Periods
Numerical computation and applications

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What is a period?

A period is the integral on a closed path of a rational function in one or several variables with rational coefficients.

“Rational coefficients” may mean

- coefficients in $\mathbb{Q}$
- coefficients in $\mathbb{C}(t)$, the period is a function of $t$. 
What is a period?

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**Etymology**
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- \( 2\pi \) is a **period** of the sine.
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- \( \arcsin(z) = \int_0^z \frac{dx}{\sqrt{1 - x^2}} \)
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- $2\pi$ is a **period** of the sine.
- $\arcsin(z) = \int_{0}^{z} \frac{dx}{\sqrt{1-x^2}}$
- $2\pi = \int_{-\infty}^{1} \frac{dx}{\sqrt{1-x^2}}$
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**Etymology**

- \( 2\pi \) is a **period** of the sine.
- \( \arcsin(z) = \int_0^z \frac{dx}{\sqrt{1-x^2}} \)
- \( 2\pi = \int_{\infty}^{-1} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi i} \oint \frac{dx dy}{y^2 - (1-x^2)} \)
Periods with a parameter

Complete elliptic integral
Periods with a parameter

Complete elliptic integral

An ellipse

- eccentricity \( t \)
- major radius 1
- perimeter \( E(t) \)

\[
\text{Euler (one.pnum/seven.pnum/three.pnum/three.pnum)} = \left(t - t^3\right)E'' + \left(1 - t^2\right)E' + tE = 0
\]

\[
\text{Liouville (one.pnum/eight.pnum/three.pnum/four.pnum)} \text{ not expressible in terms of elementary functions}
\]

Many applications in algebraic geometry, geometry of the cycles \( \leftrightarrow \) analytic properties of the periods
Periods with a parameter

An ellipse

- eccentricity $t$
- major radius 1
- perimeter $E(t)$

$$E(t) = 2 \int_{-1}^{1} \sqrt{\frac{1 - t^2 x^2}{1 - x^2}} \, dx$$
Periods with a parameter

Complete elliptic integral

An ellipse
- eccentricity \( t \)
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### An ellipse

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
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<td>Eccentricity</td>
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**Complete elliptic integral**

$$E(t) = \int_{\infty}^{1} \sqrt{\frac{1 - t^2 x^2}{1 - x^2}} \, dx$$

**Euler (1733)**

$$(t - t^3)E'' + (1 - t^2)E' + tE = 0$$
An ellipse

- **eccentricity**: \( t \)
- **major radius**: \( 1 \)
- **perimeter**: \( E(t) \)

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Periods with a parameter

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Liouville (1834) Not expressible in terms of elementary functions

since then Many applications in algebraic geometry

gallery of the cycles ↔ analytic properties of the periods
Content

Computing periods with a parameter

Volume of semialgebraic sets

Picard rank of K3 surfaces

Perspectives
Computing periods with a parameter
Differential equations as a data structure

Representation of algebraic numbers

- explicit: \( \sqrt{5} + 2 \cdot p^6 \) (also \( p^2 + p^3 \))
- implicit: \( x^4 - 10x^2 + 1 = 0 \) (with root location)

Representation of D-finite functions

An example by Bostan, Chyzak, van Hoeij, and Pech:

- explicit: \( 1 + 6 \cdot \int_0^t 2^F_{\frac{1}{3} \frac{2}{3}}(\frac{27}{2}w^{2-3w})(1-4w)(1-64w)(1-576t^3-801t^2-108t+74)y''' + 4(576t^3-801t^2-108t+74)y' = 0 \) (with initial condition)
Differential equations as a data structure

Representation of algebraic numbers

Explicit representation:
\[ \sqrt{5} + 2p + \sqrt{3}(2 + 3p) \]

Implicit representation:
\[ x^4 - 10x^2 + 1 = 0 \] (root location)

Representation of D-finite functions
An example by Bostan, Chyzak, van Hoeij, and Pech:

