On Moment Problems with Holonomic Functions

Florent Bréhard, Mioara Joldes, Jean-Bernard Lasserre
Moments of a measure

\[ m_\alpha = \int_{\mathbb{R}^n} x^{\alpha} \, d\mu \quad \text{for} \quad \alpha \in \mathbb{N}^{n_a} \]

\[ a\alpha = (\alpha_1, \ldots, \alpha_n), \quad x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \mathbb{K}[x]_d = \text{polynomials of total degree at most } d \]
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\[ m_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu = \int_G x^\alpha f(x) \, dx \quad \text{for} \quad \alpha \in \mathbb{N}^n \]

- \( G \) \( n \)-dim semi-algebraic set, with \( g \in \mathbb{K}[x] \) vanishing on \( \partial G \)
- \( f : \mathbb{R}^n \to \mathbb{R} \) D-finite = satisfies a “complete” system of PDEs

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→ Direct problem: knowing \( G \) and \( f \), find a complete system of recurrences for \( (m_\alpha) \)

\( \leadsto \) Finite determinancy of such measures
\( \leadsto \) Solved with Creative Telescoping, e.g., [Oaku2013] + Takayama’s algorithm
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→ **Inverse problem:** reconstruct \( G \) and/or \( f \), given finitely many moments \( m_\alpha \)

- Statistics
- Signal processing
- Medical imaging (MRI)
- Gravimetry
- Combinatorics
Inverse Problems

\[ (m_\alpha)_{|\alpha| \leq N} \]

Measures

Reconstruction

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Inverse Problems

Measures

\[ (m_\alpha)_{|\alpha| \leq N} \]

Reconstruction

Decision
Inverse Problems

Measures

$\left( m_\alpha \right)_{|\alpha| \leq N}$

Reconstruction

Decision or ?
→ Numerical methods, e.g.:
  ○ Convex polytopes: [GolubMilanfarVarah1999] [GravinLasserrePasechnikRobins2012]
  ○ Planar quadrature domains: [EbenfeltGustafssonKhavinsonPutinar2005]
  ○ Sublevel sets of homogeneous polynomials: [Lasserre2013]
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→ Symbolic/algebraic methods:

- A historical starting point: Prony’s method
  - reconstructing sparse exponential functions \( \sum_{\alpha \in I} \lambda_\alpha e^{\alpha x} \) from evaluations
  - link with moments of Dirac measures
- Multivariate extensions of Prony’s method, e.g., [Mourrain2018]
- Reconstructing univariate piecewise D-finite densities: [Batenkov2009]
**Exact Support and/or Density Reconstruction**

- Lasserre and Putinar's exact reconstruction algorithm (2015)

**Inverse Problem: Exponential-Polynomial Measure, Algebraic Support**

Let $G \subset \mathbb{R}^n$, bounded open set, whose algebraic boundary is included in the zero set of a polynomial $g \in \mathbb{K}[x]_d$, and $f(x) = \exp(p(x))$ with $p \in \mathbb{K}[x]_s$. Given $p$, degree $d$ and moments $m_\alpha$ up to order $|\alpha| = 3d + s$, the coefficients of $g$ can be exactly recovered.

- Key idea: Linear recurrences satisfied by the moments + Stokes’ Theorem
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Our contribution: a computer algebra approach

- generalization in the framework of holonomic distributions
  - they satisfy (as a generalized function) a “complete” system of linear PDEs/ODEs with polynomial coefficients

- exact recovery of both support and Exponential-Polynomial density $f = \exp(p)$, with explicit bound on the required number of moments

- similar algorithm for D-finite density, but no a priori bound on the required number of moments
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1 Introduction

2 Holonomic Distributions and Recurrences on Moments

3 Inverse Problem: Algorithms and Proofs
   - Exponential-Polynomial Densities
   - The General Case with D-Finite Densities

4 Limits and Perspectives
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1. Differential Ore Algebras

- Differential operators: non-commutative, spanned by $x_1, \partial_{x_1}, \ldots, x_n, \partial_{x_n}$

\[
\partial_{x_i} f = f'_{x_i} \quad (x_i f)'_{x_i} = x_i f'_{x_i} + f \quad \Rightarrow \quad \partial_{x_i} x_i = x_i \partial_{x_i} + 1
\]
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- $\mathbb{K}[x] \langle \partial_x \rangle$ polynomial Ore algebra
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\Rightarrow \quad \partial x_i x_i &= x_i \partial x_i + 1
\end{align*}
\]

- \( \mathbb{K}[x] \langle \partial_x \rangle \) polynomial Ore algebra \text{ vs } \( \mathbb{K}(x) \langle \partial_x \rangle \) rational Ore algebra

- \( \text{Ann}(f) = \{ L \in \mathbb{K}(x) \langle \partial_x \rangle \mid L f = 0 \} \) PDEs satisfied by density \( f \)
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- $\mathbb{K}[x](\partial_x)$ polynomial Ore algebra \text{ vs } $\mathbb{K}(x)(\partial_x)$ rational Ore algebra

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$\Rightarrow$ $f$ is **D-finite** iff $\mathbb{K}(x)(\partial_x)/\text{Ann}(f)$ has finite dimension over the $\partial_{x_i}$
1. Differential Ore Algebras

- Differential operators: non-commutative, spanned by $x_1, \partial x_1, \ldots, x_n, \partial x_n$

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**Example: Exponential-Polynomial Density**

\[
f(x) = c \exp(p(x)) \quad \text{with} \quad p \in \mathbb{K}_s[x] \quad \text{(e.g., Gaussian distribution)}
\]

\[
f'_{x_i} - p'_{x_i} f = 0 \quad \Rightarrow \quad \text{Ann}(f) \text{ generated by the } \partial x_i - p'_{x_i} \quad \Rightarrow \quad f \text{ is D-finite}
\]
2. Difference Ore Algebras

- Difference operators: **non-commutative**, spanned by $\alpha_1, S\alpha_1, \ldots, \alpha_n, S\alpha_n$

\[
(\alpha_i u)_\alpha = \alpha_i u_{\alpha} \quad (S\alpha_i u)_\alpha = u_{\alpha_1, \ldots, \alpha_i+1, \ldots, \alpha_n} \quad S\alpha_i \alpha_i = (\alpha_i + 1)S\alpha_i
\]

- $\text{Ann}(u) = \{ R \in K[\alpha](S\alpha) \mid R u = 0 \}$ recurrences satisfied by $u$
2. Difference Ore Algebras

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S_{\alpha_i} \alpha_i &= (\alpha_i + 1)S_{\alpha_i}
\end{align*}
\]

- $\text{Ann}(u) = \{ R \in K[\alpha] \langle S_{\alpha} \rangle \mid R u = 0 \}$ recurrences satisfied by $u$

Goals

Recurrences for the moments $m_{\alpha} = \int_G x^{\alpha} f(x) \, dx$:

- **Direct problem:** $\mathcal{I} \subseteq \text{Ann}(f)$ \implies $\mathcal{J} \subseteq \text{Ann}(m_{\alpha})$

- **Inverse problem:** Reconstruct $G$ and $\mathcal{I} \subseteq \text{Ann}(f)$ from sufficiently many $m_{\alpha}$
Holonomic Measures

- Measure $\mu = f1_G$ as a linear functional:
  
  $\langle f1_G, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)1_G(x)dx = \int_G \varphi(x)f(x)dx$

- Action of Ore polynomials: $L\mu =$ ?
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Example: Lebesgue measure over a segment

Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

$$\langle 1_G, \varphi \rangle = \int_{-1}^{1} \varphi(x)dx$$
Holonomic Measures

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Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

$$\langle \partial_x 1_G, \varphi \rangle$$
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**Example: Lebesgue measure over a segment**

Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

$$\langle \partial_x 1_G, \varphi \rangle = \langle 1_G, -\partial_x \varphi \rangle$$
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Example: Lebesgue measure over a segment

Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

$$\langle \partial_x 1_G, \varphi \rangle = \langle 1_G, -\partial_x \varphi \rangle = -\int_{-1}^{1} \varphi'(x)dx$$
Holonomic Measures

- Measure \( \mu = f \mathbf{1}_G \) as a linear functional:

\[
\langle f \mathbf{1}_G, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)\mathbf{1}_G(x)\,dx = \int_G \varphi(x)f(x)\,dx
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Example: Lebesgue measure over a segment

Let \( G = [-1, 1] \), \( f = 1 \), and \( \mu = \mathbf{1}_G \)

\[
\langle \partial_x \mathbf{1}_G, \varphi \rangle = \langle \mathbf{1}_G, -\partial_x \varphi \rangle = \varphi(-1) - \varphi(1)
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Holonomic Measures

- Measure $\mu = f1_G$ as a linear functional:

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Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

$$\langle \partial_x 1_G, \varphi \rangle = \langle 1_G, -\partial_x \varphi \rangle = \varphi(-1) - \varphi(1) \quad \Rightarrow \quad \partial_x 1_G = \delta_{-1} - \delta_1$$

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Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

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Holonomic Measures

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Let $G = [-1, 1]$, $f = 1$, and $\mu = 1_G$

