_3 \)) denote the column vectors of \( \mathbb{P} \) (resp. of \( \mathbb{P}^T \), the transpose matrix of \( \mathbb{P} \)).

**Proposition 3.** Assume there exists a positive integer \( s \) such that
\[
\delta^s(x) = x.
\]
Then \( \delta^s = I \) and the group is of order \( 2s \), where \( s \) stands for the smallest integer with property (12).

**Sketch of proof.** Each of the three following permutations

\[
x \iff y, \quad \delta \iff \delta^{-1}, \quad \mathbb{P} \iff \mathbb{P}^T,
\]
implies the two other ones.
Hence, the quantity $\rho(x, y, k) \overset{\text{def}}{=} \delta^k(x) \cdot \delta^{-k}(y)$, for any integer $k \geq 1$, remains invariant by permuting $\mathbb{P}$ with $\mathbb{P}^T$.

- Assume first $s = 2m$. Then (12) becomes $\delta^m(x) = \delta^{-m}(x)$, and

$$\rho(x, y, m) = \delta^{-m}(x) \cdot \delta^{-m}(y) = \delta^m(x) \cdot \delta^m(y),$$

where the second equality is obtained by replacing $\mathbb{P}$ by $\mathbb{P}^T$. Then, comparing with the definition of $\rho(x, y, m)$, we get $\delta^m(y) = \delta^{-m}(y)$, which yields in turn $\delta^{s}(y) = y$, whence $\delta^{s} = I$.

- If $s$ is odd, say $s = 2m + 1$, the argument works in exactly the same way. In this case

$$\rho(x, y, m) = \delta^{-(m+1)}(x) \cdot \delta^{-m}(y) = \delta^{m}(x) \cdot \delta^{m+1}(y),$$

(by exchanging again $\mathbb{P}$ with $\mathbb{P}^T$), which implies $\delta^{m+1}(y) = \delta^{-m}(y)$, that is $\delta^{s}(y) = y$, QED.
Corollary 4.

1. If there exists an integer $s$ such that $\delta^s(x) = r(x)$, where $r(x)$ represents a rational fraction of $x$, then $\delta^{2s}(x) = x$ and the group is of order $4s$.

2. If there exists an integer $s$ such that $\delta^s(x) = t(y)$, where $t(y)$ represents a rational fraction of $y$, then $\delta^{2s-1}(x) = x$ and the group is of order $4s - 2$.

In both cases, $s$ stands for the smallest integer with the corresponding property.

Proof. Note first the identities $\xi \delta^s \xi = \delta^{-s}$ and $\eta \delta^s \xi = \delta^{-s+1}$.

So, the following chain of equalities holds.

$\delta^s(x) = r(x) \implies \xi \delta^s \xi(x) = \delta^s(x) \iff \delta^{-s}(x) = \delta^s(x) \iff \delta^{2s}(x) = x.$

Similarly

$\delta^s(x) = t(y) \implies \eta \delta^s \xi(x) = \delta^s(x) \iff \delta^{-s+1}(x) = \delta^s(x) \iff \delta^{2s-1}(x) = x.$

$\implies$ In both cases, the conclusion follows from Proposition 3, in which $s$ is replaced respectively by $2s$ and $2s - 1$. ■
Lemma 5. On the algebraic curve $\{Q(x, y) = 0\}$, the following general relations hold:

$$
\begin{align*}
\eta(x) &= \frac{xv(y) - u(y)}{xw(y) - v(y)}, \\
\xi(y) &= \frac{y\tilde{v}(x) - \tilde{u}(x)}{y\tilde{w}(x) - \tilde{v}(x)},
\end{align*}
$$

(13)

where $u, v, w, h$ (resp. $\tilde{v}, \tilde{u}, \tilde{w}, \tilde{h}$) are polynomials of degree $\leq 2$. In particular, there exist affine solutions

$$
(u(y), v(y), w(y))^T = \vec{A} y + \vec{B}, \quad (\tilde{u}(x), \tilde{v}(x), \tilde{w}(x))^T = \vec{E} x + \vec{F},
$$

