Elliptic functions count walks on the square lattice with winding
Timothy Budd

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In 2001 Ira Gessel conjectured the number of walks with $2n$ steps $\in \{N, S, SW, NE\}$ in the quadrant starting and ending at 0 to be

$$16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = 2, 11, 85, 782, \ldots$$
Introduction: Gessel sequence

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- Proving this turned out to be hard, but by now we have...
  - ...a computer-aided proof. [Kauers, Koutschan, Zeilberger, '08]
  - ...a human (complex-analytic) proof. [Bostan, Kurkova, Raschel, '13]
  - ...an elementary (algebraic) proof. [Bousquet-Mélou, '15]
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- ... an elementary (algebraic) proof. [Bousquet-Mélou, '15]

As we will see, counting walks by winding angle provides a natural alternative route.
Introduction: Winding angle of a walk

To a walk $w$ on $\mathbb{Z}^2$ avoiding 0 we can naturally associate a winding angle

$$\theta_w := \sum_{i=1}^{\|w\|} \angle(w_{i-1}, 0, w_i).$$
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- Extends unambiguously to excursions from the origin.

- Natural interpretation as walks in the universal cover of $\mathbb{Z}^2 \setminus \{0\}$.

First goal today is to determine the GF for simple excursions from origin

\[
F(t, b) := \sum_{w} t^{\lvert w \rvert} e^{ib\theta_w} = 4t^2 + (12 + 4e^{-ib\pi/2} + 4e^{ib\pi/2})t^4 + \ldots
\]
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$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \ldots$$
Decomposing into walks on the slit plane

- The general idea: decompose into a sequence of walks on the slit plane.

Denote by $H(p, l)(t)$ the GF for walks $(p, 0) \to (-l, 0)$ that hit the slit from above (counted by $t$ length).

This GF can be deduced from [Bousquet-Mélou, Schaeffer, '00].

$H(l, p) = lpH(p, l)$, so introduce symmetric "matrix" $H := \sqrt{lpH(p, l)}$.

Then $\sqrt{pl}N(HN)^{pl}$ counts composite walks $(p, 0) \to (\pm l, 0)$ that alternate between axes $N$ times.

To incorporate a weight $e^{ib\theta}w$ in GF just replace $2 \to e^{ib\pi} + e^{-ib\pi}$.

$\sum$ such walks $t|w|e^{ib\theta}w = \sqrt{pl}\sum_{N=1}^{\infty} (2 \cos(\pi b))^{N}(HN)^{pl}$.
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\[ H(l, p) = lpH(p, l) \]

So introduce symmetric "matrix" \(H := (\sqrt{lpH(p, l)})\) \(p, l \geq 1\).

Then \(\sqrt{pl2\sum N(H)}\) \(pl\) counts composite walks \((p, 0) \rightarrow (\pm l, 0)\) that alternate between axes \(N\) times.

To incorporate a weight \(eib\theta\) in GF just replace 2 \(\rightarrow eib\pi + eib\pi\).

\[ \sum \text{such walks} t|w|eib\theta = \sqrt{pl\sum N(H)} \]

\[ \sqrt{pl\sum N(H)} = \sqrt{pl(2 \cos(\pi b) H - 2 \cos(\pi b) H)} \]
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$H(l, p) = l p H(p, l)$, so introduce symmetric “matrix” $H := (√lp H(p, l))_{p, l \geq 1}$.

Then $\sqrt{p} l^2 N(H N)^{pl}$ counts composite walks $(p, 0) \rightarrow (-l, 0)$ that alternate between axes $N$ times.

To incorporate a weight $e^{ibθ}w$ in GF just replace $2 \rightarrow e^{ibπ} + e^{-ibπ}$.

$$\sum \text{such walks } t |w| e^{ibθw} = \sqrt{pl} \sum_{N=1}^{∞} (2 \cos(πb))^N (H_N)^{pl} = \sqrt{pl} (2 \cos(πb) H - 2 \cos(πb) H_I)$$
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$$H(l, p) = \sqrt{p}l H(p, l),$$

so introduce symmetric "matrix" $H := (\sqrt{lp} H(p, l))_{p, l \geq 1}$.