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\langle (x^2 - 1)\partial_x 1_G, \varphi \rangle
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$$\langle (x^2 - 1)\partial_x 1_G, \varphi \rangle = \langle 1_G, -\partial_x(x^2 - 1)\varphi \rangle = [(1 - x^2)\varphi]_1^1 = 0$$
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- Ore polynomials acting on distributions: $\langle L \ T, \varphi \rangle = \langle T, L^* \varphi \rangle$

  $$x_i^* = x_i \quad \partial^*_{x_i} = -\partial_{x_i} \quad (L_1L_2)^* = L_2^*L_1^*$$
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\]

- Ore polynomials acting on distributions: $\langle L\ T, \varphi \rangle = \langle T, L^*\ \varphi \rangle$
  \[
  x_i^* = x_i \quad \partial_{x_i}^* = -\partial_{x_i} \quad (L_1L_2)^* = L_2^*L_1^*
  \]

- $\text{Ann}(T)$ in $\mathbb{K}[x]\langle \partial_x \rangle$ \quad $\Rightarrow$ \quad holonomic instead of D-finite
Again, with $G = [-1, 1]$, and using $\varphi = x^k$:

$$0 = \langle (1 - x^2) \partial_x 1_G, x^k \rangle$$
– Again, with $G = [-1, 1]$, and using $\varphi = x^k$:

\[
0 = \langle (1 - x^2) \partial_x 1_G, x^k \rangle = \langle 1_G, \partial_x (x^2 - 1)x^k \rangle = \int_{-1}^{1} ((k + 2)x^{k+1} - kx^{k-1}) \, dx
\]
Again, with $G = [-1, 1]$, and using $\varphi = x^k$:

$$0 = \langle (1 - x^2) \partial_x 1_G, x^k \rangle = \langle 1_G, \partial_x (x^2 - 1)x^k \rangle = \int_{-1}^{1} ((k + 2)x^{k+1} - kx^{k-1}) \, dx$$

$\Rightarrow$ Recurrence satisfied by the moments $(m_k)$:

$$(k + 2)m_{k+1} - km_{k-1} = 0$$

This is indeed true...

$$m_k = \int_{-1}^{1} x^k \, dx = \begin{cases} \frac{2}{k+1} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$
Using Integration by Parts

\[ \mu = f 1_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2): \]

\[ \langle \mu, \varphi \rangle = \int_{-1}^{1} \varphi f \, dx \]
Using Integration by Parts

\[ \mu = f 1_G \text{ with } G = [-1,1] \text{ and } f(x) = \exp(-x^2) : \]

\[ \int_{-1}^{1} \varphi (\partial_x - 2x)f \, dx \]
Using Integration by Parts

\[- \mu = f 1_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2) : \]

\[
0 = \int_{-1}^{1} \varphi \left( \partial_x - 2x \right) f \, dx
\]
Using Integration by Parts

\[ \mu = f 1_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2): \]

\[
0 = \int_{-1}^{1} \varphi \left( \partial_x - 2x \right) f \, dx = \int_{-1}^{1} (-\partial_x - 2x) \varphi \, f \, dx + [\varphi f]_{-1}^{1}
\]
Using Integration by Parts

\[ \mu = f \mathbf{1}_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2): \]

\[ 0 = \int_{-1}^{1} \varphi \left( 1 - x^2 \right) (\partial_x - 2x) f \, dx = 0 \]
Using Integration by Parts

\[ \mu = f 1_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2): \]

\[ 0 = \int_{-1}^{1} \varphi \left( 1 - x^2 \right) \left( \partial_x - 2x \right) f \, dx = \int_{-1}^{1} \left( \partial_x + 2x \right) \left( x^2 - 1 \right) \varphi \ f \, dx + \left[ (x^2 - 1) \varphi f \right]_{-1}^{1} = 0 \]
Using Integration by Parts

\[ \mu = f \mathbf{1}_G \text{ with } G = [-1, 1] \text{ and } f(x) = \exp(-x^2) : \]

\[ 0 = \int_{-1}^{1} \varphi \left( 1 - x^2 \right) \left( \partial_x - 2x \right) f \, dx = \int_{-1}^{1} \left( \partial_x + 2x \right) (x^2 - 1) \varphi \ f \, dx + \left[ (x^2 - 1) \varphi f \right]_{-1}^{1} = 0 \]

\[ \Rightarrow (1 - x^2)(\partial_x - 2x) \in \text{Ann}(\mu) \]
Using Integration by Parts

− \( \mu = f \mathbf{1}_G \) with \( G = [-1, 1] \) and \( f(x) = \exp(-x^2) \):

\[
0 = \int_{-1}^{1} \varphi \left( 1 - x^2 \right) \left( \partial_x - 2x \right) f \, dx = \int_{-1}^{1} \left( \partial_x + 2x \right) (x^2 - 1) \varphi \, f \, dx + \left[ (x^2 - 1) \varphi f \right]_{-1}^{1} = 0
\]

\Rightarrow (1 - x^2)(\partial_x - 2x) \in \text{Ann}(\mu)

− replace \( \varphi = x^k \) to obtain a recurrence

\[
\int_{-1}^{1} (\partial_x + 2x)(x^2 - 1) x^k \, f(x) \, dx = 0
\]
Using Integration by Parts

\(- \mu = f \mathbf{1}_G \) with \( G = [-1, 1] \) and \( f(x) = \exp(-x^2) \):

\[
0 = \int_{-1}^{1} \varphi \left( 1 - x^2 \right) \left( \partial_x - 2x \right) f \, dx = \int_{-1}^{1} \left( \partial_x + 2x \right) (x^2 - 1) \varphi \, f \, dx + \left[ (x^2 - 1) \varphi f \right]_{-1}^{1} = 0
\]

\( \Rightarrow (1 - x^2)(\partial_x - 2x) \in \text{Ann}(\mu) \)

- replace \( \varphi = x^k \) to obtain a recurrence

\[
\int_{-1}^{1} \left( 2x^{k+3} + kx^{k+1} - kx^{k-1} \right) f(x) \, dx = 0
\]
Using Integration by Parts

− \(\mu = f 1_G\) with \(G = [-1, 1]\) and \(f(x) = \exp(-x^2)\):

\[
0 = \int_{-1}^{1} \varphi \left(1 - x^2\right)(\partial_x - 2x)f \, dx = \int_{-1}^{1} (\partial_x + 2x)(x^2 - 1)\varphi \, f \, dx + \left[(x^2 - 1)\varphi f\right]_{-1}^{1} \quad = 0
\]

⇒ \((1 - x^2)(\partial_x - 2x) \in \text{Ann}(\mu)\)

− replace \(\varphi = x^k\) to obtain a recurrence

\[
\int_{-1}^{1} \left(2x^{k+3} + kx^{k+1} - kx^{k-1}\right) f(x) \, dx = 0
\]

⇒ Recurrence for the \(m_k\):

\[
2m_{k+3} + km_{k+1} - km_{k-1} = 0
\]
The General Case

\[ \mu = f \mathbf{1}_G, \quad L \in K[x] \langle \partial_x \rangle \text{ of order } r, \]

- Use Lagrange identity:

\[ \varphi (Lf) - (L^* \varphi) f = \partial_x \mathcal{L}_L (f, \varphi) \]

\[ \rightarrow \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]
The General Case

\[ \mu = f \mathbf{1}_G, \quad L \in \mathbb{K}[x]\langle \partial_x \rangle \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use **Lagrange identity**:

\[ \varphi (Lf) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L(f, \varphi) \]

\[ \rightarrow \quad \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]
The General Case

\[ \mu = f \mathbf{1}_G, \quad L \in \mathbb{K}[x]\langle \partial_x \rangle \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use Lagrange identity:

\[ \varphi (L f) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L(f, \varphi) \]

\[ \rightarrow \quad \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]

\[ \int_G \varphi (L f) \, dx - \int_G (L^* \varphi) f \, dx = \int_G \nabla \cdot \mathcal{L}_L(f, \varphi) \, dx \]
The General Case

\[ \mu = f1_G, \quad L \in K[x] \langle \partial_x \rangle \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use Lagrange identity:

\[ \varphi (Lf) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L(f, \varphi) \]

\[ \Rightarrow \quad \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]

\[ \int_G \varphi (Lf) \, dx - \int_G (L^* \varphi) f \, dx = \int_G \nabla \cdot \mathcal{L}_L(f, \varphi) \, dx \]
The General Case

\[ \mu = f_{1_G}, \quad L \in \mathbb{K}[x] \langle \partial_x \rangle \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use Lagrange identity:

\[ \varphi (Lf) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L(f, \varphi) \]

\[ \Rightarrow \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]

\[ - \int_G \varphi (Lf) \, dx = \int_G (L^* \varphi) f \, dx = \int_G \nabla \cdot \mathcal{L}_L(f, \varphi) \, dx = \int_{\partial G} \mathcal{L}_L(f, \varphi) \cdot \hat{n} \, dS \]

\[ \Rightarrow \text{ use Stokes' theorem} \]
The General Case

\[ \mu = f1_G, \quad L \in K[x, \partial_x] \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use Lagrange identity:

\[ \varphi (L f) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L (f, \varphi) \]

\[ \rightarrow \quad \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]

\[ - \int_G g^r \varphi (L f) \, dx - \int_G (L^* g^r \varphi) f \, dx = \int_G \nabla \cdot \mathcal{L}_L (f, g^r \varphi) \, dx = \int_{\partial G} \mathcal{L}_L (f, g^r \varphi) \cdot \vec{n} \, dS \]

\[ \rightarrow \quad \text{where } g = 0 \text{ on } \partial G \quad \rightarrow \quad \text{use Stokes' theorem} \]
The General Case

\[ \mu = f \chi_G, \quad L \in K[x](\partial_x) \text{ of order } r, \quad x = (x_1, \ldots, x_n) \]