(14)

with column vectors

$$
\vec{A} = (u_0, v_0, w_0)^T, \quad \vec{B} = (u_1, v_1, w_1)^T, \quad \vec{E} = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0)^T, \quad \vec{F} = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)^T.
$$
\[
\begin{align*}
\vec{A} &= (\alpha \vec{C}_2 + \beta \vec{C}_1) \times \vec{C}_3, \\
\vec{B} &= \vec{C}_1 \times (\alpha \vec{C}_3 + \beta \vec{C}_2), \\
\vec{E} &= (\tilde{\alpha} \vec{D}_2 + \tilde{\beta} \vec{D}_1) \times \vec{D}_3, \\
\vec{F} &= \vec{D}_1 \times (\tilde{\alpha} \vec{D}_3 + \tilde{\beta} \vec{D}_2),
\end{align*}
\]

(15)

where \(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}\) are arbitrary complex constants, and the operator “\(\times\)” stands for the cross vector product. In addition, when \(\mathbb{P}\) is of rank 3, none of the vectors \(\vec{A}, \vec{B}, \vec{E}, \vec{F}\) vanish. Choosing in (15) \(\alpha = \tilde{\alpha} = 0, \beta = \tilde{\beta} = 1\), gives

\[
\begin{align*}
\ u(y) &= y \Delta_{13} - \Delta_{12}, \\
\ \tilde{u}(x) &= x \Delta_{31} - \Delta_{21}, \\
\ v(y) &= y \Delta_{23} - \Delta_{22}, \\
\ \tilde{v}(x) &= x \Delta_{32} - \Delta_{22}, \\
\ w(y) &= y \Delta_{33} - \Delta_{32}, \\
\ \tilde{w}(x) &= x \Delta_{33} - \Delta_{32},
\end{align*}
\]

(16)

where \(\Delta_{ij}\) denotes the cofactor of the \((i, j)\)th entry of the matrix \(\mathbb{P}\) given in (11).
Lemma 6. Let $\gamma$ be an endomorphism defined on the algebraic surface $\mathcal{H}$, which is assumed to be invariant on the field $\mathbb{C}(x)$ of rational functions of $x$, and such that

$$\gamma(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)},$$

(17)

where $e, f, g, h$ are polynomials of degree 1 in $x$. (Note that this is always possible, as shown in Lemma 5.) Then, for $\gamma$ to be an involution, the condition $f(x) + h(x) \equiv 0$ is necessary and sufficient.

Remark. The result of Lemma 6 does not hold if polynomials $e, f, h$ are not of degree 1. For instance, one can check directly that, if $g$ or $e$ are taken to be of degree 2, then any involution $\gamma$ necessarily has the form

$$\gamma(y) = \frac{(h - b)y - c}{ay + h}.$$
Groups of Order 4

Proposition 7. The group $\mathcal{H}$ is of order 4 if, and only if,

$$\begin{vmatrix}
p_{11} & p_{10} & p_{1,-1} \\
p_{01} & p_{00} - 1 & p_{0,-1} \\
p_{-1,1} & p_{-1,0} & p_{-1,-1}
\end{vmatrix} = 0,$$

(18)

and this is the only case where the matrix $\mathbb{P}$ has rank 2.

Proof. The equality $\delta^2 = I$ can be rewritten as $\xi\eta = \eta\xi$, which by Proposition 3 is, for instance, equivalent to

$$\xi\eta(x) = \eta(x),$$

where we have used $\xi(x) = x$. So, $\eta(x)$ is left invariant by $\xi$, which implies

$$\eta(x) \in \mathbb{C}(x).$$
Finally, $\eta$ is both an involution and a conformal automorphism on $\mathbb{C}(x)$. Consequently, $\eta$ is a fractional linear transform of the type

$$\eta(x) = \frac{rx + s}{tx - r},$$

where all coefficients belong to $\mathbb{C}$. The following chain of equivalences hold.