Explicit representation:
\[ 1 + 6 \cdot \int_0^t 2F_1(\frac{1}{3}, \frac{2}{3} \left| \frac{27w(2-3w)}{(1-4w)(1-64w)} \right)^3 (1-4w)(1-64w) \) \]

Implicit representation:
\[ t(y'''' + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)y'' + 4(576t^3 - 801t^2 - 108t + 74)y') = 0 \] (init. cond.)
### Representation of algebraic numbers

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**explicit**  \( \sqrt{5} + 2\sqrt{6} \)  (also \( \sqrt{2} + \sqrt{3} \))

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### Representation of algebraic numbers

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Differential equations as a data structure

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### Differential equations as a data structure

#### Representation of algebraic numbers

**explicit** \( \sqrt{5 + 2\sqrt{6}} \) \((\text{also } \sqrt{2} + \sqrt{3})\)

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#### Representation of D-finite functions

*An example by Bostan, Chyzak, van Hoeij, and Pech (2011)*

**explicit** \( 1 + 6 \cdot \int_0^t \frac{t \, _2F_1\left(\frac{1}{3}, \frac{2}{3} \mid \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1 - 4w)(1 - 64w)} \, dw \)
Differential equations as a data structure

**Representation of algebraic numbers**

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**Representation of D-finite functions**

*An example by Bostan, Chyzak, van Hoeij, and Pech (2011)*

- **explicit** \( 1 + 6 \cdot \int_0^t \frac{2F_1 \left( \frac{1}{3}, \frac{2}{3} ; 2 \right) \left( \frac{27w(2-3w)}{(1-4w)^3} \right)}{(1 - 4w)(1 - 64w)} \, dw \)

- **implicit** \( t(t - 1)(64t - 1)(3t - 2)(6t + 1)y''' + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)y'' 
+ 4(576t^3 - 801t^2 - 108t + 74)y' = 0 \) (+ init. cond.)
What can we compute?

- addition, multiplication, composition with algebraic functions
- power series expansion
- equality testing, given differential equations and initial conditions
- numerical analytic continuation with certified precision (D. V. Chudnovsky and G. V. Chudnovsky, van der Hoeven, Mezzarobba)

More on this later.
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Differential equations as a data structure

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More on this later.
The Picard-Fuchs equation

\[ R(t, x_1, \ldots, x_n) \] a rational function
The Picard-Fuchs equation

\( R(t, x_1, \ldots, x_n) \) a rational function

\( \gamma \subset \mathbb{C}^n \) a \( n \)-cycle (\( n \)-dim. compact submanifold) which avoids the poles of \( R \), for \( t \in U \subset \mathbb{C} \)
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**define** \( y(t) \triangleq \oint_{\gamma} R(t, x_1, \ldots, x_n) dx_1 \cdots dx_n \), for \( t \in U \)
The Picard-Fuchs equation

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**wanted** a differential equation \( a_r(t)y^{(r)} + \cdots + a_1(t)y' + a_0(t)y = 0 \), with polynomial coefficients
The Picard-Fuchs equation

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**wanted** a differential equation \[ a_r(t)y^{(r)} + \cdots + a_1(t)y' + a_0(t)y = 0, \]
with polynomial coefficients

One equation fits all cycles, the **Picard-Fuchs equation**.
A computational handle

Perimeter of an ellipse

\[ E(t) = \oint \sqrt{\frac{1 - t^2 x^2}{1 - x^2}} \, dx = \frac{1}{2\pi i} \oint \frac{1}{1 - \frac{1-t^2 x^2}{(1-x^2)y^2}} \, dx \, dy \]

Picard-Fuchs equation \( (t - t^3)E'' + (1 - t^2)E' + tE = 0 \)
A computational handle

Perimeter of an ellipse

**recall** \( E(t) = \oint \sqrt{\frac{1 - t^2 x^2}{1 - x^2}} \, dx = \frac{1}{2 \pi i} \oint \frac{1}{1 - \frac{1-t^2 x^2}{(1-x^2)y^2}} \, dx \, dy \)