- Use Lagrange identity:

\[ \varphi (Lf) - (L^* \varphi) f = \nabla \cdot \mathcal{L}_L(f, \varphi) \]

\[ \Rightarrow \mathcal{L}_L \text{ bilinear concomitant in } f, \varphi \text{ with derivatives of order } \leq r - 1 \]

\[ = 0 \]

\[ \int_G g^r \varphi(Lf) \, dx - \int_G (L^* g^r \varphi) f \, dx = \int_G \nabla \cdot \mathcal{L}_L(f, g^r \varphi) \, dx = \int_{\partial G} \mathcal{L}_L(f, g^r \varphi) \cdot \bar{n} \, dS \]

\[ \Rightarrow \text{ if } L \in \text{Ann}(f) \quad \Rightarrow \text{ where } g = 0 \text{ on } \partial G \quad \Rightarrow \text{ use Stokes’ theorem} \]

\[ \Rightarrow \overline{L} = g^r L \in \text{Ann}(\mu) \]
Translate \( \bar{L} = g^r L \in \text{Ann}(\mu) \) into a recurrence on \((m_\alpha)\):

\[
\begin{align*}
  x_i & \rightarrow S_{\alpha_i} \\
  \partial_{x_i} & \rightarrow -\alpha_i S_{\alpha_i}^{-1}
\end{align*}
\]
Translate $\overline{L} = g^r L \in \mathcal{A}nn(\mu)$ into a recurrence on $(m_{\alpha})$:

\[
\begin{align*}
    x_i &\rightarrow S_{\alpha_i} \\
    \partial x_i &\rightarrow -\alpha_i S_{\alpha_i}^{-1}
\end{align*}
\]

Direct Problem

1. $\{L_1, \ldots, L_k\} \subseteq \mathcal{A}nn(f)$ D-finite
2. $\{\overline{L}_1, \ldots, \overline{L}_k\} \subseteq \mathcal{A}nn(\mu)$
3. Translate into $\{R_1, \ldots, R_k\} \subseteq \mathcal{A}nn(m_{\alpha})$
4. Gröbner basis algo on $\{R_1, \ldots, R_k\}$
Translate $\bar{L} = g^r L \in \mathcal{A}nn(\mu)$ into a recurrence on $(m_\alpha)$:

$$
\begin{align*}
    x_i & \rightarrow S_{\alpha_i} \\
    \partial x_i & \rightarrow -\alpha_i S_{\alpha_i}^{-1}
\end{align*}
$$

**Direct Problem**

1. $\{L_1, \ldots, L_k\} \subseteq \mathcal{A}nn(f)$ D-finite
2. $\{\bar{L}_1, \ldots, \bar{L}_k\} \subseteq \mathcal{A}nn(\mu)$
3. Translate into $\{R_1, \ldots, R_k\} \subseteq \mathcal{A}nn(m_\alpha)$
4. Gröbner basis algo on $\{R_1, \ldots, R_k\}$

**Theorem**

If $f(x) = \exp(p(x))$ and $g = 0$ on $\partial G$ s.t.

$$
\{x \in \mathbb{C}^n \mid g(x) = 0 \text{ and } \nabla g(x) = 0\} = \emptyset,
$$

then the recurrences system is holonomic.

$\Rightarrow$ Conjecture for the general case?
Translate $\bar{L} = g^r L \in \text{Ann}(\mu)$ into a recurrence on $(m_\alpha)$:

$\begin{align*}
\text{Direct Problem} & \\
\forall i & : \ x_i \to S_{\alpha_i} \\
\forall i & : \ \partial x_i \to -\alpha_i S_{\alpha_i}^{-1}
\end{align*}$

$\begin{align*}
\text{Inverse Problem} & \\
\text{1.} & : \ \{L_1, \ldots, L_k\} \subseteq \text{Ann}(f) \text{ D-finite} \\
\text{2.} & : \ \{\bar{L}_1, \ldots, \bar{L}_k\} \subseteq \text{Ann}(\mu) \\
\text{3.} & : \ \text{Translate into } \{R_1, \ldots, R_k\} \subseteq \text{Ann}(m_\alpha) \\
\text{4.} & : \ \text{Gröbner basis algo on } \{R_1, \ldots, R_k\}
\end{align*}$

**Theorem**

If $f(x) = \exp(p(x))$ and $g = 0$ on $\partial G$ s.t. 
$
\{x \in C^n \mid g(x) = 0 \text{ and } \nabla g(x) = 0\} = \emptyset,
$
then the recurrences system is holonomic.

⇒ Conjecture for the general case?

⇒ Reconstruct $\bar{L}_i$, then $g$ and $L_i$ from the given moments $m_\alpha$

⇒ Translation $\bar{L}_i \leftrightarrow R_i$ is linear

⇒ Holonomicity not needed
1 Introduction

2 Holonomic Distributions and Recurrences on Moments

3 Inverse Problem: Algorithms and Proofs
   - Exponential-Polynomial Densities
   - The General Case with D-Finite Densities

4 Limits and Perspectives
To reconstruct $g$ vanishing on $\partial G$ and $L \in \text{Ann}(f)$ of order $r$:

1. Make an ansatz $\tilde{L}$ for $\bar{L} = g^r L \in \text{Ann}(\mu)$

2. Find the coefficients of $\tilde{L}$ by solving the linear system:

$$
\langle \tilde{L} \mu, x^\alpha \rangle = \langle \mu, \tilde{L}^* x^\alpha \rangle = \int_G (\tilde{L}^* x^\alpha) f(x) \, dx = 0, \quad |\alpha| \leq N \quad (LS_N)
$$

requiring moments $m_\alpha$ for $|\alpha| \leq N + \ldots$

3. Extract $g$ and $L$ from $\tilde{L}$ using (numerical) GCDs
Inverse Problem — Roadmap and Issues

- To reconstruct $g$ vanishing on $\partial G$ and $L \in \text{Ann}(f)$ of order $r$:

  1. Make an ansatz $\tilde{L}$ for $L = g^r L \in \text{Ann}(\mu)$
  2. Find the coefficients of $\tilde{L}$ by solving the linear system:

\[
\langle \tilde{L} \mu, x^\alpha \rangle = \langle \mu, \tilde{L}^* x^\alpha \rangle = \int_G (\tilde{L}^* x^\alpha) f(x) \, dx = 0, \quad |\alpha| \leq N \quad (LS_N)
\]

  requiring moments $m_\alpha$ for $|\alpha| \leq N + \ldots$

  3. Extract $g$ and $L$ from $\tilde{L}$ using (numerical) GCDs

- Issues to be handled:

  ○ **False** solutions in $(LS_N)$: $\tilde{L} \notin \text{Ann}(\mu)$?
  ○ How many moments $m_\alpha$: a priori bounds on $N$?
  ○ Can $g$ and $L$ be always extracted from $\tilde{L} \in \text{Ann}(\mu)$?
Outline

1 Introduction

2 Holonomic Distributions and Recurrences on Moments

3 Inverse Problem: Algorithms and Proofs
   - Exponential-Polynomial Densities
   - The General Case with D-Finite Densities

4 Limits and Perspectives
Reconstruction of Exp-Poly Densities

\[ \mu = f1_G \text{ with } f(x) = \exp(p(x)) \text{ for } p \in \mathbb{K}[x]_s \text{ and } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G \]

\[ \overline{L}_i = g(\partial x_i - p'_{x_i}) \in \text{Ann}(\mu) \]
Reconstruction of Exp-Poly Densities

\[ \mu = f \mathbf{1}_G \] with \( f(x) = \exp(p(x)) \) for \( p \in \mathbb{K}[x]_s \) and \( g \in \mathbb{K}[x]_d \) vanishing on \( \partial G \)

\[ \overline{L}_i = g \partial x_i - gp'_{x_i} \in \text{Ann}(\mu) \]
Reconstruction of Exp-Poly Densities

\[ \mu = f_1 \mathbf{1}_G \] with \( f(x) = \exp(p(x)) \) for \( p \in \mathbb{K}[x]_s \) and \( g \in \mathbb{K}[x]_d \) vanishing on \( \partial G \)

\[ \overline{L}_i = g \partial x_i - \underbrace{g p'_{x_i}}_{h_i} \in \text{Ann}(\mu) \]

**Algorithm RECONSTRUCTEXPPOLY**

**Input:** Moments \( m_{\alpha} \) of \( \mu \) for \( |\alpha| \leq N + d + s - 1 \)

**Output:** Polynomials \( \widetilde{g} \) and \( \widetilde{p} \)
Reconstruction of Exp-Poly Densities

\[- \mu = f \mathbf{1}_G \text{ with } f(x) = \exp(p(x)) \text{ for } p \in \mathbb{K}[x]_s \text{ and } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G \]

\[ \bar{L}_i = g \partial x_i - g p'_{x_i} \in \text{Ann}(\mu) \]