$$\eta(x) = \frac{rx + s}{tx - r} \iff tx.\eta(x) = r(x + \eta(x)) + s$$

$$\iff 1, x + \eta(x), x.\eta(x) \text{ are linearly dependent on } \mathbb{C}$$

$$\iff 1, -\frac{\tilde{b}(y)}{\tilde{a}(y)}, \frac{\tilde{c}(y)}{\tilde{a}(y)} \text{ are linearly dependent on } \mathbb{C}$$

$$\iff \tilde{a}(y), \tilde{b}(y), \tilde{c}(y) \text{ are also linearly dependent on } \mathbb{C},$$

where equation (2) has been used in the form

$$Q(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y).$$
Groups of Order 6

Proposition 8. $\mathcal{H}$ is of order 6 if, and only if,

$$\begin{vmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{vmatrix} = 0,$$

(19)

where the $\Delta_{ij}$'s have be given in Lemma 5.

Sketch of proof. In this case $(\xi\eta)^3 = I$, which is equivalent to

$$\eta\xi\eta = \xi\eta\xi.$$  

(20)
Applying (20) for instance to $x$, we get

$$\xi \eta(x) = \eta \xi \eta(x),$$

which shows that $\xi \eta(x)$ is invariant with respect to $\eta$.

Similarly, $\eta \xi(y)$ is invariant with respect to $\xi$. Hence (20) is plainly equivalent to

$$\begin{cases} 
\xi \eta(x) = P(y), \\
\eta \xi(y) = R(x),
\end{cases}$$

where $P$ and $R$ are rational. Then

$$y = R(\xi \eta(x)) = R \circ P(y),$$

or, equivalently,

$$R \circ P = I,$$

(21)

so that $P$ and $R$ are fractional linear transforms.
Thus (21) yields the relation

\[ \xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s}. \] (22)

Hence, there is a linear dependence on \( \mathbb{C} \) between the 4 elements 1, \( \xi(y) \), \( \eta(x) \), \( \xi(y)\eta(x) \), with 4 unknown constants (in fact three by homogeneity). Starting from equation (13), we choose \( \eta(x) \) by means of (16),

\[ \eta(x) = \frac{y(x\Delta_{23} - \Delta_{13}) - x\Delta_{22} + \Delta_{12}}{y(x\Delta_{33} - \Delta_{23}) - x\Delta_{32} + \Delta_{22}}. \] (23)

Instantiating now (23) in (22), we obtain

\[ \xi(y) = \frac{y[p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23})] + p(\Delta_{12} - x\Delta_{22}) + q(\Delta_{22} - x\Delta_{32})}{y[r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23})] + r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32})}. \] (24)
Then, according to (17),

\[ \xi(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)}, \]

where

\[
\begin{align*}
    e(x) &= p(x\Delta_{22} - \Delta_{12}) + q(x\Delta_{32} - \Delta_{22}), \\
    f(x) &= p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23}), \\
    g(x) &= r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23}), \\
    h(x) &= r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32}),
\end{align*}
\]

(25)

and we can compare system (25) with the solution presented in equation (15) of Lemma 5.

The final step is to analyze the feasibility of a global linear system formed of 8 equations with 6 unknown variables. . .
Criterion for Groups of Order $4m$

Proposition 9. The group $\mathcal{H}$ is of order $4m$ if and only if the Weierstrass $\wp$ function with periods $(\omega_1, \omega_2)$ satisfies the equation

$$\wp(m\omega_3) = \wp(\omega_2/2).$$  \hspace{1cm} (26)

Proof. Recalling that $\delta = \eta \xi$, we have here $\delta^{2m} = I$, that is

$$(\xi \eta)^m = (\eta \xi)^m.$$ \hspace{1cm} (27)

By applying equation (27) at $x$ (or even at an arbitrary element of $\mathbb{C}(x)$), and replacing $\xi(x)$ by $x$ in the right-hand side, we obtain

$$\xi \delta^m(x) = \delta^m(x),$$

showing that the involution $\delta^m(x)$ is invariant with respect to $\xi$. 

