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Multi-Summation in Refined Difference Fields

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November 8, 2012

Indefinite summation

Simplify

$$\sum_{k=0}^{a}\big(1+(n-2k)\,S_1(k)\big)\binom{n}{k} = \text{?},$$

where $S_1(k) := \displaystyle\sum_{i=1}^{k}\frac{1}{i} \quad (= H_k).$

GIVEN $f(k) = \left(1 + (n - 2k)\, S_1(k)\right) \binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

GIVEN $f(k) = \left(1 + (n - 2k)\, S_1(k)\right)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Sigma computes

$$g(k) = (k\, S_1(k) - 1)\binom{n}{k}$$

GIVEN $f(k) = \big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

Summing the telescoping equation over k from 0 to a gives

$$\boxed{\sum_{k=0}^{a}\big(1 + (n-2k)\,S_1(k)\big)\binom{n}{k}} = \boxed{g(a+1) - g(0)}$$

$$= 1 + (n-a)S_1(a)\binom{n}{a}.$$

GIVEN $f(k) = \left(1 + (\mathbf{n} - 2k)\, S_1(k)\right)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(\mathbf{c}) = \mathbf{c} \quad \forall \mathbf{c} \in \mathbb{Q}(\mathbf{n}),$$

GIVEN $f(k) = \big(1 + (n - 2\mathbf{k})\, S_1(k)\big) \binom{n}{k}$

FIND $g(k)$:

$$f(k) = g(k+1) - g(k)$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(\mathbf{k}) = \mathbf{k} + \mathbf{1}, \qquad\qquad \mathcal{S}\, k = k + 1,$$

GIVEN $f(k) = \left(1 + (n-2k)\,\mathbf{S_1(k)}\right)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(\mathbf{h})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\,k = k+1,$$

$$\sigma(\mathbf{h}) = \mathbf{h} + \frac{\mathbf{1}}{\mathbf{k+1}}, \qquad \mathcal{S}\,S_1(k) = S_1(k) + \frac{1}{k+1},$$

GIVEN $f(k) = \left(1 + (n - 2k)\, S_1(k)\right) \binom{\mathbf{n}}{\mathbf{k}}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(\mathbf{b})$$

Karr 1981

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(\mathbf{b}) = \frac{\mathbf{n} - \mathbf{k}}{\mathbf{k} + \mathbf{1}}\,\mathbf{b}, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K} \quad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

such that

$$\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1) \qquad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \qquad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)(t_3)$$

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\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= a_3\, t_3 + f_3, \quad a_3 \in \mathbb{K}(t_1,t_2)^*, \qquad f_3 \in \mathbb{K}(t_1,t_2)
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3) \quad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)(t_3)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= a_3\, t_3 + f_3, \quad a_3 \in \mathbb{K}(t_1,t_2)^*, && f_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

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such that

$$\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

GIVEN $f(k) = \left(1 + (n-2k)\, S_1(k)\right)\binom{n}{k}$

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

$$\downarrow$$

GIVEN $f := (1 + (n-2k)h)b \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

GIVEN $f(k) = \left(1 + (n - 2k)\, S_1(k)\right) \binom{n}{k}$

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$\downarrow$ Sigma

$$g = (k\,h - 1)b$$

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$\downarrow$

GIVEN $f := (1 + (n-2k)h)b \in \mathbb{F}$.
FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

$h \equiv S_1(k)$
$b \equiv \binom{n}{k}$

$\downarrow$ Sigma

$$g = (k\,h - 1)b$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} = ?$$

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$\alpha=1$:

$$\sum_{k=0}^{a}\big(1+(n-2k)S_1(k)\big)\binom{n}{k}=1+(n-a)S_1(a)\binom{n}{a}$$

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$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k} = 1$$

$\alpha = 2$:

$$\sum_{k=0}^{a}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^2 = \frac{(a-n)^2(1+2nS_1(a))}{n^2}\binom{n}{a}$$

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$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=?$$

$\alpha=4$:
$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=?$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=?$$

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$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=(-1)^n\binom{2n}{n}$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=?$$

Telescoping

GIVEN

$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1 + 5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{\qquad f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

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$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\big(1+5(n-2k)S_1(k)\big)\binom{n}{k}^5}_{=:\,f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

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$$\text{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1 + 5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

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FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_3(n)\,f(n+3,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + \dots + c_3(n)\,f(n+3,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

☺ No solutions implies that the sequences

$$\langle S_1(a)\rangle_{a\geq 0}, \quad \langle \binom{n}{a} \rangle_{a\geq 0}, \quad \langle \sum_{k=0}^{a} f(n,k)\rangle_{a\geq 0}, \dots, \langle \sum_{k=0}^{a} f(n+3,k)\rangle_{a\geq 0} \in \mathbb{Q}(n)^{\mathbb{N}}$$

are algebraically independent over the field of rational sequences.