**Algorithm** \textsc{ReconstructExpPoly}

**Input:** Moments \( m_\alpha \) of \( \mu \) for \( |\alpha| \leq N + d + s - 1 \)

**Output:** Polynomials \( \bar{g} \) and \( \bar{p} \)

1. Build ansatz \( \bar{L}_i = \bar{g} \partial x_i - \bar{h}_i \) for \( 1 \leq i \leq n \)

2. Compute coefficients of \( \bar{g}, \bar{h}_i \) with nontrivial solution of

\[ \langle \mu, \bar{L}_i^* x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N \quad (LS_N) \]

3. \( \bar{p} \leftarrow \sum_{i=1}^{n} \int_0^{x_i} \bar{p}_i(0, \ldots, t_i, x_{i+1}, \ldots, x_n) \, dt_i \quad \text{where} \quad \bar{p}_i = \bar{h}_i / \bar{g} \)
Reconstruction of Exp-Poly Densities

\[ \mu = f \mathbf{1}_G \text{ with } f(x) = \exp(p(x)) \text{ for } p \in \mathbb{K}[x]_s \text{ and } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G \]

\[ \overline{L}_i = g \partial_{x_i} - g p'_{x_i} \in \text{Ann}(\mu) \]

Algorithm RECONSTRUCTEXP POLY

Input: Moments \( m_\alpha \) of \( \mu \) for \( |\alpha| \leq N + d + s - 1 \)
Output: Polynomials \( \tilde{g} \) and \( \tilde{p} \)

1. Build ansatz \( \tilde{L}_i = \tilde{g} \partial_{x_i} - \tilde{h}_i \) for \( 1 \leq i \leq n \)
2. Compute coefficients of \( \tilde{g}, \tilde{h}_i \) with nontrivial solution of

\[ \langle \mu, \tilde{L}_i x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N \]

(LS\(_N\))

3. \( \tilde{p} \leftarrow \sum_{i=1}^{n} \int_{0}^{x_i} \tilde{p}_i(0, \ldots, t_i, x_{i+1}, \ldots, x_n) \, dt_i \) where \( \tilde{p}_i = \tilde{h}_i/\tilde{g} \)

Theorem — Correctness of RECONSTRUCTEXP POLY

If \( N \geq 3d + s - 1 \), then RECONSTRUCTEXP POLY computes:

- \( \tilde{g} = \lambda g \) with \( \lambda \neq 0 \)
- \( \tilde{p} = p - p(0) \)
Theorem — Correctness of \textsc{ReconstructExpPoly}

If $N \geq ???$, then \textsc{ReconstructExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.
Theorem — Correctness of \texttt{RECONSTRUCTExpPoly}

If $N \geq \text{???}$, then \texttt{RECONSTRUCTExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.
1. Reconstruction of $p$

2. Reconstruction of $g$
If $N \geq ???$, then RECONSTRUCT_EXP_POLY computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

**Proof.**

1. Reconstruction of $p$

   For all $\varphi \in \mathbb{K}[x]_N$:

   $$0 = \langle \tilde{L}\mu, \varphi \rangle$$

2. Reconstruction of $g$
Theorem — Correctness of $\text{ReconstructExpPoly}$

If $N \geq ???$, then $\text{ReconstructExpPoly}$ computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$

   For all $\varphi \in \mathbb{K}[x]_N$:

   $$0 = \langle L\mu, \varphi \rangle = \int_G \varphi (\tilde{g} \partial x_i - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \tilde{e}_i \cdot \tilde{n} \, dS$$

2. Reconstruction of $g$
Theorem — Correctness of \textsc{ReconstructExpPoly}

If $N \geq \ldots$, then \textsc{ReconstructExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$ for all $\varphi \in \mathbb{K}[x]_N$:

$$0 = \langle \mathcal{L}_\mu, \varphi \rangle = \int_G \varphi (\tilde{g} p'_{x_i} - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \vec{e}_i \cdot \vec{n} \, dS$$

2. Reconstruction of $g$
Theorem — Correctness of \textsc{ReconstructExpPoly}

If $N \geq 3d + s - 1$, then \textsc{ReconstructExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$

   For all $\varphi \in \mathbb{K}[x]_N$:

   $$0 = \langle \mathcal{L}_\mu, \varphi \rangle = \int_G \varphi (\tilde{g} p'_x - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \bar{e}_i \cdot \bar{n} \, dS$$

   $$= 0$$

   Take $\varphi = (\tilde{g} p'_x - \tilde{h}_i) g^2$ of degree $3d + s - 1$

2. Reconstruction of $g$
Theorem — Correctness of \textsc{ReconstructExpPoly}

If $N \geq 3d + s - 1$, then \textsc{ReconstructExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$

   For all $\varphi \in \mathbb{K}[x]_N$:

   $$0 = \langle \mathcal{L}_\mu, \varphi \rangle = \int_G \varphi (\tilde{g} p_x' - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \vec{e}_i \cdot \vec{n} \, dS$$

   \[=\]

   \[
   \Rightarrow \text{Take } \varphi = (\tilde{g} p_x' - \tilde{h}_i) g^2 \text{ of degree } 3d + s - 1
   \]

   \[
   \Rightarrow \text{Hence } (\star) = 0 \Rightarrow g^2 (\tilde{g} p_x' - \tilde{h}_i)^2 f = 0 \text{ on } G \Rightarrow p_x' = \tilde{h}_i / \tilde{g}
   \]

2. Reconstruction of $g$
Reconstruction of Exp-Poly Densities — Proof

**Theorem — Correctness of RECONSTRUCTEXPPOLY**

If $N \geq 3d + s - 1$, then $\text{RECONSTRUCTEXPPOLY}$ computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

**Proof.**

1. Reconstruction of $p$ for all $\varphi \in K[X]_N$:

   $$0 = \langle \tilde{L}_\mu, \varphi \rangle = \int_G \varphi (\tilde{g} p'_x - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \vec{e}_i \cdot \vec{n} \, dS$$

   \[(\ast)\]

   \[\Rightarrow\ T\ake \ \varphi = (\tilde{g} p'_x - \tilde{h}_i) g^2 \text{ of degree } 3d + s - 1\]

   \[\Rightarrow\ Henc\e \ (\ast) = 0 \quad \Rightarrow \quad g^2 (\tilde{g} p'_x - \tilde{h}_i)^2 f = 0 \text{ on } G \quad \Rightarrow \quad p'_x = \tilde{h}_i / \tilde{g}\]

2. Reconstruction of $g$
Theorem — Correctness of \textsc{ReconstructExpPoly}

If \( N \geq 3d + s - 1 \), then \textsc{ReconstructExpPoly} computes:

- \( \tilde{g} = \lambda g \) with \( \lambda \neq 0 \)
- \( \tilde{p} = p - p(0) \)

**Proof.**

1. Reconstruction of \( p \)

   For all \( \varphi \in \mathbb{K}[x]_N \):
   
   \[
   0 = \langle \tilde{L}_\mu, \varphi \rangle = \left( \int_G \varphi (\tilde{g} p'_{x_i} - \tilde{h}_i) f \, dx \right) + \int_{\partial G} \tilde{g} \varphi f \, \tilde{e}_i \cdot \tilde{n} \, dS = 0
   \]

   \( \Rightarrow \) Take \( \varphi = (\tilde{g} p'_{x_i} - \tilde{h}_i) g^2 \) of degree \( 3d + s - 1 \)

   \( \Rightarrow \) Hence \((*) = 0 \) \( \Rightarrow \) \( g^2(\tilde{g} p'_{x_i} - \tilde{h}_i)^2 f = 0 \) on \( G \) \( \Rightarrow \) \( p'_{x_i} = \tilde{h}_i/\tilde{g} \)

2. Reconstruction of \( g \)

   For all \( \varphi \in \mathbb{K}[x]_N \):

   \[
   \int_{\partial G} \tilde{g} \varphi f \, \tilde{e}_i \cdot \tilde{n} \, dS = 0
   \]
Theorem — Correctness of \texttt{RECONSTRUCT\_EXP\_POLY}

If $N \geq 3d + s - 1$, then \texttt{RECONSTRUCT\_EXP\_POLY} computes:

\begin{itemize}
  \item $\tilde{g} = \lambda g$ with $\lambda \neq 0$
  \item $\tilde{p} = p - p(0)$
\end{itemize}

Proof.