Hence $\delta^m(x)$ is an element of $\mathbb{C}(x)$, so that

$$\delta^m(x) = F(x) = \frac{xf - e}{xg - f},$$

(28)

where $F(x)$ is a simple fractional linear transform, with constants $e, f, g$, to be determined. Hence, equation (28) implies the existence of a linear dependence between the functions

$$x\delta^m(x), \ x + \delta^m(x), \ 1.$$  

(29)

Lemma 10. For the group to be of order $4m$, a necessary and sufficient condition is that the three functions

$$x(\omega - m\omega_3/2)x(\omega + m\omega_3/2), \ x(\omega - m\omega_3/2) + x(\omega + m\omega_3/2), \ 1,$$

(30)

be linearly dependent, $\forall \omega \in \mathbb{C}$.  

$\blacksquare$
Recall that \( \varphi'^2 = 4\varphi^3 - g_2\varphi - g_3 \), and let, for arbitrary \( u, v \),

\[
A(u, v) \overset{\text{def}}{=} \varphi(u + v) + \varphi(u - v), \quad B(u, v) \overset{\text{def}}{=} \varphi(u + v)\varphi(u - v)
\]

Then, setting for now \( X \overset{\text{def}}{=} \varphi(u), \ Y \overset{\text{def}}{=} \varphi(v) \), we have

** Lemma 11. **

\[
A(u, v) = \frac{(X + Y)(4XY - g_2) - 2g_3}{2(X - Y)^2}, \quad (31)
\]

\[
B(u, v) = \frac{(XY)^2 + \frac{g_2}{2}XY + g_3(X + Y) + \frac{g_2^2}{16}}{(X - Y)^2}. \quad (32)
\]
Let
\[ S(u, v) \overset{\text{def}}{=} x(u + v) + x(u - v), \quad P(u, v) \overset{\text{def}}{=} x(u + v)x(u - v). \quad (33) \]

Since
\[ x(\omega) = p + \frac{q}{\wp(\omega) - r}, \quad (34) \]
where \( p, q, r \) are known constants [see equation (6)], we have

\[
\begin{aligned}
S &= \frac{2pB + (q - 2pr)A + 2r(pr - q)}{B - rA + r^2}, \\
P &= \frac{p^2B + p(q - pr)A + (pr - q)^2}{B - rA + r^2}.
\end{aligned}
\quad (35)
\]

Taking \( u = \omega, v = m\omega_3/2 \), the claims involving (29) and (30) are merely equivalent to the existence of a non-trivial linear relation

\[ eS + fP + g = 0, \quad \forall X \in \mathbb{C}. \quad (36) \]

In other words \( S, P, 1 \), considered as functions of \( X \), are linearly dependent.
Here $Y = \wp(m\omega_3/2)$, and the independence condition reads

$$\Delta(Y) \stackrel{\text{def}}{=} \begin{vmatrix} 4Y & 4Y^2 - g_2 & -(g_2Y + 2g_3) \\ 2Y^2 & g_2Y + 2g_3 & 2g_3Y + g_2^2/8 \\ 1 & -2Y & Y^2 \end{vmatrix} = 0,$$

(37)

which yields exactly (26), by using the factorization of $\Delta(Y)$ as the product of 3 polynomials of degree 2 in $Y$.

However, the computation of $\wp(m\omega_3/2)$, via the recursive relationship

$$\wp((l + 1)\omega_3/2) + \wp((l - 1)\omega_3/2) =$$

$$\frac{(\wp(l\omega_3/2) + \wp(\omega_3/2))(4\wp(l\omega_3/2)\wp(\omega_3/2) - g_2) - 2g_3}{2(\wp(l\omega_3/2) - \wp(\omega_3/2))^2},$$

is hardly exploitable. . .
• **Case** $m = 2k$. Applying the operator $\delta^{-k}$ in (29) amounts to saying that

$$
\delta^k(x) \cdot \xi \delta^k(x), \ \delta^k(x) + \xi \delta^k(x), \ 1,
$$

are linearly dependent. But $\delta^k(x) \cdot \xi \delta^k(x)$ and $\delta^k(x) + \xi \delta^k(x)$ are elements of $\mathbb{C}(x)$, and by (34), (31), (32), they are in fact ratios of polynomials of degree 2 in $x$ with the same denominator.