Then

$$\sqrt{pl^2} N(HN) pl$$

counts composite walks $(p, 0) \rightarrow (\pm l, 0)$ that alternate between axes $N$ times.

To incorporate a weight $e^{i b \theta}$ in GF just replace $2 \rightarrow e^{i b \pi} + e^{-i b \pi}$.

$$\sum \text{walks } t \left| w \right| e^{i b \theta w} = \sqrt{pl \infty} \sum_{N=1} (2 \cos(\pi b)) N(H^N) pl$$
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\[ H^{(p,l)}(t) = l^p H^{(p,l)}(t) \]

\[ \sqrt{p} l H^{(p,l)}(t) = \sum_{|w|} t^{\text{length}} e^{ib\theta} = \sqrt{p} l \sum_{N=1}^{\infty} (2\cos(\pi b))^N (H_N(t)^2 - 2\cos(\pi b) H_N(t)) \]
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- $H^{(l,p)} = \frac{1}{p} H^{(p,l)}$, so introduce symmetric “matrix” $H := \left( \sqrt{\frac{l}{p}} H^{(p,l)} \right)_{p,l \geq 1}$
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- Then \( \sqrt{\frac{p}{l}} 2^N (H^N)_{pl} \) counts composite walks \((p,0) \rightarrow (\pm l,0)\) that alternate between axes \(N\) times.
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\[ \sum_{\text{such walks}} t^{|w|} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2\cos(\pi b))^N (H^N)_{pl} \]
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- This GF can be deduced from [Bousquet-Mélou, Schaeffer, '00].
- $H^{(l,p)} = \frac{l}{p} H^{(p,l)}$, so introduce symmetric “matrix” $\mathcal{H} := \left( \sqrt{\frac{l}{p}} H^{(p,l)} \right)_{p,l \geq 1}$.
- Then $\sqrt{\frac{p}{l}} 2^N (\mathcal{H}^N)_{pl}$ counts composite walks $(p,0) \to (\pm l,0)$ that alternate between axes $N$ times.
- To incorporate a weight $e^{ib\theta_w}$ in GF just replace $2 \to e^{ib\pi} + e^{-ib\pi}$.

$$
\sum_{\text{such walks}} t^{\text{length}} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2 \cos(\pi b))^N (\mathcal{H}^N)_{pl} = \sqrt{\frac{p}{l} \left( \frac{2 \cos(\pi b) \mathcal{H}}{l - 2 \cos(\pi b) \mathcal{H}} \right)_{pl}}
$$
Relation with planar maps

- Planar map = a multigraph properly embedded in the plane up to homeomorphism. Take it to be rooted on the outer face.
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\[ W(p, l)(q_1, q_2, \ldots) \] is the GF for planar maps with outer degree \( p \geq 1 \), a marked face of degree \( l \geq 1 \), weighted by \( \prod \text{faces} q^{\text{degree}} \).

For quasi-bipartite maps (\( q_1 = q_3 = \cdots = 0 \)) it takes a universal form (see e.g. [Collet, Fusy, '12])

\[ W(p, l) = \frac{1}{l}^{\alpha(l)} \frac{1}{p}^{\alpha(p)} \left( \rho q^4 \right)^{p + l} / 2 \]

Remarkably \( H(p, l)(t) = W(p, l)(\rho q \rightarrow \rho(t)) := 1 - \sqrt{1 - 16t^2 / 8t^2 - 1} \).
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$H(p, l)(t) = W^{(p,l)}(t) \bigg|_{\rho q \to \rho(t)} := 1 - \sqrt{1 - 16t^2} - \frac{8t^2}{2}$
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$\begin{align*}
W^{(p,l)}(q_1, q_2, \ldots) &= 1 - \frac{\sqrt{1 - 16t^2}}{8t^2 - 1} \\
\end{align*}$
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Remarkably