1. Reconstruction of $p$

\begin{equation}
0 = \langle \tilde{L}_\mu, \varphi \rangle = \int_G \varphi (\tilde{g} p'_{x_i} - \tilde{h}_i) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \hat{e}_i \cdot \hat{n} \, dS
\end{equation}

\begin{itemize}
  \item Take $\varphi = (\tilde{g} p'_{x_i} - \tilde{h}_i) g^2$ of degree $3d + s - 1$
  \item Hence $(\ast) = 0 \Rightarrow g^2 (\tilde{g} p'_{x_i} - \tilde{h}_i)^2 f = 0 \text{ on } G \Rightarrow p'_{x_i} = \tilde{h}_i / \tilde{g}$
\end{itemize}

2. Reconstruction of $g$

\begin{equation}
\int_{\partial G} \tilde{g} \varphi f \, \hat{e}_i \cdot \hat{n} \, dS = 0
\end{equation}

\begin{itemize}
  \item $g'_{x_i} / \| \nabla g \|$
Theorem — Correctness of \textsc{reconstructExpPoly}

If $N \geq 3d + s - 1$, then \textsc{reconstructExpPoly} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$

   for all $\varphi \in \mathbb{K}[x]_N$:

   \[ 0 = \langle \mathcal{L}_\mu, \varphi \rangle = \int_G \varphi \left( \tilde{g} p'_x - \tilde{h}_i \right) f \, dx + \int_{\partial G} \tilde{g} \varphi f \, \hat{e}_i \cdot \hat{n} \, dS \]

   $\hspace{1cm} (\ast) \hspace{1cm}$

   $\Rightarrow$ Take $\varphi = (\tilde{g} p'_x - \tilde{h}_i) g^2$ of degree $3d + s - 1$

   $\Rightarrow$ Hence $(\ast) = 0$ \Rightarrow $g^2 (\tilde{g} p'_x - \tilde{h}_i)^2 f = 0$ on $G$ \Rightarrow $p'_x = \tilde{h}_i / \tilde{g}$

2. Reconstruction of $g$

   for all $\varphi \in \mathbb{K}[x]_N$:

   \[ \int_{\partial G} \tilde{g} \varphi f \, \hat{e}_i \cdot \hat{n} \, dS = 0 \]

   $\hspace{1cm} = g'_x / \| \nabla g \|

   $\Rightarrow$ Take $\varphi = \tilde{g} g'_x$ of degree $2d - 1$
Theorem — Correctness of \texttt{RECONSTRUCTEXP\textsc{POLY}}

If $N \geq 3d + s - 1$, then \texttt{RECONSTRUCTEXP\textsc{POLY}} computes:

- $\tilde{g} = \lambda g$ with $\lambda \neq 0$
- $\tilde{p} = p - p(0)$

Proof.

1. Reconstruction of $p$

   For all $\varphi \in K[x]_N$:
   
   $$0 = (\tilde{L}_\mu, \varphi) = \int_G \varphi \left( (\tilde{g} p'_{x_i} - \tilde{h}_i) f dx \right) + \int_{\partial G} \tilde{g} \varphi f \mathbf{e}_i \cdot \mathbf{n} dS$$

   $(\ast)$

   \[ \Rightarrow \text{Take } \varphi = (\tilde{g} p'_{x_i} - \tilde{h}_i) g^2 \text{ of degree } 3d + s - 1 \]

   \[ \Rightarrow \text{Hence } (\ast) = 0 \Rightarrow g^2 (\tilde{g} p'_{x_i} - \tilde{h}_i)^2 f = 0 \text{ on } G \Rightarrow p'_{x_i} = \tilde{h}_i / \tilde{g} \]

2. Reconstruction of $g$

   For all $\varphi \in K[x]_N$:

   $$\int_{\partial G} \tilde{g} \varphi f \mathbf{e}_i \cdot \mathbf{n} dS = 0$$

   \[= \frac{g'_{x_i}}{\| \nabla g \|} \] \(\ast\)

   \[ \Rightarrow \text{Take } \varphi = \tilde{g} g'_{x_i} \text{ of degree } 2d - 1 \Rightarrow \tilde{g}^2 g'_{x_i} \frac{f}{\| \nabla g \|} = 0 \text{ on } \partial G \Rightarrow \tilde{g} = 0 \text{ on } \partial G \]
Example — Algebraic Support, Gaussian Measure

→ Reconstruction of:

\[ f(x, y) = \exp(-x^2 + xy - y^2/2) \quad \text{and} \quad g(x, y) = (x^2 - 9/10)^2 + (y^2 - 11/10)^2 - 1 \]

Moments \((m_{ij})_{i+j\leq18}\) with 10 digits of accuracy
Reconstruction of:

\[ f(x, y) = \exp(-x^2 + xy - y^2/2) \quad \text{and} \quad g(x, y) = (x^2 - 9/10)^2 + (y^2 - 11/10)^2 - 1 \]

Moments \((m_{ij})_{i+j \leq 18}\) with 4, 6, 8 digits of accuracy
Reconstruction of:

\[ f(x, y) = \exp(-x^2 + xy - y^2/2) \quad \text{and} \quad g(x, y) = (x^2 - 9/10)^2 + (y^2 - 11/10)^2 - 1 \]

Moments \((m_{ij})_{i+j \leq 18}\) with 8 digits of accuracy
Outline

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2 Holonomic Distributions and Recurrences on Moments
   - Exponential-Polynomial Densities
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4 Limits and Perspectives
\[ \mu = f 1_G \text{ with } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G, \text{ and } \{L_1, \ldots, L_n\} \text{ rectangular system for } f:\]

\[ L_i = q_{i_1} \partial_{x_i}^{r_i} + \cdots + q_{i_1} \partial_{x_i} + q_{i_0} \in \text{Ann}(f) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle \]
\[ \mu = f \mathbf{1}_G \text{ with } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G, \text{ and } \{L_1, \ldots, L_n\} \text{ rectangular system for } f: \]

\[ L_i = g^{r_i}(q_{i1} \partial_{x_i} + \cdots + q_{i1} \partial_{x_i} + q_{i0}) \in \text{Ann}(\mu) \cap \mathbb{K}[x](\partial_{x_i}) \quad h_{ij} = g^{r_i} q_{ij} \in \mathbb{K}[x]_s \]
\[ \mu = f 1_G \text{ with } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G, \text{ and } \{ L_1, \ldots, L_n \} \text{ rectangular system for } f: \]
\[ \bar{L}_i = g^{r_i} (q_{ir_i} \partial_{x_i}^{r_i} + \cdots + q_{i1} \partial_{x_i} + q_{i0}) \in \text{Ann}(\mu) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle \quad h_{ij} = g^{r_i} q_{ij} \in \mathbb{K}[x]_s \]

Algorithm **ReconstructDensity**

**Input:** Moments \( m_\alpha \) of \( \mu \) for \( |\alpha| \leq N + s \)

**Output:** A rectangular system \( \{ \bar{L}_1, \ldots, \bar{L}_n \} \) for \( f \)
Density and Support Reconstruction in the General Case

\[ \mu = f 1_G \text{ with } g \in \mathbb{K}[x]_d \text{ vanishing on } \partial G, \text{ and } \{L_1, \ldots, L_n\} \text{ rectangular system for } f:\]

\[ L_i = g^{r_i}(q_{i_1} x_i + \cdots + q_{i_1} x_i + q_{i_0}) \in \text{Ann}(\mu) \cap \mathbb{K}[x] (\partial x_i) \quad h_{ij} = g^{r_i} q_{ij} \in \mathbb{K}[x] s \]

Algorithm \textsc{ReconstructDensity}

**Input:** Moments \( m_{\alpha} \) of \( \mu \) for \(|\alpha| \leq N + s\)

**Output:** A rectangular system \( \{\widetilde{L}_1, \ldots, \widetilde{L}_n\} \) for \( f \)

1. Build ansatz \( \widetilde{L}_i = h_{i_1} x_{i_1} + \cdots + h_{i_0} \) for \( 1 \leq i \leq n \)

2. Compute coefficients of \( \widetilde{h}_{ij} \) with nontrivial solution of

\[ \langle \mu, \widetilde{L}_i^{*} x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N \]

3. Extract (numerical) GCD polynomial factor in \( \widetilde{L}_i \)
\[ \mu = f1_G \text{ with } g \in K[x]_d \text{ vanishing on } \partial G, \text{ and } \{L_1, \ldots, L_n\} \text{ rectangular system for } f: \]
\[ \bar{L}_i = g^{r_i}(q_{i1}\partial_{x_i} + \cdots + q_{i0}) \in \text{Ann}(\mu) \cap K[x] \langle \partial x_i \rangle \quad h_{ij} = g^{r_i} q_{ij} \in K[x]_s \]

**Algorithm ReconstructDensity**

**Input:** Moments \( m_{\alpha} \) of \( \mu \) for \( |\alpha| \leq N + s \)

**Output:** A rectangular system \( \{\bar{L}_1, \ldots, \bar{L}_n\} \) for \( f \)

1. Build ansatz \( \bar{L}_i = \bar{h}_{ir_i} \partial_{x_i} + \cdots + \bar{h}_{i0} \) for \( 1 \leq i \leq n \)
2. Compute coefficients of \( \bar{h}_{ij} \) with nontrivial solution of
   \[ \langle \mu, \bar{L}_i^* x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N \]
3. Extract (numerical) GCD polynomial factor in \( \bar{L}_i \)

**Algorithm ReconstructSupport**

**Input:** Rectangular \( \{L_1, \ldots, L_n\} \) and \( m_{\alpha} \) for \( |\alpha| \leq N + dr + \max_{ij} \{\deg(q_{ij}) - j\} \)

**Output:** Polynomial \( \bar{g} \in K[x]_d \)
Density and Support Reconstruction in the General Case

- \( \mu = f 1_G \) with \( g \in \mathbb{K}[x]_d \) vanishing on \( \partial G \), and \( \{L_1, \ldots, L_n\} \) rectangular system for \( f \):

\[
\overline{L}_i = g^{r_i}(q_{ir_i} \partial_{x_i}^{r_i} + \cdots + q_{i1} \partial_{x_i} + q_{i0}) \in \text{Ann}(\mu) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle \quad h_{ij} = g^{r_i} q_{ij} \in \mathbb{K}[x]
\]

