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Accelerating the moment-SOS hierarchy for volume approximation

Joint PolSys SpecFun Seminar

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Outline

1. Semialgebraic volume approximation

2. Original moment-SOS hierarchy

3. Accelerated moment-SOS hierarchy

Series on Optimization and its Applications – Vol. 4

The Moment-SOS Hierarchy

Lectures in Probability, Statistics, Computational Geometry, Control and Nonlinear PDEs

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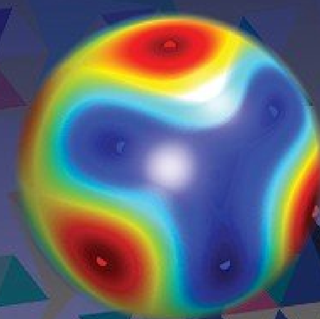
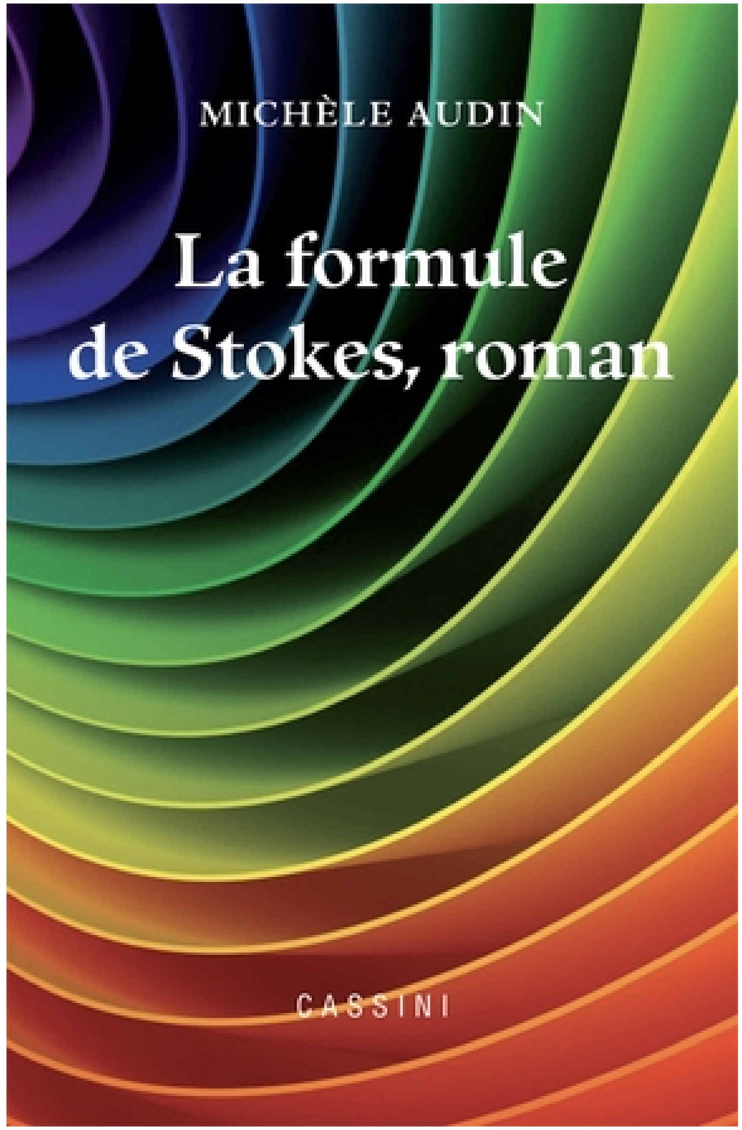


World Scientific


MICHÈLE AUDIN
La formule de Stokes, roman
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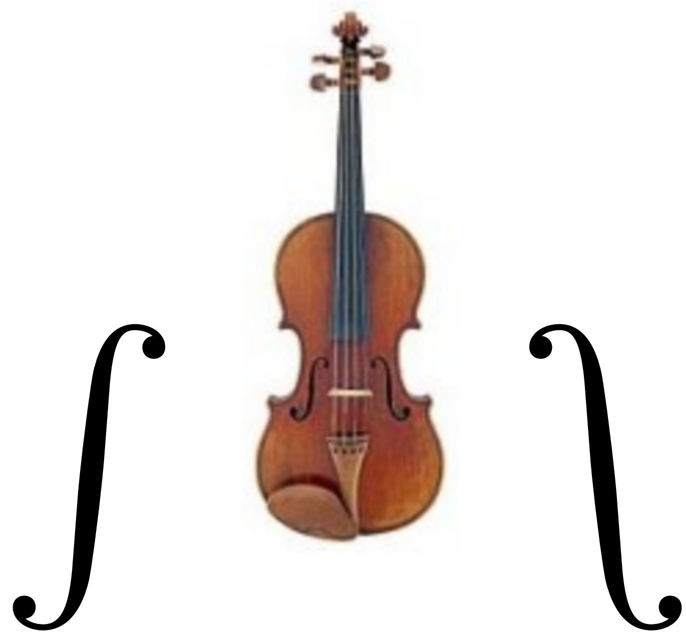