For more details see: Parameterized telescoping proves algebraic independence of sums, Ann. Comb. 2010

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(1+5(n-2k)S_1(k))\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n),\dots,c_4(n) \in \mathbb{Q}[n]$

$$g(n,k) := \binom{n}{k}^5 \frac{p_1(k,n,S_1(k))}{(k-n-4)^5(k-n-3)^5(k-n-2)^5(k-n-1)^5},$$

$$g(n,k+1) := \binom{n}{k}^5 \frac{p_2(k,n,S_1(k))}{(k-n-3)^5(k-n-2)^5(k-n-1)^5}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^5}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1)-g(n,0)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n)+ \\ c_1(n)\,\left[\mathsf{SUM}(n+1)-f(n+1,n+1)\right] \\ \vdots \\ c_4(n)\,\left[\mathsf{SUM}(n+4)-f(n+4,n+1)-f(n+4,n+2)-\dots-f(n+4,n+4)\right]. \end{array}}$$

Zeilberger's creative telescoping paradigm

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FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n), c_3(n), c_4(n)$

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+\dots+c_4(n)\,f(n+4,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to n gives:

Sigma

$$\boxed{g(n,n+1)-g(n,0)} =$$

$$\boxed{\begin{aligned}&c_0(n)\,\mathsf{SUM}(n)+\\&c_1(n)\left[\mathsf{SUM}(n+1)-f(n+1,n+1)\right]\\&\vdots\\&c_4(n)\left[\mathsf{SUM}(n+4)-f(n+4,n+1)-f(n+4,n+2)-\dots-f(n+4,n+4)\right].\end{aligned}}$$

A family of identities discovered by S. Ahlgren (see Paule/Schneider 03):

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

extended → Krattenthaler/Rivoal 07

$\alpha=1$:
$$\sum_{k=0}^{n}\left(1+(n-2k)S_1(k)\right)\binom{n}{k}=1$$

$\alpha=2$:
$$\sum_{k=0}^{n}\left(1+2(n-2k)S_1(k)\right)\binom{n}{k}^{2}=0$$

$\alpha=3$:
$$\sum_{k=0}^{n}\left(1+3(n-2k)S_1(k)\right)\binom{n}{k}^{3}=(-1)^n$$

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$$\sum_{k=0}^{n}\left(1+4(n-2k)S_1(k)\right)\binom{n}{k}^{4}=(-1)^n\binom{2n}{n}$$

$\alpha=5$:
$$\sum_{k=0}^{n}\left(1+5(n-2k)S_1(k)\right)\binom{n}{k}^{5}=(-1)^n\sum_{j=0}^{n}\binom{n}{j}^{2}\binom{n+j}{j}$$

Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

and

$$b(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(S_3(n) + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

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$$(n+1)^3 A(n) - (2n+3)\left(17n^2+51n+39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

Van der Poorten (1979) points out that Henri Cohen and Don Zagier showed this fact by

"some rather complicated but ingenious explanations"

based on the creative telescoping method.

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satisfy both the recurrence relation

$$(n+1)^3A(n)-(2n+3)\left(17n^2+51n+39\right)A(n+1)+(n+2)^3A(n+2)=0.$$

$a(n)$-case: trivial exercise by Zeilberger's algorithm (1991)

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satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2+51n+39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

$b(n)$-case: skilful application of computer algebra

1. Generalization of the Cohen/Zagier method in the WZ-setting (Zeilberger, 1993)
2. Multi-summation + holonomic closure properties (Chyzak/Salvy, 1998)

Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

and

$$b(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(S_3(n) + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

$b(n)$-case: plain sailing (and not plane sailing) by Sigma

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:
$$\sum_{k=0}^{a}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=\frac{(a+1)S_1(a)+1}{\binom{n}{a}}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{a}(1-2(n-2k)S_1(k))\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{(a+1)(-a+2n+2(a+1)(n+2)S_1(a)+3)}{(n+2)^2\binom{n}{a}^{-2}}$$