Algorithm \texttt{RECONSTRUCTDENSITY}

\begin{itemize}
  \item \textbf{Input:} Moments \( m_{\alpha} \) of \( \mu \) for \( |\alpha| \leq N + s \)
  \item \textbf{Output:} A rectangular system \( \{\overline{L}_1, \ldots, \overline{L}_n\} \) for \( f \)
\end{itemize}

1. Build ansatz \( \overline{L}_i = \overline{h}_{ir_i} \partial_{x_i}^{r_i} + \cdots + \overline{h}_{i0} \) for \( 1 \leq i \leq n \)
2. Compute coefficients of \( \overline{h}_{ij} \) with nontrivial solution of
   \[
   \langle \mu, (\overline{L}_i)^* x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N
   \]
3. Extract (numerical) GCD polynomial factor in \( \overline{L}_i \)

Algorithm \texttt{RECONSTRUCTSUPPORT}

\begin{itemize}
  \item \textbf{Input:} Rectangular \( \{L_1, \ldots, L_n\} \) and \( m_{\alpha} \) for \( |\alpha| \leq N + dr + \max_{ij}\{\deg(q_{ij}) - j\} \)
  \item \textbf{Output:} Polynomial \( \overline{g} \in \mathbb{K}[x]_d \)
\end{itemize}

1. Compute coefficients of ansatz \( \overline{h} \in \mathbb{K}[x]_{dr} \) with nontrivial solution of
   \[
   \langle \mu, (\overline{h} L_i)^* x^\alpha \rangle = 0, \quad 1 \leq i \leq n, \quad |\alpha| \leq N
   \]
2. \( \overline{g} \leftarrow \) (numerical) GCD of \( \{\overline{h}, \overline{h}_{x_1}', \ldots, \overline{h}_{x_n}'\} \)
Density and Support Reconstruction in the General Case

Theorem — Correctness of \( \text{ReconstructDensity} \)

For \( N \) large enough, the rectangular system \( \{\tilde{L}_1, \ldots, \tilde{L}_n\} \) computed by \( \text{ReconstructDensity} \) is in \( \text{Ann}(f) \).
Theorem — Correctness of \texttt{ReconstructDensity}

For \( N \) large enough, the rectangular system \( \{\tilde{L}_1, \ldots, \tilde{L}_n\} \) computed by \texttt{ReconstructDensity} is in \( \text{Ann}(f) \).

Theorem — Correctness of \texttt{ReconstructSupport}

\texttt{ReconstructSupport} computes \( \tilde{g} = \lambda g \) with \( \lambda \neq 0 \) whenever \( q_{ir} \neq 0 \) on \( \partial G \) and \( N \geq (2r - 1)d + (d - 1)b + s \) where:

- \( r = \max_{1 \leq i \leq n} r_i \), orders of the \( L_i \)
- \( b = r \mod 2 \)
- \( s = \max_{1 \leq i \leq n} \{\deg(q_{ir})\} \) maximal degree of the head coefficients
Theorem — Correctness of \texttt{RECONSTRUCT\_SUPPORT}

\texttt{RECONSTRUCT\_SUPPORT} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

\begin{align*}
N / \text{uni2A7E} & (2r-1)d + (d-1)b + s \\
\circ q_{ir} & \neq 0 \text{ on } \partial G
\end{align*}

Proof.

$\Rightarrow$ Contradiction: $\tilde{h}_0 = 0$ on $\partial G$, hence $g / \text{divides.alt0}$ $h_0$. 

\begin{align*}
\text{Proof.} \quad & \\
\text{Take } \phi = q_{ir} h_0 g^{r-1-k} g' x_i b \text{ of deg } \text{uni2A7D} (2r-1)d + (d-1)b + s, \text{ so that } g^{r-1} / \text{divides.alt0} \tilde{h} \phi \Rightarrow 0 = \text{integral.disp} \partial G \partial r-1 x_i (q_{ir} \tilde{h} \phi) g' x_i / \text{parallel.alt1} \nabla g / \text{parallel.alt1} f dS = (r-1)! / \text{integral.disp} \partial G / \text{parenleft.alt3} g' x_i r+b 2q_{ir} h_0 / \text{parenright.alt3} 2f / \text{parallel.alt1} \nabla g / \text{parallel.alt1} dS
\end{align*}
Theorem — Correctness of \textsc{ReconstructSupport}

\textsc{ReconstructSupport} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

Proof.
\begin{itemize}
  \item $0 = \langle \tilde{h} L_i \mu, \varphi \rangle$
  \item $\text{for } \varphi \in K[x]^N$
\end{itemize}
Theorem — Correctness of \texttt{ReconstructSupport}

\texttt{ReconstructSupport} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

\[ \text{Suppose for contradiction that } \tilde{h} = gh \text{ with } g \neq 0 \text{ and } k < r \]

\[ = \int_G \varphi \tilde{h}(L_i f) dx - \int_{\partial G} \mathcal{L}_{L_i}(f, \tilde{h}) \tilde{e}_i \cdot \tilde{n} dS \quad \text{for } \varphi \in K[x]_N \]
Theorem — Correctness of \textsc{ReconstructSupport}

\textsc{ReconstructSupport} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

$$N / \text{uni2A7E} (2r - 1)d + (d - 1)b + s \neq 0 \text{ on } \partial G$$

Proof. \quad $0 = \langle \tilde{h} L_i \mu, \varphi \rangle = \int_G \varphi \tilde{h}(L_i f) dx - \int_{\partial G} \mathcal{L}_i (f, \tilde{h} \varphi) \hat{e}_i \cdot \hat{n} dS \quad \text{for } \varphi \in \mathbb{K}[x]_N$
Theorem — Correctness of RECONSTRUCTSUPPORT

RECONSTRUCTSUPPORT computes \( \tilde{g} = \lambda g \) with \( \lambda \neq 0 \) whenever:

\[
\text{Proof.} \quad 0 = \langle \tilde{h} L_i \mu, \varphi \rangle = \int_G \varphi \tilde{h} (L_i f) \, dx - \int_{\partial G} \mathcal{L}_i (f, \tilde{h} \varphi) \bar{e}_i \cdot \bar{n} \, dS \quad \text{for } \varphi \in K[x]_N
\]

\[- \text{ Suppose for contradiction that } \tilde{h} = g^k h_0 \text{ with } g \nmid h_0 \text{ and } k < r \]
Theorem — Correctness of \textsc{ReconstructSupport}

\textsc{ReconstructSupport} computes \( \tilde{g} = \lambda g \) with \( \lambda \neq 0 \) whenever:

\[
0 = \langle \tilde{h} L_i \mu, \varphi \rangle = \int_G \varphi \tilde{h}(L_i f) \, dx - \int_{\partial G} \mathcal{L}_{L_i}(f, \tilde{h} \varphi) \, \tilde{e}_i \cdot \tilde{n} \, dS \quad \text{for } \varphi \in K[x]_N
\]

Proof.

- Suppose for contradiction that \( \tilde{h} = g^k h_0 \) with \( g \nmid h_0 \) and \( k < r \)

\[
\mathcal{L}_{L_i}(f, \tilde{h} \varphi) = f \left[ q_{i_1} \tilde{h} \varphi - \partial_{x_i}(q_{i_2} \tilde{h} \varphi) + \cdots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{i_r} \tilde{h} \varphi) \right]
+ \partial_{x_i}(f) \left[ q_{i_2} \tilde{h} \varphi - \partial_{x_i}(q_{i_3} \tilde{h} \varphi) + \cdots + (-1)^{r-2} \partial_{x_i}^{r-2}(q_{i_r} \tilde{h} \varphi) \right]
+ \cdots
+ \partial_{x_i}^{r-1}(f) q_{i_r} \tilde{h} \varphi.
\]
Theorem — Correctness of \textsc{ReconstructSupport}

\textsc{ReconstructSupport} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geq (2r - 1)d + (d - 1)b + s$

Proof.

- $0 = \langle \tilde{h} L_i \mu, \varphi \rangle = \int_G \varphi \tilde{h}(L_i f) dx - \int_{\partial G} \mathcal{L}_{L_i}(f, \tilde{h}\varphi) \bar{e}_i \cdot \vec{n} dS$ for $\varphi \in K[x]_N$

- Suppose for contradiction that $\tilde{h} = g^k h_0$ with $g \not| h_0$ and $k < r$

\[
\mathcal{L}_{L_i}(f, \tilde{h}\varphi) = f \left[ q_{i1} \tilde{h}\varphi - \partial_{x_i}(q_{i2} \tilde{h}\varphi) + \ldots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{ir} \tilde{h}\varphi) \right] \\
+ \partial_{x_i}(f) \left[ q_{i2} \tilde{h}\varphi - \partial_{x_i}(q_{i3} \tilde{h}\varphi) + \ldots + (-1)^{r-2} \partial_{x_i}^{r-2}(q_{ir} \tilde{h}\varphi) \right] \\
+ \ldots \\
+ \partial_{x_i}^{r-1}(f) \ q_{ir} \tilde{h}\varphi.
\]

$\rightarrow$ Take $\varphi = q_{ir} h_0 g^{r-1-k} g_{x_i}^l b$ of deg $\leq (2r - 1)d + (d - 1)b + s$, so that $g^{r-1} | \tilde{h}\varphi$
Theorem — Correctness of RECONSTRUCTSUPPORT

RECONSTRUCTSUPPORT computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geq (2r - 1)d + (d - 1)b + s$

Proof.