$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

1 - Semialgebraic volume approximation

Given a polynomial g, we want to compute the volume or Lebesgue measure of the compact basic semialgebraic set

$$\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \geq 0\}$$

included in the unit Euclidean ball $\mathbf{B}$.

Existing algorithmic approaches include:
- sampling (for convex sets),
- computer algebra (real algebraic geometry, symbolic integration, numerical analytic continuation),
- moment-SOS hierarchy (convex optimization),

see the recent overview:

[P. Lairez. Computing with integrals in nonlinear algebra. Online lecture #5, 23 Mar 21]

2 - Original moment-SOS hierarchy

Linear optimization problems in duality:

$$\begin{array}{ll} \max_\mu & \int_{\mathbf{X}} \mu = \mu(\mathbf{X}) \\ \text{s.t.} & 1 - \mu \in \mathscr{C}(\mathbf{B})'_+ \\ & \mu \in \mathscr{C}(\mathbf{X})'_+ \end{array} \qquad \begin{array}{ll} \inf_v & \int_{\mathbf{B}} v = \|v\|_{\mathscr{L}^1(\mathbf{B})} \\ \text{s.t.} & v \in \mathscr{C}(\mathbf{B})_+ \\ & v - 1 \in \mathscr{C}(\mathbf{X})_+ \end{array}$$

The value of both problems is vol $\mathbf{X}$

It can be approximated with the moment-SOS hierarchy

[D. Henrion, J. B. Lasserre, C. Savorgnan. Approximate volume and integration for basic semialgebraic sets. SIAM Review 51(4), 2009]

The key idea behind the moment-SOS hierarchy is to replace the cone $\mathscr{C}(\mathbf{X})_+$ of positive continuous functions on

$$\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \geq 0\} \subset \mathbf{B} := \{\mathbf{x} \in \mathbb{R}^n : b(\mathbf{x}) := 1 - \mathbf{x}'\mathbf{x} \geq 0\}$$

with the truncated quadratic module

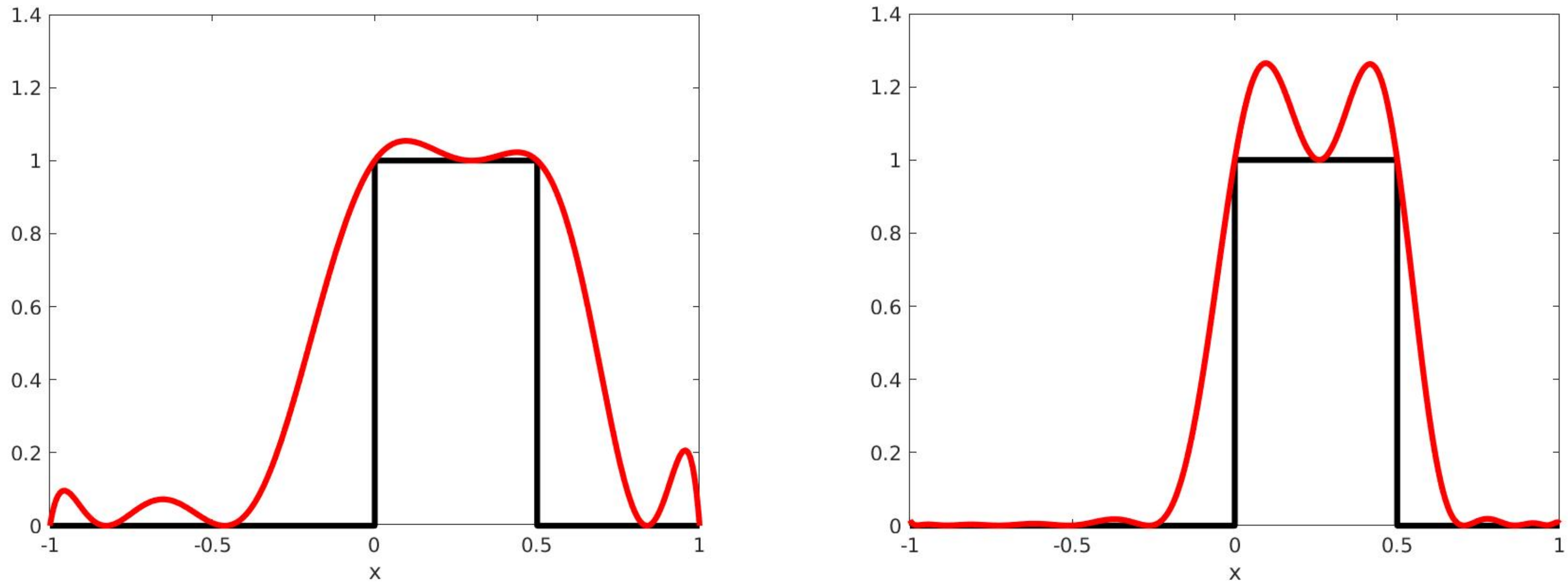
$$\mathscr{Q}(\mathbf{X})_d := \{p \in \mathbb{R}[\mathbf{x}]_d : p = s_p + s_b b + s_g g \text{ for SOS } s_p, s_b, s_g\}$$