In addition, letting $\zeta_j(x) \overset{\text{def}}{=} \delta^j(x) + \xi \delta^j(x)$, the following recursive scheme holds.

$$
\begin{cases}
\zeta_0(x) = 2x, & \zeta_1(x) = \delta^{-1}(x) + \delta(x), \\
\zeta_j(x) = \zeta_{j-1}(\zeta_1(x)) - \zeta_{j-2}(x), & \forall j \geq 2.
\end{cases}
$$

(39)

• **Case** $m = 2k - 1$. Upon applying here the operator $\delta^{-k+1}$ in (29) and using the identity $\delta^{-k+1}(x) = \eta \delta^k(x)$, we obtain that

$$
\delta^k(x) \cdot \eta \delta^k(x), \ \delta^k(x) + \eta \delta^k(x), \ 1
$$

are linearly dependent. Moreover, $\delta^k(x) \cdot \eta \delta^k(x)$ and $\delta^k(x) + \eta \delta^k(x)$ are elements of $\mathbb{C}(y)$, namely ratios of polynomials of degree 2 in $y$ with the same denominator.
Proposition 12. The group $\mathcal{H}$ is of order 8 if, and only if, the third order determinant

$$
\begin{vmatrix}
2 \Delta_{22} \Delta_{32} & 2 (\Delta_{22}^2 - \Delta_{12} \Delta_{31} + \Delta_{21} \Delta_{23}) & 2 \Delta_{12} \Delta_{22} \\
- (\Delta_{21} \Delta_{33} + \Delta_{31} \Delta_{23}) & + \Delta_{11} \Delta_{33} + \Delta_{31} \Delta_{13} & - (\Delta_{11} \Delta_{23} + \Delta_{21} \Delta_{13}) \\
\Delta_{32}^2 - \Delta_{31} \Delta_{33} & -2 \Delta_{32} \Delta_{22} + \Delta_{31} \Delta_{23} + \Delta_{21} \Delta_{33} & \Delta_{22}^2 - \Delta_{21} \Delta_{23} \\
\Delta_{22}^2 - \Delta_{21} \Delta_{23} & -2 \Delta_{22} \Delta_{12} + \Delta_{11} \Delta_{23} + \Delta_{13} \Delta_{21} & \Delta_{12}^2 - \Delta_{11} \Delta_{13}
\end{vmatrix}
$$

is equal to zero, where $\Delta_{ij}$ denotes the cofactor of the $(i, j)^{th}$ entry of the matrix $\mathcal{P}$ given in (11).

Example: Gessel’s walk.
Criterion for Groups of Order $4m - 2$

Here

$$\delta^{2m-1} = I,$$

which by Proposition 3 and Corollary 4 is equivalent to

$$\eta(\delta^m(x)) = \delta^m(x),$$

that is

$$\delta^m(x) = G(y) \in \mathbb{C}(y).$$

Similarly, upon applying (41) to $y$, we get

$$\delta^{-m}(y) = \delta^{m+1}(y) = \xi(\delta^{-m}(y)),$$

whence

$$\delta^{-m}(y) = F(x) \in \mathbb{C}(x).$$
Applying now $\delta^{-m}$ to both members of (42) yields

$$x = \delta^{-m}(G(y)) = G(\delta^{-m}(y)) = G \circ F(x),$$

which shows that $G \circ F = I$, and hence $G$ and $F$ are simple fractional linear transforms.

Setting for instance

$$G(y) = -\frac{py + q}{ry + s},$$

where $p, q, r, s$ are arbitrary complex constants, the problem is to achieve the linear relation

$$r \ y \delta^m(x) + s \ \delta^m(x) + py + q = 0 \mod Q(x, y), \quad (43)$$

which is necessary and sufficient for the group to be of order $4m - 2$. 
Final results

For the group to be finite, there is a unique condition tantamount to the cancellation of a determinant, the elements of which are intricate functions of the coefficients of the transition matrix $\mathbb{P}$, but nonetheless recursively computable.