The other direction:

$$\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = (n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha} =?}$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1} = (n+1)S_1(n)+1$$

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$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$\alpha=-3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3} = ?$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-1$:

$$\sum_{k=0}^{n}\left(1-(n-2k)S_1(k)\right)\binom{n}{k}^{-1}=(n+1)S_1(n)+1$$

$\alpha=-2$:

$$\sum_{k=0}^{n}\left(1-2(n-2k)S_1(k)\right)\binom{n}{k}^{-2}$$

$$=\frac{(n+1)^2}{(n+2)^2}+\frac{\left(n+2\left(n^2+3n+2\right)S_1(n)+3\right)(n+1)}{(n+2)^2}$$

$\alpha=-3$:

$$\sum_{k=0}^{n}\left(1-3(n-2k)S_1(k)\right)\binom{n}{k}^{-3}=5(-1)^nS_{-3}(n)(n+1)^3$$

$$-6(-1)^nS_{-2,1}(n)(n+1)^3+6S_1(n)(n+1)+1$$

The other direction:

$$\boxed{\sum_{k=0}^{n}\left(1+\alpha(n-2k)S_1(k)\right)\binom{n}{k}^{\alpha}=?}$$

$\alpha=-4$:

$$\sum_{k=0}^{n}\left(1-4(n-2k)S_1(k)\right)\binom{n}{k}^{-4}=\frac{(10(n+1)S_1(n)+3)(n+1)}{2n+3}$$
$$+\frac{(-1)^n\binom{2n}{n}^{-1}(n+1)^5}{(4n(n+2)+3)}\left(\frac{7}{2}\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}}{i^3}-5\sum_{i=1}^{n}\frac{(-1)^i\binom{2i}{i}S_1(i)}{i^2}\right)$$

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_d(n)A(n+d)+\dots+a_0(n)A(n)=h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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NOTE: By construction, the solutions are highly nested.

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FIND all solutions expressible by indefinite nested products and sums
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3. Indefinite summation for simplification

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

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A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1

$$\sum_{k=1}^{n} \frac{\displaystyle\sum_{j=1}^{k} \frac{\displaystyle\sum_{i=1}^{j} \frac{1}{i}}{j}}{k}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

h

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \quad \longrightarrow \quad S$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s),\sigma)$ with

$$\sigma(k)=k+1$$

$$\sigma(h)=h+\frac{1}{k+1}$$

$$\sigma(s)=s+\frac{\sigma(h)}{k+1}$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} \quad \longrightarrow \quad t$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s)(t), \sigma)$ with

$$\sigma(k) = k + 1$$

$$\sigma(h) = h + \frac{1}{k+1}$$

$$\sigma(s) = s + \frac{\sigma(h)}{k+1}$$

$$\sigma(t) = t + \frac{\sigma(s)}{k+1}$$

No simplification ☹

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$h$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s)(t),\sigma)$ with

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$\Pi\Sigma$-field $(\mathbb{Q}(k)(h),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$\frac{1}{2}(h^2+h_2)$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s)(t),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$
$$\sigma(s)=s+\frac{\sigma(h)}{k+1}$$
$$\sigma(t)=t+\frac{\sigma(s)}{k+1}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(h_2),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$
$$\sigma(h_2)=h_2+\frac{1}{(k+1)^2}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \qquad \frac{1}{6}h^3+\frac{1}{2}h_2^2h+\frac{1}{3}h_3$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s)(t),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$
$$\sigma(s)=s+\frac{\sigma(h)}{k+1}$$
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$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$
$$\sigma(h_2)=h_2+\frac{1}{(k+1)^2}$$
$$\sigma(h_3)=h_3+\frac{1}{(k+1)^3}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s)(t),\sigma)$ with

$$\begin{aligned}\sigma(k)&=k+1\\ \sigma(h)&=h+\frac{1}{k+1}\\ \sigma(s)&=s+\frac{\sigma(h)}{k+1}\\ \sigma(t)&=t+\frac{\sigma(s)}{k+1}\end{aligned}$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(h_2)(h_3),\sigma)$ with

$$\begin{aligned}\sigma(k)&=k+1\\ \sigma(h)&=h+\frac{1}{k+1}\\ \sigma(h_2)&=h_2+\frac{1}{(k+1)^2}\\ \sigma(h_3)&=h_3+\frac{1}{(k+1)^3}\end{aligned}$$

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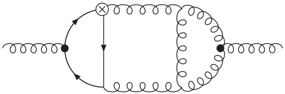
FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

A(n)=combined solutions.