$H^{(p,l)}(t) = W^{(p,l)}(\rho)_{\mid \rho q \rightarrow \rho(t)} := 1 - \sqrt{1 - 16 t^2} / 8$
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- For quasi-bipartite maps \((q_1 = q_3 = \cdots = 0)\) it takes a universal form (see e.g. [Collet, Fusy, '12])

\[
W^{(p,l)} = \frac{1}{l} \frac{2}{p + 1} \alpha(l) \alpha(p) \left(\frac{p q_4}{4}\right)^{(p + l)/2} \\
\alpha(p) := \frac{p!}{\left\lfloor \frac{p}{2} \right\rfloor! \left\lfloor \frac{p-1}{2} \right\rfloor!}
\]
Relation with planar maps

- Planar map = a multigraph properly embedded in the plane up to homeomorphism. Take it to be rooted on the outer face.
- $W^{(p,l)}(q_1, q_2, \ldots)$ is the GF for planar maps with outer degree $p \geq 1$, a marked face of degree $l \geq 1$, weighted by $\prod_{\text{faces}} q_{\text{degree}}$.
- For quasi-bipartite maps ($q_1 = q_3 = \cdots = 0$) it takes a universal form (see e.g. [Collet, Fusy, '12])

$$W^{(p,l)} = \frac{1}{l} \frac{2}{p+l} \alpha(l) \alpha(p) \left( \frac{p_q}{4} \right)^{(p+l)/2} \alpha(p) := \frac{p!}{\left\lfloor \frac{p}{2} \right\rfloor! \left\lfloor \frac{p-1}{2} \right\rfloor!}$$

- Remarkably $H^{(p,l)}(t) = W^{(p,l)} \bigg|_{\rho_q \to \rho(t) := \frac{1 - \sqrt{1 - 16t^2}}{8t^2} - 1}$
A bijective explanation

Proposition

For any step set $\mathcal{S} \subset \{\text{steps}\}$, there exists a bijection

$$\Phi^{(p, l)} : \{\mathcal{S}\text{-walks} (p, 0) \to (-l, 0) \text{ hitting slit from above}\}$$

$$\mapsto \left\{ \begin{array}{l}
\text{"$\mathcal{S}\text{-walk-decorated maps}"$ with root face degree } p \\
\text{and marked face degree } l
\end{array} \right\}$$

A $\mathcal{S}\text{-walk-decorated map}$ is a rooted planar map with a marked face together with:

- for each face (except root or marked) of degree $k$ an excursion $$(0, 0) \to (k - 2, 0)$$ above or below $x$-axis.
- for each vertex an excursion $$(0, 0) \to (-2, 0)$$ above $x$-axis.
A bijective explanation

**Proposition**

For any step set $\mathcal{S} \subset \mathbb{R}$, there exists a bijection

$$
\Phi^{(p,l)} : \{\mathcal{S}\text{-walks } (p,0) \rightarrow (-l,0) \text{ hitting slit from above}\} \\
\rightarrow \left\{ \right. \\
\left. \text{“}\mathcal{S}\text{-walk-decorated maps” with root face degree } p \\
\text{and marked face degree } l \right\}
$$

- A $\mathcal{S}$-walk-decorated map is a rooted planar map with a marked face together with...
A bijective explanation

Proposition

For any step set $\mathcal{S} \subset \mathbb{R}$, there exists a bijection

$$\Phi^{(p,l)} : \{\mathcal{S}\text{-walks } (p, 0) \rightarrow (-l, 0) \text{ hitting slit from above}\} \rightarrow \left\{ \text{"$\mathcal{S}$-walk-decorated maps" with root face degree $p$ and marked face degree $l"} \right\}$$

- A $\mathcal{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree $k$ an excursion $(0, 0) \rightarrow (k - 2, 0)$ above or below $x$-axis.
A bijective explanation

**Proposition**

For any step set \( \mathcal{S} \subset \mathbb{N} \), there exists a bijection

\[
\Phi^{(p,l)} : \{ \mathcal{S}-walks \ (p,0) \rightarrow (-l,0) \text{ hitting slit from above} \} \rightarrow \left\{ \text{“}\mathcal{S}\text{-walk-decorated maps” with root face degree } p \right. \\
\left. \text{and marked face degree } l \right\}
\]

- A \( \mathcal{S}\)-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree \( k \) an excursion \((0,0) \rightarrow (k-2,0)\) above or below \( x \)-axis.
  - for each vertex an excursion \((0,0) \rightarrow (-2,0)\) above \( x \)-axis
A bijective explanation