which admits an explicit semidefinite representation

Semidefinite optimization can be used to generate a converging sequence of upper bounds $v_d \geq v_{d+1} \geq \cdots \geq v_\infty = \text{vol}\, \mathbf{X}$

Convergence is guaranteed by Putinar's Positivstellensatz, yet...
in practice it is slow and subject to numerical issues

The dual problem is $\inf \|v\|_{\mathscr{L}^1(\mathbf{B})}$ s.t. $v \geq 1_{\mathbf{X}}$ so its polynomial SOS approximation suffers from the Gibbs effect



3 - Accelerated moment-SOS hierarchy

It was observed experimentally that adding **redundant linear constraints** on the primal moment relaxation accelerates the hierarchy and the quality of the bounds

[J. B. Lasserre. Computing Gaussian and exponential measures of semi-algebraic sets. Adv. Appl. Math. 91, 2017], [T. Weisser. Computing Approximations and Generalized Solutions using Moments and Positive Polynomials. PhD thesis, Univ. Toulouse, 2018]

The constraints come from a special case of **Stokes' Theorem**

In the remainder of the talk, let us generalize these constraints and explain **why** they accelerate the hierarchy

Stokes' Theorem reads

$$\int_{\partial\Omega} w = \int_{\Omega} dw$$

with $\Omega := \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) > 0\} = \text{int}\,\mathbf{X}$ with $\mathscr{C}^1$ boundary

In particular for given $\mathbf{u} \in \mathscr{C}^1(\bar{\Omega})^n$ and $w(\mathbf{x}) := \mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_\Omega(\mathbf{x})\, d\sigma(\mathbf{x})$ with $\mathbf{n}_\Omega$ the outward pointing unit normal and σ the Hausdorff boundary measure on $\partial\Omega$, it holds

$$\int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_\Omega(\mathbf{x})\, d\sigma(\mathbf{x}) = \int_{\Omega} \text{div}\, \mathbf{u}(\mathbf{x})\, d\mathbf{x}$$

which links the Hausdorff measure and the Lebesgue measure

[J. B. Lasserre, V. Magron. Computing the Hausdorff Boundary Measure of Semialgebraic Sets. SIAM J. Applied Algebra & Geometry 4(3), 2020]

Since $0 \neq \textbf{grad}\, g(\mathbf{x}) = -\|\textbf{grad}\, g(\mathbf{x})\| \mathbf{n}_{\mathbf{X}}(\mathbf{x})$ we get

$$\int_{\mathbf{X}} \text{div}\, \mathbf{u}(\mathbf{x})\, d\mu(\mathbf{x}) = -\int_{\partial \mathbf{X}} \mathbf{u}(\mathbf{x}) \cdot \textbf{grad}\, g(\mathbf{x})\, d\nu(\mathbf{x})$$

or equivalently in the sense of distributions

$$\textbf{grad}\, \mu = (\textbf{grad}\, g)\, \nu$$

where μ is the Lebesgue measure on $\mathbf{X}$
and ν is the Hausdorff measure on $\partial\mathbf{X}$ with density $\|\textbf{grad}\, g(\mathbf{x})\|^{-1}$

If $\mathbf{u} \in \mathbb{R}[\mathbf{x}]^n$ these are linear constraints on moments of μ and ν

The constraints are redundant for the infinite-dimensional linear optimization problem on measures and moments...