**Picard-Fuchs equation** \((t - t^3) E'' + (1 - t^2) E' + t E = 0\)

**Computational proof**

\[
(t - t^3) \frac{\partial^2 R}{\partial t^2} + (1 - t^2) \frac{\partial R}{\partial t} + t R = \\
\frac{\partial}{\partial x} \left( -\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2} \right) + \frac{\partial}{\partial y} \left( \frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2} \right)
\]
given \( R(t, x_1, \ldots, x_n) \), a rational function
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find \( a_0, \ldots, a_r \in \mathbb{Q}[t] \), with \( a_r \neq 0 \) and \( r \) minimal

\( C_1, \ldots, C_n \in \mathbb{Q}(t, x_1, \ldots, x_n) \) with \( \text{poles}(C_i) \subseteq \text{poles}(R) \),
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such that

\[
a_r(t) \frac{\partial^r R}{\partial t^r} + \cdots + a_1(t) \frac{\partial R}{\partial t} + a_0(t) R = \sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i}.
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existence Grothendieck (1966), Monsky (1972), etc.

see also Picard (1902) for \( n \leq 3 \)
given $R(t, x_1, \ldots, x_n)$, a rational function

find $a_0, \ldots, a_r \in \mathbb{Q}[t]$, with $a_r \neq 0$ and $r$ minimal

$C_1, \ldots, C_n \in \mathbb{Q}(t, x_1, \ldots, x_n)$ with $\text{poles}(C_i) \subseteq \text{poles}(R)$,

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algorithms Almkvist, Apagodu, Bostan, Chen, Christol, Chyzak, van Hoeij, Kauers, Koutschan, Lairez, Lipshitz, Movasati, Nakayama, Nishiyama, Oaku, Salvy, Singer, Takayama, Wilf, Zeilberger, etc.

(People who wrote a paper that solves the problem.)
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Problem (mostly) solved!
Computing binomial sums with periods

Example

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ? \]
Computing binomial sums with periods

Example

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\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ?
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basic block
\[
\binom{n}{k} = \left( \frac{1}{2\pi i} \oint \frac{(1 + x)^n}{x^{k+1}} \, dx \right)
\]
Computing binomial sums with periods

Example

\[\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ?\]

**Basic block**
\[
\binom{n}{k} = \frac{1}{2\pi i} \oint \frac{(1 + x)^n}{x^k} \frac{dx}{x}
\]

**Product**
\[
\binom{2n}{k}^3 = \frac{1}{(2\pi i)^3} \oint \frac{(1 + x_1)^{2n}}{x_1^k} \frac{(1 + x_2)^{2n}}{x_2^k} \frac{(1 + x_3)^{2n}}{x_3^k} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}
\]

Generating functions of binomial sums are periods!
Computing binomial sums with periods

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\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ?
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**Example**

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\]

**summation**
\[
y(t) = \frac{1}{(2\pi i)^3} \oint \frac{\left(x_1 x_2 x_3 - t \prod_{i=1}^{3} (1 + x_i)^2\right) dx_1 dx_2 dx_3}{\left(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^{3} (1 + x_i)^2 \prod_{i=1}^{3} (1 + x_i)^2\right)}
\]

where \(y(t)\) is the generating function of the l.h.s.
Computing binomial sums with periods

Example

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ? \]

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**Summation**

\[ y(t) = \frac{1}{(2\pi i)^3} \oint \frac{(x_1 x_2 x_3 - t \prod_{i=1}^{3} (1 + x_i)^2) dx_1 dx_2 dx_3}{(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^{3} (1 + x_i)^2)(1 - t \prod_{i=1}^{3} (1 + x_i)^2)} \]

where \( y(t) \) is the generating function of the l.h.s.