- The determinant is of order 3, for groups of order $4m$, $m \geq 1$.

- The determinant is of order 4, for groups of order $4m - 2$, $m \geq 1$.

- The condition depends on the entries of the matrix $\mathbb{P}$ in a polynomial way, as shown in the next three theorems.
Theorem 13. For any integer \( s \geq 1 \), we have

\[
\delta^s(x) = \frac{y U_s(x) + V_s(x)}{W_s(x)} \mod Q(x, y),
\]  

(44)

where \( U_s, V_s, W_s \) are second degree polynomials.

Theorem 14. The finiteness of the group is always equivalent to the cancellation of a single constant, which depends on the entries of \( \mathbb{P} \) in a polynomial way. In other words, the group is finite if and only if the non-negative \( (p_{ij})'s \) belong to the intersection of some algebraic hypersurface with the hyperplane \( \sum p_{ij} = 1 \).

Theorem 15. For the group \( \mathcal{H} \) to be finite, the necessary and sufficient condition is \( \det(\Omega) = 0 \), where \( \Omega \) is a matrix of order 3 (resp. 4) when the group is of order \( 4m \) (resp. \( 4m+2 \)).
Sketch of proofs.

Let
\[ \delta(x) - \xi \delta(x) \stackrel{\text{def}}{=} 2H, \quad X = \wp(\omega), \ Y = \wp(s\omega_3). \]

Then
\[ \delta(x) = \frac{S(\omega, s\omega_3)}{2} + H, \]

with \( S(\omega, s\omega_3) \) given by (35).

\[
H = \frac{q[\wp(s\omega_3 - \omega) - \wp(s\omega_3 + \omega)]}{2[\wp(s\omega_3 + \omega) - r][\wp(s\omega_3 - \omega) - r]} = \frac{q \wp'(\omega) \wp'(s\omega_3)}{D(X, Y)} = \frac{2q^2 \wp'(s\omega_3)[2a(x)y + b(x)]}{(x - p)^2D(X, Y)}
\]

where, by using (31), (32), (35),

\[
D(X, Y) = 2(X - Y)^2(B - rA + r^2)
\]

is a polynomial of second degree in \( X \) and \( Y \).
On the other hand, by construction, we can a priori write

\[ \delta^s(x) = M_s(x)y + N_s(x) \mod Q(x, y), \]  

(45)

where \( M_s \) and \( N_s \) are rational fractions whose numerators and denominators are polynomials of (a priori) unknown degrees, but with coefficients given in terms of polynomials of the entries of \( P \). The decomposition (45) is unique, so that, comparing with (44), we have

\[
\begin{cases}
M_s(x) = \frac{4q^2 \varphi'(s\omega_3)a(x)}{W_s(x)}, \\
N_s(x) = \frac{V_s(x)}{W_s(x)},
\end{cases}
\]

where \( V_s, W_s \) are the second degree polynomials given by Theorem 13.
By homogeneity, we can always rewrite

\[ M_s(x) = \frac{A_s a(x)}{F_s(x)}, \]

where \( A_s, K_s \) are real constants with

\[ A_s = K_s [4q^2 \phi'(s\omega_3)], \quad F_s(x) = K_s W_s(x). \]

• By Corollary 4, the group is of order \( 4s \) if and only if \( M_s \equiv 0 \), that is \( A_s = 0 \), where now \( A_s \) depends only on the entries of \( \mathbb{P} \) in a complicated polynomial form. It is also equivalent to \( \phi'(s\omega_3) = 0 \).

• When the group is of order \( 4s + 2 \), exchange the role of \( x \) and \( y \) by uniformizing \( y(\omega) \). Then, \textit{mutatis mutandis}, this yields

\[ \delta^s(x) = \frac{\tilde{\tilde{A}}_s \tilde{\tilde{a}}(y)x + \tilde{\tilde{V}}_s(y)}{\tilde{F}_s(y)} \text{ mod } Q(x, y), \]

where \( \tilde{\tilde{F}}_s, \tilde{\tilde{V}}_s \) are second degree polynomials. . .
As for combinatorics..?