...but they are not necessarily redundant for the truncated semi-definite relaxations on quasi-moments in the hierarchy

Updated linear optimization problems in duality:

$$\begin{array}{ll} \max_{\mu,\nu} & \int_{\mathbf{X}} \mu \\ \text{s.t.} & 1-\mu \in \mathscr{C}(\mathbf{B})'_+ \\ & \mu \in \mathscr{C}(\mathbf{X})'_+ \\ & \nu \in \mathscr{C}(\partial\mathbf{X})'_+ \\ & \textbf{grad}\,\mu = (\textbf{grad}\,g)\,\nu \end{array} \qquad \begin{array}{ll} \inf_{v,\mathbf{u}} & \int_{\mathbf{B}} v \\ \text{s.t.} & v \in \mathscr{C}(\mathbf{B})_+ \\ & \mathbf{u} \in \mathscr{C}^1(\mathbf{X})^n \\ & v-1-\text{div}\,\mathbf{u} \in \mathscr{C}(\mathbf{X})_+ \\ & -\mathbf{u}\cdot\textbf{grad}\,g \in \mathscr{C}(\partial\mathbf{X})_+ \end{array}$$

The value of both problems is still equal to vol $\mathbf{X}$

Note however that the dual constraint becomes $v \geq 1 + \text{div}\,\mathbf{u}$ so that v is not enforced anymore to approximate from above a discontinuous function

Indeed we can prove that the dual infimum is **attained** and hence that there is no Gibbs effect anymore

Let Ω_i, $i = 1, \ldots, N$ denote the connected components of Ω

Theorem: The dual infimum is attained at

$$v^*(\mathbf{x}) := g(\mathbf{x}) \sum_{i=1}^{N} \frac{\int_{\Omega_i} d\mathbf{x}}{\int_{\Omega_i} g(\mathbf{x}) d\mathbf{x}} 1_{\Omega_i}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{B}$$

and

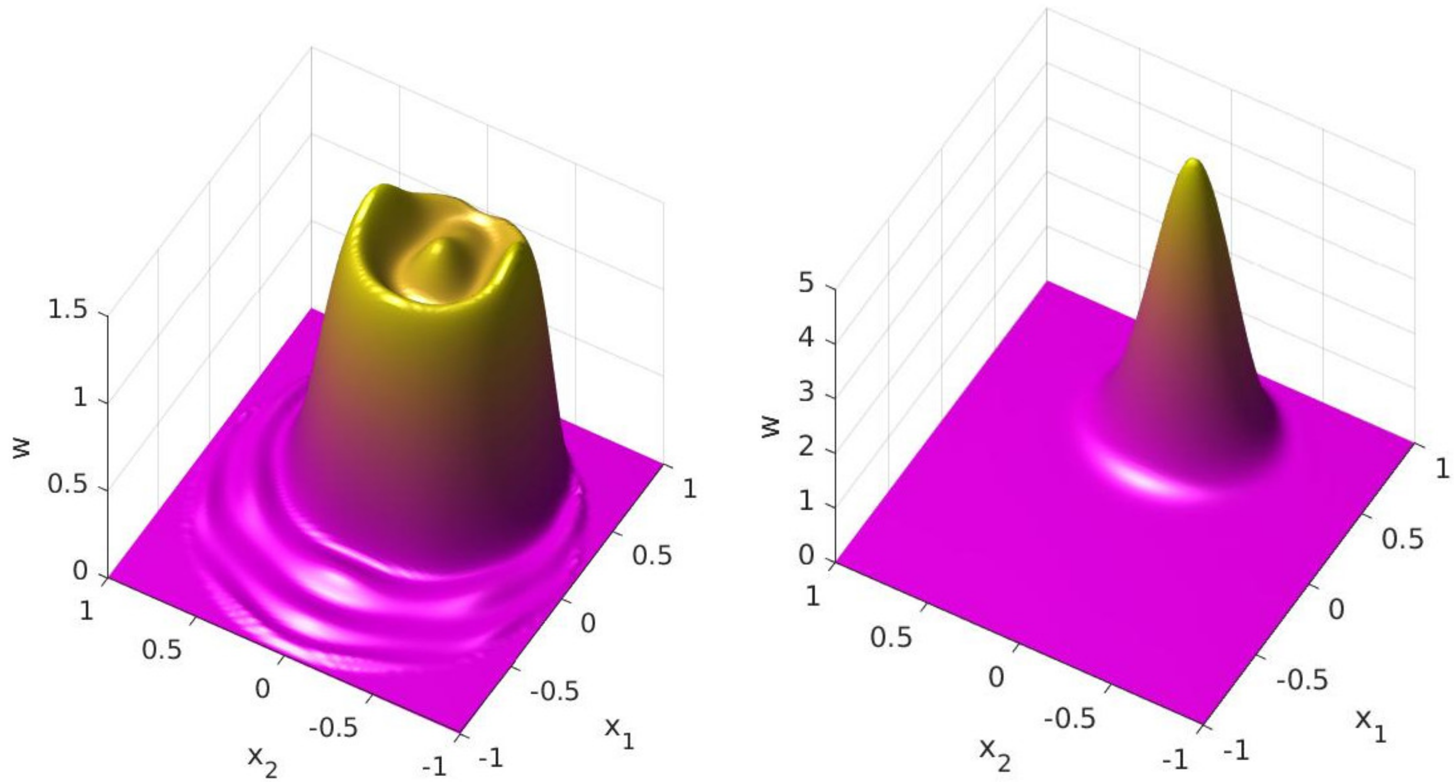
$$\mathbf{u}^*(\mathbf{x}) := \mathbf{grad}\, u(\mathbf{x})$$