**Proposition**

For any step set $\mathcal{S} \subset \mathbb{R}$, there exists a bijection

$$\Phi^{(p,l)} : \{ \text{$\mathcal{S}$-walks } (p,0) \rightarrow (-l,0) \text{ hitting slit from above} \}$$

$$\longleftrightarrow \left\{ \begin{array}{l}
\text{“$\mathcal{S}$-walk-decorated maps” with root face degree $p$}
\text{and marked face degree $l$}
\end{array} \right\}$$

- A $\mathcal{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree $k$ an excursion $(0,0) \rightarrow (k-2,0)$ above or below x-axis.
  - for each vertex an excursion $(0,0) \rightarrow (-2,0)$ above x-axis

- Substituting in $W^{(p,l)}(q_i)$ the GFs
  $$q_k \rightarrow \begin{pmatrix} q_0 & \cdots & q_{k-2} & q_{k} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

leads to $H^{(p,l)}(t)$. 
An example
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Φ(6,2)
From walks to (rigid) loop-decorated maps
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Recall such walks $t \mid w \mid e \mid \theta \mid w = \sqrt{p_l} \sum_{N=1}^{\infty} (2 \cos(\pi b))^N (H_N)$

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees $p_l, l$ carrying a weight $(2 \cos(\pi b))^{#\text{loops}+1} \prod_{\text{regular faces}} q^\text{degree}$.
From walks to (rigid) loop-decorated maps

Recall

\[ \sum_{t} \left| w \right| e^{ib\theta} w = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2 \cos(\pi b H_{N}))^{#\text{loops} + 1} \prod_{\text{regular faces}} q^{\text{degree}} \]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \((2 \cos(\pi b H))\).
From walks to (rigid) loop-decorated maps

Recall

\[ \sum_{w|e|b}\theta^w \approx \sqrt{p_l} \sum_{N=1}^{\infty} (2 \cos(\pi b)) N^2 (H_N) \]

\[ \sqrt{p_l} (2 \cos(\pi b) H_{I-2} \cos(\pi b)) \]

▶

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \( (2 \cos(\pi b)) \#\text{loops}+1 \prod \text{regular faces} q^\text{degree} \)
Recall \[ \sum_{w|e|} \theta_w \sqrt{p_l} \sum_{N=1}^{\infty} (2 \cos(\pi b))^N (H_N) \] follows. Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \((2 \cos(\pi b))^{#\text{loops}+1}\) for each regular face of degree \( q \).
From walks to (rigid) loop-decorated maps

Recall \[ \sum_{\sum_{|w|}} t^{|w|} e^{ib\theta}w = \sqrt{p}\sum_{N=1}^{\infty} (2\cos(\pi b))^{N}(H_{N}) \]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \(p,l\) carrying a weight \((2\cos(\pi b))^{#\text{loops}+1}\prod_{\text{regular faces}} q^{\text{degree}}\).
From walks to (rigid) loop-decorated maps

\[ \sum_{t \mid w \mid e}^{\theta} w = \sqrt{\prod_{\infty} N = 1} (2 \cos(\pi b))^N \]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \((2 \cos(\pi b))^{#\text{loops} + 1} \prod_{\text{regular faces}} q^{\text{degree}}\).
From walks to (rigid) loop-decorated maps

\[ \sum_t |w| e^{ib \theta w} = \sqrt{p_{\infty}} \sum_{N=1}^{\infty} (2 \cos(\pi b H N) + 1) \prod \text{regular faces} q^{\text{degree}} \]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \( (2 \cos(\pi b)) \times \#\text{loops} + 1 \times \prod \text{regular faces} q^{\text{degree}} \).
From walks to (rigid) loop-decorated maps

\[ \sum_{\theta^w eib} t |w| eib \theta^w = \sqrt{p l} \sum_{N=1}^{\infty} (2 \cos(\pi b))^{N} (H_N) \]

\[ \sqrt{p l} (2 \cos(\pi b) H_{I-2} - 2 \cos(\pi b) H_{I}) \]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight \((2 \cos(\pi b))^{#\text{loops}+1} \prod \text{regular faces } q \text{ degree}\)

\[ p = 6 \]

\[ l = 2 \]
From walks to (rigid) loop-decorated maps

Recall

\[
\sum_{\text{such walks}} t^{|w|} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2 \cos(\pi b))^N (\mathcal{H}^N)_{pl} = \sqrt{\frac{p}{l}} \left( \frac{2 \cos(\pi b) \mathcal{H}}{l - 2 \cos(\pi b) \mathcal{H}} \right)_{pl}
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\]

Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees \( p, l \) carrying a weight

\[
(2 \cos(\pi b))^{#\text{loops}+1} \prod_{\text{regular faces}} q_{\text{degree}}
\]
Planar maps coupled to a rigid $O(n)$ loop model

- Rigid $O(n)$ model: a planar map + disjoint loops, that intersect solely quadrangles through opposite sides. Enumerated with

  \[
  \text{weight} \quad n \# \text{loops} \quad g \# \text{loop faces} \quad \prod_{\text{regular faces}} q_{\text{degree}}
  \]

- An exact solution of a closely related model was obtained by [Eynard, Kristjansen, '95] in terms of elliptic functions.
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- An exact solution of a closely related model was obtained by [Eynard, Kristjansen, '95] in terms of elliptic functions.
- Made more precise in [Borot, Eynard, '09], and in [Borot, Bouttier, Guitter, '11] for this “rigid” setting.
Planar maps coupled to a rigid $O(n)$ loop model

- **Rigid $O(n)$ model:** A planar map + disjoint loops, that intersect solely quadrangles through opposite sides. Enumerated with

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- Recently in [Borot, Bouttier, Duplantier, '16] (for triangulations) exact statistics for the nesting of loops was obtained, i.e. distribution of $\#$ loops surrounding a marked vertex/face.
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- Recently in [Borot, Bouttier, Duplantier, '16] (for triangulations) exact statistics for the nesting of loops was obtained, i.e. distribution of $\#$ loops surrounding a marked vertex/face.

- Importantly: the form of the GF $G^{(p,l)}(n, g, q)$ is universal and is not affected by suppressing loops that do not surround the marked face.
We know that (with \( n = 2 \cos(\pi b) \) and appropriate \( g, q \))

\[
\sqrt{\frac{p}{l}} \left( \frac{\mathcal{H}}{l - n\mathcal{H}} \right)_{pl} = G^{(p,l)}(n, g, q)
\]
We know that (with $n = 2 \cos(\pi b)$ and appropriate $g, q$)

$$
\sum_{p, l \geq 1} x_1^p x_2^l \sqrt{\frac{p}{l}} \left( \frac{\mathcal{H}}{l - n\mathcal{H}} \right)_{pl} = \sum_{p, l \geq 1} x_1^p x_2^l \mathcal{G}^{(p, l)}(n, g, q)
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Adapting GF from [Borot, Bouttier, Duplantier, '16] and computing a series expansion:

\[
= 4 \sum_{m=1}^{\infty} \frac{1}{q^m + q^{-m} - n} \cos(2\pi m \nu(x_2)) x_1 \frac{\partial}{\partial x_1} \cos(2\pi m \nu(x_1)) \left( \frac{m}{q^{-m} - q^m} \right)
\]

where \( q = q(4t) = t^2 + 8t^4 + \cdots \) is the nome of modulus \( 4t \) and

\[
\nu(x) := \text{cd}^{-1}(\frac{-x}{\sqrt{\rho}}, \rho)/(4K(\rho)), \quad \rho(t) = \frac{1 - \sqrt{1 - 16t^2}}{8t^2} - 1
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\]

Proposition (Diagonalization of $\mathcal{H}$)
\[
\mathcal{H} = U^T \cdot \Lambda_q \cdot U \text{ in the sense of operators on } \ell^2(\mathbb{R}) \text{ with }
\]
\[
\Lambda_q = \text{diag} \left( \frac{1}{q^m + q^{-m}} \right)_{m \geq 1}, \quad U_{mp} = \sqrt{\frac{4p}{m(q^{-m} - q^m)}} [x^p] \cos(2\pi m \nu(x))
\]
Refinement: increase winding angle resolution

- Up to now: decomposed walk into sequence of walks on slit plane, each numerated by $\sqrt{\frac{p}{l}} \mathcal{H}_{pl}$.