- $0 = \langle \tilde{h}L_i\mu, \varphi \rangle = \int_G \varphi \tilde{h}(L_i f) \, dx - \int_{\partial G} \mathcal{L}_{L_i}(f, \tilde{h}\varphi) \tilde{e}_i \cdot \tilde{n} \, dS$ for $\varphi \in K[x]_N$

- Suppose for contradiction that $\tilde{h} = g^k h_0$ with $g \nmid h_0$ and $k < r$

\[
\mathcal{L}_{L_i}(f, \tilde{h}\varphi) = f \left[ q_{i1} \tilde{h}\varphi - \partial_{x_i}(q_{i2} \tilde{h}\varphi) + \cdots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{ir} \tilde{h}\varphi) \right] \\
+ \partial_{x_i}(f) \left[ q_{i2} \tilde{h}\varphi - \partial_{x_i}(q_{i3} \tilde{h}\varphi) + \cdots + (-1)^{r-2} \partial_{x_i}^{r-2}(q_{ir} \tilde{h}\varphi) \right] \\
+ \cdots \\
+ \partial_{x_i}^{r-1}(f) q_{ir} \tilde{h}\varphi.
\]

$\rightarrow$ Take $\varphi = q_{ir} h_0 g^{r-1-k} g'_{x_i} b$ of deg $\leq (2r - 1)d + (d - 1)b + s$, so that $g^{r-1} \mid \tilde{h}\varphi$

$\rightarrow 0 = \int_{\partial G} \partial_{x_i}^{r-1}(q_{ir} \tilde{h}\varphi) \frac{g'_{x_i}}{\|\nabla g\|} f \, dS$
Theorem — Correctness of \textsc{ReconstructSupport}

\textsc{ReconstructSupport} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geq (2r - 1)d + (d - 1)b + s$

Proof.

$0 = \langle hL_{i}\mu, \varphi \rangle = \int_{G} \varphi \tilde{h}(L_{i}f)dx - \int_{\partial G} L_{L_{i}}(f, \tilde{h}\varphi)\bar{e}_{i} \cdot \bar{n}dS$

for $\varphi \in K[x]_{N}$

- Suppose for contradiction that $\tilde{h} = g^{k}h_{0}$ with $g \nmid h_{0}$ and $k < r$

$$\mathcal{L}_{L_{i}}(f, \tilde{h}\varphi) = f\left[q_{i1}\tilde{h}\varphi - \partial_{x_{i}}(q_{i2}\tilde{h}\varphi) + \cdots + (-1)^{r-1}\partial_{x_{i}}^{r-1}(q_{ir}\tilde{h}\varphi)\right] + \partial_{x_{i}}(f)\left[q_{i2}\tilde{h}\varphi - \partial_{x_{i}}(q_{i3}\tilde{h}\varphi) + \cdots + (-1)^{r-2}\partial_{x_{i}}^{r-2}(q_{ir}\tilde{h}\varphi)\right] + \cdots + \partial_{x_{i}}^{r-1}(f)q_{ir}\tilde{h}\varphi.$$

Take $\varphi = q_{ir}h_{0}g^{r-1-k}g'_{x_{i}}^{b}$ of deg $\leq (2r - 1)d + (d - 1)b + s$, so that $g^{r-1} \mid \tilde{h}\varphi$

$$0 = \int_{\partial G} \partial_{x_{i}}^{r-1}(q_{ir}\tilde{h}\varphi)\frac{g'_{x_{i}}}{\|\nabla g\|}f dS = (r - 1)! \int_{\partial G} \left(g'_{x_{i}}\frac{r+b}{2} q_{ir}h_{0}\right)^{2} \frac{f}{\|\nabla g\|} dS$$
**Theorem — Correctness of** \texttt{RECONSTRUCTSUPPORT}

\texttt{RECONSTRUCTSUPPORT} computes $\tilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geq (2r - 1)d + (d - 1)b + s$
- $q_{ir} \neq 0$ on $\partial G$

**Proof.**

- $0 = \langle \tilde{h}L_i \mu, \varphi \rangle = \int_G \varphi \tilde{h}(L_i f) \, dx - \int_{\partial G} \mathcal{L}_{L_i}(f, \tilde{h}\varphi) \bar{e}_i \cdot \tilde{n} \, dS$ for $\varphi \in K[x]_N$

- Suppose for contradiction that $\tilde{h} = g^k h_0$ with $g \notdivides h_0$ and $k < r$

\[
\mathcal{L}_{L_i}(f, \tilde{h}\varphi) = f \left[ q_{i1} \tilde{h}\varphi - \partial_{x_i}(q_{i2} \tilde{h}\varphi) + \cdots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{ir} \tilde{h}\varphi) \right] \\
+ \partial_{x_i}(f) \left[ q_{i2} \tilde{h}\varphi - \partial_{x_i}(q_{i3} \tilde{h}\varphi) + \cdots + (-1)^{r-2} \partial_{x_i}^{r-2}(q_{ir} \tilde{h}\varphi) \right] \\
+ \cdots \\
+ \partial_{x_i}^{r-1}(f) q_{ir} \tilde{h}\varphi.
\]

$\rightarrow$ Take $\varphi = q_{ir} h_0 g^{r-1-k} g'_{x_i} \bar{b}$ of deg $\leq (2r - 1)d + (d - 1)b + s$, so that $g^{r-1} \mid \tilde{h}\varphi$

$\rightarrow 0 = \int_{\partial G} \partial_{x_i}^{r-1}(q_{ir} \tilde{h}\varphi) \frac{g'_{x_i}}{\| \nabla g \|} f \, dS = (r - 1)! \int_{\partial G} \left( g'_{x_i} \frac{r+b}{2} q_{ir} h_0 \right)^2 \frac{f}{\| \nabla g \|} \, dS$

$\Rightarrow$ **Contradiction:** $h_0 = 0$ on $\partial G$, hence $g \mid h_0$
Express Catalan numbers as moments of a measure $\mu$:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \int_I x^n f(x) \, dx$$
Express **Catalan numbers** as moments of a measure $\mu$:

$$ C_n = \frac{1}{n+1} \binom{2n}{n} = \int x^n f(x) \, dx $$

$$(n + 2) C_{n+1} - (4n + 2) C_n = 0$$
→ Express **Catalan numbers** as moments of a measure \( \mu \):

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\]

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(n + 2) C_{n+1} - (4n + 2) C_n = 0
\]

→ Reverse translation \( x \leftarrow S_n \) and \( \partial x \leftarrow -S_n^{-1}(n + 1) \):

\[
(n + 2) S_n - (4n + 2)
\]
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$$C_n = \frac{1}{n+1} \binom{2n}{n} = \int_I x^n f(x) \, dx$$

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Reverse translation $x \leftrightarrow S_n$ and $\partial x \leftrightarrow -S_n^{-1}(n + 1)$:

$$S_n(n + 1) - 4(n + 1) + 2$$
Express **Catalan numbers** as moments of a measure \( \mu \):

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Reverse translation \( x \leftrightarrow S_n \) and \( \partial_x \leftrightarrow -S_n^{-1}(n + 1) \):

\[
S_n^2 S_n^{-1}(n + 1) - 4S_n S_n^{-1}(n + 1) + 2
\]
Express **Catalan numbers** as moments of a measure $\mu$:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \overset{?}{=} \int_I x^n f(x) \, dx$$

$$(n + 2)C_{n+1} - (4n + 2)C_n = 0$$

Reverse translation $x \leftarrow S_n$ and $\partial_x \leftarrow -S_n^{-1}(n + 1)$:

$$\underbrace{S_n^2}_{x^2} \underbrace{S_n^{-1}(n + 1)}_{-\partial_x} - 4 \underbrace{S_n}_{x} \underbrace{S_n^{-1}(n + 1)}_{-\partial_x} + 2$$
→ Express **Catalan numbers** as moments of a measure $\mu$:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \int_I x^n f(x) \, dx$$

$$(n + 2) C_{n+1} - (4n + 2) C_n = 0$$

- Reverse translation $x \leftarrow S_n$ and $\partial_x \leftarrow -S^{-1}_n(n + 1)$:

$$S^2_n S^{-1}_n(n + 1) - 4 S_n S^{-1}_n(n + 1) + 2$$

$$\Rightarrow (4x - x^2) \partial_x + 2 \in \text{Ann}(\mu)$$
Express Catalan numbers as moments of a measure $\mu$:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \int_{I} x^n f(x) dx$$

$$(n + 2)C_{n+1} - (4n + 2)C_n = 0$$

Reverse translation $x \mapsto S_n$ and $\partial_x \mapsto -S_{n}^{-1}(n + 1)$:

$$\begin{align*}
S_n^2 - \underbrace{S_{n}^{-1}(n + 1) - 4 S_n S_{n}^{-1}(n + 1)}_{x^2 - \partial_x} + 2 \\
\underbrace{x}_{x^2} - \underbrace{\partial_x}_{-\partial_x}
\end{align*}$$

$$\Rightarrow \quad (4x - x^2) \partial_x + 2 \in \text{Ann}(\mu) \quad g = 1 ?$$

$$C_n = \lambda \int_{-\infty}^{+\infty} x^n \sqrt{\frac{4-x}{x}} dx$$
→ Express **Catalan numbers** as moments of a measure $\mu$:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \int x^n f(x) \, dx$$

$$(n+2)C_{n+1} - (4n+2)C_n = 0$$

← Reverse translation $x \leftarrow S_n$ and $\partial_x \leftarrow -S_n^{-1}(n+1)$:

$$S_n^2 S_n^{-1}(n+1) - 4 S_n S_n^{-1}(n+1) + 2$$

$$+2 \quad +2 \quad \binom{2n}{n}$$

$$\Rightarrow (4x - x^2) \partial_x + 2 \in \text{Ann}(\mu) \quad \text{?}$$

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}} \, dx \quad ?$$
Outline

1 Introduction

2 Holonomic Distributions and Recurrences on Moments

3 Inverse Problem: Algorithms and Proofs
   ■ Exponential-Polynomial Densities
   ■ The General Case with D-Finite Densities

4 Limits and Perspectives
Some Limits and Perspectives

- A priori bounds for $N$ in the general case with unknown D-finite density?