where u solves the Poisson PDE

$$\begin{aligned} -\Delta u(\mathbf{x}) &= 1 - v^\star(\mathbf{x}), & \mathbf{x} \in \Omega \\ \textbf{grad}\, u(\mathbf{x}) \cdot \mathbf{n}_\Omega(\mathbf{x}) &= 0, & \mathbf{x} \in \partial\Omega \end{aligned}$$

Examples

$$\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^2 : 1/4 - (x_1 - 1/2)^2 - x_2^2 \geq 0\} \subset \mathbf{B}$$



Degree 16 approximation without and with Stokes constraints

n	d	without Stokes	with Stokes
3	4	88% (0.03s)	18% (0.04s)
3	8	57% (0.16s)	1.0% (0.44s)
3	12	47% (1.97s)	0.0% (4.63s)
3	16	43% (23.9s)	0.0% (30.1s)
3	20	41% (142s)	0.0% (206s)

Relative errors (%) and CPU times (secs) for solving moment relaxations of increasing degrees d approximating the volume of a ball of dimension $n = 3$

n	d	without Stokes	with Stokes
1	10	17% (0.05s)	0.0% (0.03s)
2	10	35% (0.09s)	0.2% (0.25s)
3	10	56% (0.52s)	0.3% (1.19s)
4	10	72% (9.74s)	0.4% (22.8s)
5	10	79% (150s)	0.6% (669s)

n	d	without Stokes	with Stokes
6	4	190% (0.25s)	45.1% (1.03s)
7	4	203% (0.32s)	60.0% (4.88s)
8	4	221% (0.42s)	78.6% (8.45s)
9	4	245% (1.15s)	102% (45.1s)
10	4	278% (3.10s)	131% (176s)

Relative errors (%) and CPU times (secs) for solving the degree $d = 10$ (top) and $d = 4$ (bottom) moment relaxation approximating the volume of a ball of increasing dimensions n

Open problem

If there is a measure μ on $\mathbf{X}$ and ν on $\partial\mathbf{X}$ such that

$$\textbf{grad}\,\mu = (\textbf{grad}\,g)\,\nu$$

then the same measure μ satisfies also

$$\textbf{grad}(g\mu) = (\textbf{grad}\,g)\,\mu$$

which was the original constraint introduced by Lasserre

This holds in particular if μ is the Lebesgue measure on $\mathbf{X}$

Are there more general linear constraints on the moments of the Lebesgue measure on $\mathbf{X}$?

Thanks for your attention !

For more details please refer to

Matteo Tacchi, Jean Bernard Lasserre, Didier Henrion.
Stokes, Gibbs and volume computation of semi-algebraic sets.
hal-02947268, arXiv:2009.12139, September 2020

and

Matteo Tacchi. Moment-SOS hierarchy for large scale set approximation. Application to power systems transient stability analysis. PhD thesis. Univ. Toulouse, June 2021

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