- Why not decompose into walks on half plane?

- Denote GF for half-plane walks $(p, 0) \rightarrow (0, l)$ by $\sqrt{p} \mathcal{J}l$. Then $2^H = (2^J) (J + J \cdot 2^H)$, $J = \sqrt{4^H I + 2^H}$.

- Hence $J$ has same eigenmodes as $H$ but eigenvalues are $\frac{1}{2} q m + q - m$ instead of $\frac{1}{2} q m + q$. Such an operation $q \rightarrow \sqrt{q}$ on elliptic functions is called a "Landen transformation" and is thus connected to angle doubling.
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- Up to now: decomposed walk into sequence of walks on slit plane, each numerated by $\sqrt{P_l H}$. 
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![Diagram of walk decomposition](image)
Refinement: increase winding angle resolution

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![Graphical representation of walks on a half plane]

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Hence $J$ has same eigenmodes as $\mathcal{H}$ but eigenvalues are $\frac{1}{2} q m + \frac{1}{2} - m$ instead of $\frac{1}{2} q m + \frac{1}{2} - m$. Such an operation $q \rightarrow \sqrt{q}$ on elliptic functions is called a "Landen transformation" and is thus connected to angle doubling.
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2 \mathcal{H} = (2 \mathcal{J}) (\mathcal{J} + \mathcal{J} \cdot 2 \mathcal{H})
\]

Hence \( \mathcal{J} \) has same eigenmodes as \( \mathcal{H} \) but eigenvalues are \( 1 + \eta m / 2 + \eta - m / 2 \) instead of \( 1 + \eta m + \eta - m \). Such an operation \( q \rightarrow \sqrt{q} \) on elliptic functions is called a "Landen transformation" and is thus connected to angle doubling.
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- Hence $J$ has same eigenmodes as $H$ but eigenvalues are $\frac{1}{q^{m/2} + q^{-m/2}}$ instead of $\frac{1}{q^m + q^{-m}}$. Such an operation $q \rightarrow \sqrt{q}$ on elliptic functions is called a “Landen transformation” and is thus connected to angle doubling.
Winding angle of excursions

- Wish to enumerate excursions from origin by length and winding angle:

\[ F(t, b) := \sum_w t^{|w|} e^{ib \theta_w} \]

\[ = 4t^2 + (12 + 4e^{-ib \frac{\pi}{2}} + 4e^{ib \frac{\pi}{2}})t^4 + \ldots \]
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- This maps excursions 4-to-2 onto sequences of half-plane walks with \( p = l = 2 \) and a restriction on first and last step.
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- Enumerated by

\[ F(t, b) = 2 \sum_{N \geq 1} (2 \cos (\frac{\pi b}{2}))^{N-1} \left[ (J^N)_{22} - \ldots \right] \]
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\[ F(t, b) = 2 \sum_{N \geq 1} (2 \cos \left( \frac{\pi b}{2} \right) )^{N-1} \sum_{p, l \geq 0} (-1)^{p+l} \sqrt{\frac{p}{l}} (J^N)_{2p, 2l} \]
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  = \sec \left(\frac{\pi b}{2}\right) \left[ 1 - \frac{\pi \tan \left(\frac{\pi b}{4}\right)}{2K(4t)} \frac{\theta'_{1\left(\frac{\pi b}{4}, \sqrt{q}\right)}}{\theta_{1\left(\frac{\pi b}{4}, \sqrt{q}\right)}} \right]
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Application: walks in cones

Theorem (Excursions in the $\frac{n\pi}{4}$-cone.)

For any set of integers $-n < m - n < p < m < n$ the generating function $F_{n,m,p}(t)$ for excursions from the origin with winding angle $\frac{p\pi}{2}$ staying strictly inside angular region $(\frac{p+m-n}{4}\pi, \frac{p+m}{4}\pi)$ is given by

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$(n, m, p) = (5, 2, -1)$
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The proof uses the reflection principle.
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Thanks to a hint of Killian Raschel: for $b \in \mathbb{Q}$, $F(t, b)$ is expressible in Jacobi elliptic functions at rational angles, which are algebraic in $t$. 
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which is algebraic, i.e. $P(t, F_{n,m,p}(t)) = 0$ for some $P(t, x) \in \mathbb{Z}[t, x]$.