- Full determination of the density, including initial conditions

- Extracting the component of $V(g)$ corresponding to $\partial G$
Is there an explicit bound $N_0$ on $N$ s.t. for ansatz $\widetilde{L}$ of $\overline{L} = g' L$:

$$\langle \overline{L} \mu, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathbb{K}[x]_N \quad \Rightarrow \quad \widetilde{L} \mu = 0 \quad \text{when } N \geq N_0$$?
Is there an explicit bound $N_0$ on $N$ s.t. for ansatz $\tilde{L}$ of $L = g' L$:

$$\langle \tilde{L} \mu, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathbb{K}[x]_N \quad \Rightarrow \quad \tilde{L} \mu = 0 \quad \text{when } N \geq N_0$$

The proof of the Exp-Poly density case doesn’t generalize:

$$\langle \tilde{L} \mu, \varphi \rangle = \int_G \varphi (\tilde{L} f) \, dx - \int_{\partial G} L \tilde{L} (f, \varphi) \cdot \tilde{n} \, dS$$
Is there an explicit bound $N_0$ on $N$ s.t. for ansatz $\tilde{L}$ of $\bar{L} = g^r L$:

$$\langle \tilde{L} \mu, \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in \mathbb{K}[x]_N \quad \Rightarrow \quad \tilde{L} \mu = 0 \quad \text{when} \quad N \geq N_0$$

The proof of the Exp-Poly density case doesn’t generalize:

$$\langle \tilde{L} \mu, \varphi \rangle = \int_G \varphi(\bar{L} f) \, dx - \int_{\partial G} L_{\bar{L}}(f, \varphi) \cdot \tilde{n} \, dS$$
Is there an explicit bound $N_0$ on $N$ s.t. for ansatz $\tilde{\mathcal{L}}$ of $\mathcal{L} = g^r L$:

\[
\langle \mathcal{L} \mu, \phi \rangle = 0 \quad \text{for all } \phi \in \mathbb{K}[x]_N \quad \Rightarrow \quad \mathcal{L} \mu = 0 \quad \text{when } N \geq N_0 \quad ?
\]

The proof of the Exp-Poly density case doesn’t generalize:

\[
\langle \tilde{\mathcal{L}} \mu, \phi \rangle = \int_G \phi(\mathcal{L} f) \, dx - \int_{\partial G} \mathcal{L} \tilde{\mathcal{L}}(f, \phi) \cdot \vec{n} \, dS
\]

Such a bound $N_0$ depending only on the structure of $\tilde{\mathcal{L}}$ cannot exist:

Example [Batenkov2009] — Legendre Polynomials $P_n$ over $[-1, 1]$

$P_n(x)$ annihilated by $L_n = \partial_x \left( (1 - x^2) \partial_x \right) + n(n + 1) \quad \Rightarrow \quad$ common ansatz $\tilde{\mathcal{L}}$

but $m_k^{(n)} = \int_{-1}^{1} x^k P_n(x) \, dx = 0 \quad \text{for } k < n \quad \text{and} \quad m_n^{(n)} > 0$

→ Explicit bounds depending on upper bounds on the coefficients of $\tilde{\mathcal{L}}$?
Algorithm \textsc{ReconstructDensity} only computes a system \( \mathcal{I} = \{ \mathcal{L}_1, \ldots, \mathcal{L}_n \} \) but not the initial conditions that fully characterize \( f \).
Reconstructing Initial Conditions of the Density

\[ f(x, y) = \lambda_1 e^{p_1(x, y)} \]

\[ p_1 = -\frac{1}{2} \begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix}^T \Sigma_1^{-1} \begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix} \]
Reconstructing Initial Conditions of the Density

\[ f(x, y) = \lambda_1 e^{p_1(x, y)} \]

\[
p_1 = -\frac{1}{2} \begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix}^T \Sigma_1^{-1} \begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix}
\]

\[
\lambda_1 = \frac{1}{2\pi\sqrt{|\Sigma|}}
\]
Reconstructing Initial Conditions of the Density

\[ f(x, y) = \lambda_1 e^{p_1(x,y)} + \lambda_2 e^{p_2(x,y)} + \lambda_3 e^{p_3(x,y)} \]

\[ p_i = -\frac{1}{2} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix}^T \Sigma_i^{-1} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix} \]
Reconstructing Initial Conditions of the Density

\[ f(x, y) = \lambda_1 e^{p_1(x,y)} + \lambda_2 e^{p_2(x,y)} + \lambda_3 e^{p_3(x,y)} \]

\[ p_i = -\frac{1}{2} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix}^T \Sigma_i^{-1} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix} \quad \lambda_i = ??? \]
Algorithm \textsc{ReconstructDensity} only computes a system \( \mathcal{I} = \{ \tilde{L}_1, \ldots, \tilde{L}_n \} \) but not the initial conditions that fully characterize \( f \)

Solution: compute initial moments for a basis of solution densities of \( \mathcal{I} \)

- Optimization techniques, e.g., [HenrionLasserreSavorgnan2009]
- Computer algebra, e.g., [LairezMezzarobbaElDin2019]
Isolation of the Topological Boundary

\[ I(\partial G) = g(x, y) \]

where \( g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)(x^2 + (y - 2)^2 - 1) \).
Isolation of the Topological Boundary

\[ I(\partial G) = (g) \quad \text{with} \quad g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x-2)^2 + y^2 - 1)(x^2 + (y-2)^2 - 1) \]
\( I(\partial G) = (g) \) with 
\[
g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x-2)^2 + y^2 - 1)(x^2 + (y-2)^2 - 1)
\]

\( \tilde{g} \) reconstructed using 6 digits accuracy for the moments \( (m_\alpha) \)
Isolation of the Topological Boundary

\[
I(\partial G) = (g) \quad \text{with} \quad g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)(x^2 + (y - 2)^2 - 1)
\]

\(\tilde{g}\) reconstructed using 4 digits accuracy for the moments \(m_{\alpha}\)
Isolation of the Topological Boundary

\[ I(\partial G) = (g) \quad \text{with} \quad g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)(x^2 + (y - 2)^2 - 1) \]

\( \tilde{g} \) reconstructed using 2 digits accuracy for the moments \((m_\alpha)\)
\( l(\partial G) = (g) \) with \( g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)(x^2 + (y - 2)^2 - 1) \)

\( \widetilde{g} \) reconstructed using 1 digit accuracy for the moments \( (m_\alpha) \)
Isolation of the Topological Boundary

\[ I(\partial G) = (g) \quad \text{with} \quad g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x - 2)^2 + y^2 - 1)(x^2 + (y - 2)^2 - 1) \]

\( \tilde{g} \) reconstructed using 2 digits accuracy for the moments \( (m_\alpha) \)
Isolation of the Topological Boundary

\[ I(\partial G) = (g) \quad \text{with} \quad g(x, y) = (x^2 + y^2 - 9)(x^2 + y^2 - 1)((x-2)^2 + y^2 - 1)(x^2 + (y-2)^2 - 1) \]

\[ \partial G \approx \{ (x, y) \mid g(x, y) = 0 \quad \text{and} \quad \mathbb{E}[\tilde{g}(x, y)^2] \leq \epsilon \} , \quad \tilde{g} \leftarrow \text{randomly perturbed} \ (\tilde{m}_\alpha) \]
Conclusion and Perspectives

Contributions:

- Extension of [LasserrePutinar2015] to reconstruction of unknown Exp-Poly density and unknown semi-algebraic support
  → Explicit bound for the number $N$ of required moments
- Reconstruction algorithm for unknown holonomic density and unknown semi-algebraic support
- Numerical experiments using least-squares approximation when approximate moments are known

Future work:

- Generic bounds for $N$ depending on the magnitude of the coefficients
- Numerical aspects: robustness w.r.t. approximate moments, or nonpolynomial boundary
- Isolation of the topological boundary via perturbation techniques
- Application to problems involving combinatorial sequences
Thanks!