**Simplification**

\[ y(t) = \frac{1}{(2\pi i)^2} \oint \frac{x_1 x_2 dx_1 dx_2}{x_1^2 x_2^2 - t (1 + x_1)^2 (1 + x_2)^2 (1 - x_1 x_2)^2} \]
Computing binomial sums with periods

Example

\[\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n^3}\]

**Basic block**

\[\binom{n}{k} = \frac{1}{2\pi i} \oint \frac{(1+x)^n}{x^k} \frac{dx}{x}\]

**Product**

\[\binom{2n}{k}^3 = \frac{1}{(2\pi i)^3} \oint \frac{(1+x_1)^{2n}}{x_1^k} \frac{(1+x_2)^{2n}}{x_2^k} \frac{(1+x_3)^{2n}}{x_3^k} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}\]

**Summation**

\[y(t) = \frac{1}{(2\pi i)^3} \oint \frac{(x_1 x_2 x_3 - t \prod_{i=1}^{3} (1 + x_i)^2)}{(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^{3} (1 + x_i)^2) (1 - t \prod_{i=1}^{3} (1 + x_i)^2)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}\]

where \(y(t)\) is the generating function of the l.h.s.

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\[y(t) = \frac{1}{(2\pi i)^2} \oint \frac{x_1 x_2 dx_1 dx_2}{x_1^2 x_2^2 - t (1 + x_1)^2 (1 + x_2)^2 (1 - x_1 x_2)^2}\]

**Integration**

\[t(27t + 1) y'' + (54t + 1) y' + 6y = 0, \text{ i.e. } 3(3n + 2)(3n + 1) u_n + (n + 1)^2 u_{n+1} = 0\]

Generating functions of binomial sums are periods!
Computing binomial sums with periods

Example

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n^3}
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\]

Summation:

\[
y(t) = \frac{1}{(2i\pi)^3} \oint \frac{(x_1 x_2 x_3 - t \prod_{i=1}^{3} (1 + x_i)^2) dx_1 dx_2 dx_3}{(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^{3} (1 + x_i)^2) \left(1 - t \prod_{i=1}^{3} (1 + x_i)^2\right)}
\]

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Simplification:

\[
y(t) = \frac{1}{(2i\pi)^2} \oint \frac{x_1 x_2 dx_1 dx_2}{x_1^2 x_2^2 - t(1 + x_1)^2 (1 + x_2)^2 (1 - x_1 x_2)^2}
\]

Integration:

\[
t(27t + 1) y'' + (54t + 1) y' + 6y = 0, \text{ i.e. } 3(3n+2)(3n+1) u_n + (n+1)^2 u_{n+1} = 0
\]

Conclusion: Generating functions of binomial sums are periods!
Theorem + Algorithm (Bostan, Lairez, and Salvy 2016)

One can decide the equality between binomial sums.
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One can decide the equality between binomial sums.
Computing binomial sums with periods

**Theorem + Algorithm (Bostan, Lairez, and Salvy 2016)**

One can decide the equality between binominal sums.

**Theorem (Bostan, Lairez, and Salvy 2016)**

$(u_n)_{n \geq 0}$ is a binomial sum **if and only if** $u_n = a_n$,..., $n$, for some rational power series $\sum_I a_I x^I$. 
Volume of semialgebraic sets

joint work with Mezzarobba and Safey El Din
**Numerical analytic continuation**

**input**  A linear differential equation $L(f) = 0$
Initial conditions at a point $a \in \mathbb{C}$
Another point $b \in \mathbb{C}$
$\varepsilon > 0$

```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
[0.78539816339744831 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1], path=[0,i+1,2*i,i-1,0,1])
[3.9269908169872415 +/- 4.81e-17] + [ +/- 4.63e-21]*I
```
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**complexity**  Quasilinear in $\log \frac{1}{\epsilon}$

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Numerical analytic continuation

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**complexity**  
Quasilinear in $\log \frac{1}{\epsilon}$

**implementation**  
Package `ore_algebra-analytic` by Mezzarobba

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```
A numeric integral

\[ \{ x^2 + y^2 + z^2 \leq 1 - 2^{10} (x^2 y^2 + y^2 z^2 + z^2 x^2) \} \]

What is the volume of this shape?
A numeric integral

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  - Monte-Carlo
A numeric integral

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\{ x^2 + y^2 + z^2 \leq 1 - 2^{10} \left( x^2 y^2 + y^2 z^2 + z^2 x^2 \right) \}
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- Exponential complexity with respect to precision
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What is the volume of this shape?