Let \( f(i, j, k) \) denote the number of paths starting from \((0, 0)\) and ending at \((i, j)\) at time \(k\) (or after \(k\) steps). Then the corresponding CGF

\[
F(x, y, z) = \sum_{i,j,k \geq 0} f(i, j, k) x^i y^j z^k
\]

satisfies the functional equation

\[
K(x, y, z) F(x, y, z) = c(x) F(x, 0, z) + \tilde{c}(y) F(0, y, z) + c_0(x, y, z),
\]

where

\[
K(x, y; z) = xy \left[ \sum_{(i,j) \in S} x^i y^j - 1/z \right].
\]

Note that here the group depends on \(z\) . . .
About the genus 0 case

Here the Riemann surface \( \mathcal{H} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\} \) is of genus 0 (the Riemann Sphere) and admits a uniformization in terms of simple rational functions.

For all non-singular random walks, \( S \) has genus 0 if, and only if, one of the following relations holds:

\[
\begin{align*}
M_x &= M_y = 0, \\
p_{10} &= p_{11} = p_{01} = 0, \\
p_{10} &= p_{1,-1} = p_{0,-1} = 0, \\
p_{-1,0} &= p_{-1,-1} = p_{0,-1} = 0, \\
p_{01} &= p_{-1,0} = p_{-1,1} = 0.
\end{align*}
\]

(48) \hspace{2cm} (49) \hspace{2cm} (50) \hspace{2cm} (51) \hspace{2cm} (52)

Define the drift \( \vec{\mathbf{M}} = (\sum ip_{ij}, \sum jp_{ij}) \) and \( \theta = \arccos(-r) \), where \( r \) denotes the correlation coefficient.
Theorem. [Fayolle-Raschel, MPRF 2011]

(a) When $\vec{M} = 0$, the group $\mathcal{H}$ is finite if and only if $\theta/\pi$ is rational, in which case its order is equal to

$$2 \inf \{ \ell \in \mathbb{Z}^*_+ : \ell \theta/\pi \in \mathbb{Z} \}.$$ 

(b) When $\vec{M} \neq 0$, the order of $\mathcal{H}$ is always infinite in the four remaining cases.

Sketch of proof of part (a). The main idea consists in working by continuity from the genus 1 case! Now letting $\vec{M} \to 0$, so that $x_2, x_3 \to 1$, we have

$$\begin{aligned}
\omega_1 &\to i\infty, \\
\omega_2 &\to \alpha_2 = \pi \left[ C(x_4 - 1)(1 - x_1) \right]^{1/2}, \\
\omega_3 &\to \alpha_3 = \int_{x_0(y_1)}^{x_1} \frac{dx}{(1 - x)[C(x - x_1)(x - x_4)]^{1/2}}, \\
\theta &\to \lim_{\vec{M} \to 0} \frac{\omega_2}{\omega_3} = \frac{\alpha_2}{\alpha_3}.
\end{aligned}$$ (53)
Sketch of proof of part (b).

By symmetry, it suffices to consider the case $p_{10} = p_{1,-1} = p_{0,-1} = 0$. Then

$$\begin{align*}
\omega_1 &\to i\alpha_1, \text{ with } \alpha_1 \in (0, \infty), \\
\omega_2 &\to \infty, \\
\omega_3 &\to \alpha_3 \in (0, \infty).
\end{align*}$$

Hence, the limit group can be interpreted as the group of transformations

$$\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \alpha_3 \rangle$$

on $\mathbb{C}/(\alpha_1 \mathbb{Z})$. This group is obviously infinite, and so is $\mathcal{H}$. 


Thank you for your attention!

But, what to do now?

*The trick will be to avoid the pitfalls, seize the opportunities, and get back home by six o’clock.*