▶ The proof uses the reflection principle.

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Application: walks in cones (Gessel case)

- Special case: \((n, m, p) = (3, 2, 0)\)
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\frac{1}{t^2} F_{3,2,0}(t) = \frac{1}{4t^2} F(t, \frac{4}{3})
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= 1 + 2t^2 + 11t^4 + 85t^6 + \cdots,
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which is an algebraic series.
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- Can reproduce the known formula

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\sum_{n=0}^{\infty} t^{2n} \frac{(5/6)^n(1/2)^n}{(2)_n(5/3)_n} = \frac{1}{2t^2} \left[ {}_2F_1\left( -\frac{1}{2}, -\frac{1}{6}; \frac{2}{3}; (4t)^2 \right) - 1 \right].
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by checking that both solve same algebraic equation…
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  by checking that both solve same algebraic equation... or by comparing modular properties of both as suggested by Alin Bostan.
Application: winding field of a random loop

- Consider a uniform loop of length $2n$ on $\mathbb{Z}^2$.

$$\text{What is the expected area of squares with winding angle } 2\pi k?$$

It can be expressed in terms of the spectrum of $H$ and is
$$2n k \left( \frac{2n n}{2} \right)^2 \left( t^2 n \right)^{-2k-2} \sim n^4 \pi k^2$$

The $n \to \infty$ asymptotics reproduces result of Brownian motion. [Garban, Ferreras,'06]
Application: winding field of a random loop

- Consider a uniform loop of length $2n$ on $\mathbb{Z}^2$.
- One may color each square according to the total winding angle of the loop around it.

What is the expected area of squares with winding angle $2\pi k$?

It can be expressed in terms of the spectrum of $H$ and is

$$2n^k \left[ t \right]^{q - 2k - q^2} \sim n^{4\pi k^2}$$

The $n \to \infty$ asymptotics reproduces result of Brownian motion. [Garban, Ferreras, '06]
Application: winding field of a random loop

- Consider a uniform loop of length $2n$ on $\mathbb{Z}^2$.
- One may color each square according to the total winding angle of the loop around it.
- What is the expected area of squares with winding angle $2\pi k$?

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[Garban, Ferreras,'06]
Jacobi elliptic functions are characteristic functions

- Let $n_p \geq 1$ be a geometric random variable with parameter $p \in (0, 1)$.
- Let $\theta_{n_p}$ be the winding angle around $(-\frac{1}{2}, \frac{1}{2})$ of an SSRW at time $n_p - \frac{1}{2}$.
- Denote by $[\cdot]_{\pi\mathbb{Z}}$ resp. $[\cdot]_{\pi(\mathbb{Z}+\frac{1}{2})}$ rounding to nearest integer resp. half-integer multiple of $\pi$.

Then

\[
E \exp\left( ib [\theta_{n_p}]_{\pi\mathbb{Z}}\right) = \text{cn}(u; p)
\]

\[
E \exp\left( ib [\theta_{n_p-1}\pi Z]\right) = \text{dn}(u; p)
\]

with $\text{cn}$, $\text{dn}$ Jacobi elliptic functions.
Jacobi elliptic functions are characteristic functions

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- Then

  $$E \exp \left( ib \left[ \theta_{n_p} \right]_{\pi (\mathbb{Z} + \frac{1}{2})} \right) = \text{cn}(u; p), \quad u := K(p)b$$

  $$E \exp \left( ib \left[ \theta_{n_p} - 1 \right]_{\pi \mathbb{Z}} \right) = \text{dn}(u; p),$$

with $\text{cn}$, $\text{dn}$ Jacobi elliptic functions.
Concluding remarks

- It is still mysterious why some of the generating functions are so simple.
  - Is there a combinatorial explanation of: Landen transformation ↔ angle doubling?
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  - Note: This does not necessarily help in counting such walks in the quadrant, since the reflection principle relies on symmetry in the steps.
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Thanks for your attention!