- Basic question
- Few algorithms
  - Monte-Carlo
- Exponential complexity with respect to precision
- Difficult certification on precision
## Proposition

For any generic $f \in \mathbb{R}[x_1, \ldots, x_n]$, 

$$
\text{vol}\{f \leq 0\} \triangleq \int_{\{f \leq 0\}} dx_1 \cdots dx_n = \frac{1}{2\pi i} \oint \text{Tube}\{f=0\} \frac{x_1}{f} \frac{\partial f}{\partial x_1} dx_1 \cdots dx_n.
$$

NB. $\text{vol}\{f \leq 0\} = \int_{-\infty}^{\infty} \text{vol}\{f \leq 0\} \cap \{x_n = t\} dt$
Proposition

For any generic $f \in \mathbb{R}[x_1, \ldots, x_n],$

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**proof** Stokes formula + Leray tube map
Volumes are periods

**Proposition**

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**proof**  Stokes formula + Leray tube map

**not so useful**  There is no parameter.
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**better say**  For a generic \( t \),

\[
\text{vol} \{ f \leq 0 \} \cap \{ x_n = t \} = \frac{1}{2\pi i} \oint x_1 \left. \frac{\partial f}{\partial x_1} \right|_{x_n = t} dx_1 \cdots dx_{n-1}
\]

satisfies a Picard-Fuchs equation!
Volumes are periods

**Proposition**

For any generic $f \in \mathbb{R}[x_1, \ldots, x_n]$,

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satisfies a Picard-Fuchs equation!

**NB.** $\text{vol} \{ f \leq 0 \} = \int_{-\infty}^{\infty} \text{vol} \{ f \leq 0 \} \cap \{ x_n = t \} dt$
The “volume of a slice” function

\( \{y_1, y_2\} \), basis of the solution space of the Picard-Fuchs equation

\[
\[ 0 \cdot y_1 + 0 \cdot y_2 \\
1.0792353 \ldots \cdot y_1 - 40.100605 \ldots \cdot y_2 \\
0 \cdot y_1 + 0 \cdot y_2
\]

volume of the slice

\( z \) coordinate

\( -1 \)

\( 1 \)
An algorithm for computing volumes

**input** \( f \in \mathbb{R}[x_1, \ldots, x_n] \) generic
An algorithm for computing volumes

\textbf{input} \quad f \in \mathbb{R}[x_1, \ldots, x_n] \text{ generic}

\textbf{symbolic integration} \quad \text{Compute a differential equation for } y(t) \triangleq \text{vol}\{ f \leq 0 \} \cap \{ x_n = t \}.
An algorithm for computing volumes

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**symbolic integration**  Compute a differential equation for \( y(t) \triangleq \text{vol} \{ f \leq 0 \} \cap \{ x_n = t \} \).

**bifurcations**  Spot singular points where \( y(t) \) may not be analytic.
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  • identify $y|_I$ in the solution space of the PF equation,
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- identify \( y|_I \) in the solution space of the PF equation,
- compute \( \int_I y(t) \).

The complexity is quasi-linear with respect to the precision! (To get twice as many digits, you need only twice as much time.)
An algorithm for computing volumes

input \( f \in \mathbb{R}[x_1, \ldots, x_n] \) generic

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- identify \( y|_I \) in the solution space of the PF equation,
- compute \( \int_I y(t) \).

**return** \( \text{vol}\{f \leq 0\} = \sum_I \int_I y(t) \).

The complexity is quasi-linear with respect to the precision! (To get twice as many digits, you need only twice as much time.)
An algorithm for computing volumes

\textbf{input} \quad f \in \mathbb{R}[x_1, \ldots, x_n] \text{ generic}

\textbf{symbolic integration} \quad \text{Compute a differential equation for } y(t) \triangleq \text{vol}\{f \leq 0\} \cap \{x_n = t\}.

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\textbf{return} \quad \text{vol}\{f \leq 0\} = \sum_I \int_I y(t).

\textit{The complexity is quasi-linear with respect to the precision!}

(To get twice as many digits, you need only twice as much time.)
A hundred digits (within a minute)

\[
\text{vol} \left( \begin{array}{c} \text{\textbullet} \\ \text{\textbullet} \\ \text{\textbullet} \end{array} \right) = 0.1085754214603609377395033959942076198109178744466074754444758229932853606730329281949434744140640661366242346279598087781034932346781568... 
\]

Computation of the Picard group of quartic surfaces

joint work with Emre Sertöz
**The Picard group**

**quartic surface**  \( X = V(f) \subseteq \mathbb{P}^3 \) smooth, where \( f \in \mathbb{C}[w, x, y, z] \) is homogeneous of degree 4.

**Picard group**  \( \text{Pic} \, X = \{ [\gamma] \mid \gamma \text{ algebraic curve} \} \subset H_2(X, \mathbb{Z}) \cong \mathbb{Z}^{22} \)

**example 1**  \( \text{Pic(very generic quartic surface)} = \mathbb{Z} \cdot (\text{hyperplane section}) \)

**example 2**  \( \text{Pic} \, V(w^4 + x^4 + y^4 + z^4) \cong \mathbb{Z}^{20}, \text{generated by the 48 lines} \)

**How to compute it?**  Symbolic approach is difficult because computing elements of \( \text{Pic} \, X \) explicitly involves solving huge polynomial systems. And we do not even have an *a priori* degree bound.
Lefschetz (1,1)-theorem

\[ X = V(f) \subset \mathbb{P}^3 \text{ smooth quartic surface} \]

**periods** \( \gamma_1, \ldots, \gamma_{22} \) basis of \( H_2(X, \mathbb{Z}) \)

\[ \eta_i = \oint_{\text{tube}(\gamma_i)} \frac{dx dy dz}{f(1, x, y, z)} \in \mathbb{C} \]

Efficiently computable at high precision thanks to Picard-Fuchs equations and numerical analytic continuation!

**theorem** \( \text{Pic} X = \{(a_1, \ldots, a_{22}) \in \mathbb{Z}^{22} \mid a_1 \eta_1 + \cdots + a_{22} \eta_{22} = 0\} \)

The Picard group is the lattice of integer relations between the periods of the quartic surface.

**algorithm** Compute the periods with high precision (typically 1000 digits). Use LLL to recover \( \text{Pic} X \).
**How to certify the computation?**

**goal** For $M > 0$, compute $\epsilon_M > 0$ such that for all $a \in \mathbb{Z}^r$,

$$\|a\| \leq M \text{ and } \left| \sum a_i \eta_i \right| < \epsilon_M \Rightarrow \sum a_i \eta_i = 0.$$  

For contradiction assume that $0 < \left| \sum a_i \eta_i \right| \ll 1$.

**perturbation** There exists $\tilde{f}$ near $f$ such that the periods $\tilde{\eta}_i$ of $V(\tilde{f})$ satisfy $\sum_i a_i \tilde{\eta}_i = 0$.

**Lefschetz** Then $V(\tilde{f})$ contains an algebraic curve of a certain type whereas $V(f)$ does not.

**algebraic condition** There is an explicit polynomial with integer coefficients such that $P(f) \neq 0$ and $P(\tilde{f}) = 0$.

**separation** If $f$ has integer coefficients, then $|P(f)| \geq 1$ so $\tilde{f}$ cannot be too close to $f$. 

Perspectives
Quelques objectifs liés à ces questions

**T21** *Calcul des périodes et des volumes*
Plus efficace, plus général

**T22** *Calcul symbolique des intégrales à bord*
Elles interviennent dans le calcul des volumes et en arithmétique

**T23** *Calcul efficaces de bases de Gröbner différentielles*
Outil important pour l’analyse algébrique