Evaluation of Feynman diagrams

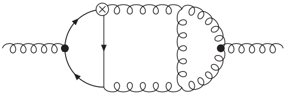
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

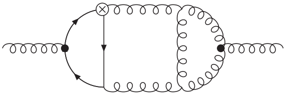
$\longrightarrow$

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

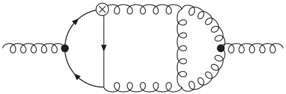
DESY

$$\sum f(n, \epsilon, k)$$

multi sums

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles ⟶ $\int \Phi(n, \epsilon, x)dx$ Feynman integrals

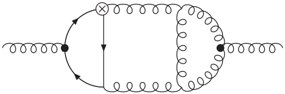
↓ **DESY**

$\sum f(n, \epsilon, k)$ multi sums

simple sum expressions ⟵ **symbolic summation** multi sums

Evaluation of Feynman diagrams

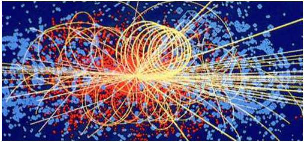
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n,\epsilon,x)dx$$

Feynman integrals



Evaluations required for the LHC experiment at CERN

DESY

processable by physicists

symbolic summation

simple sum expressions

$$\sum f(n,\epsilon,k)$$

multi sums

Warming up example

A warm up example

GIVEN
$$F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}
+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

GIVEN $F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).$$

FIND the first coefficients of the ϵ-expansion

$$F(N)=F_0(n)+\varepsilon F_1(n)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$$

$$\times \underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)}_{f(n,k,j)}.$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k,j) = \qquad f_0(n,k,j) + \qquad \varepsilon f_1(n,k,j) + \dots$$

Arose in the context of

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$$\times\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\Bigg).$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}(=H_n)$$

Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$$\uparrow$$

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(n,k,j) = g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

creative telescoping

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{\frac{1}{2} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

Simplify

$$\underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

$$= \frac{S_1(n)^2+S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$

$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$

$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$
$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$
$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$
$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$
$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

Guessing and Finding

(J. Blümlein, M. Kauers, S. Klein, CS; Comput. Phys. Comm. 180, pp. 2143-2165. 2009; arXiv 0902.4091)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$
$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$\downarrow$ difficult, unsolved task

Initial values $F_0(i)$, $i = 1,\dots,5114$

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\boxed{F_0(n)}\varepsilon^{0}+\dots$$

$\downarrow$ difficult, unsolved task

Initial values $F_0(i)$, $i=1,\dots,5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n)+a_1(n)F_0(n+1)+\dots+a_{35}(n)F_0(n+35)=0$$

$$a_0(n)F_0(n)+a_1(n)F_0(n+1)+\cdots+\boxed{a_{35}(n)}F_0(n+35)=0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \dots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \dots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$A_0 =$ 4640944309211313672503980223716264124200407085993854002412460315194
95765021269344971048446299722216293405285738333200767150194016391501666
27950213807356109710952045603966273388757782697588602201277983560532017
37487592671445911325765145271945214255462153147308420597210761595329365
51563452998613135384718911305253299053198893606401464021608911620974192
09001668029951620780182947258262939450801154511774527832503874341661898
89167522107378468797979810265385510643937043867557563467523740406094658
99100467933353731959645624977524424672990654427732309881685346483771128
69020837147452024401528169079406933665344476181260243344172097691636706
62803059675535809027169693064474147719610219849628486896079642312975136
20776876867741883488363846944854496482629372436829699055391369178850397
00381638011612302679580897488076647721311930634735316787779620757659951
5202809978299053753901432067359626151

(885 decimal digits)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1, x_2, \dots, x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1, x_2, \dots, x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$\downarrow$ difficult, unsolved task

Initial values $F_0(i)$, $i = 1, \dots, 5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$
$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$\downarrow$ still difficult, unsolved task

Initial values $F_0(i)$, $i = 1, \dots, 450$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+7) = 0$$

$\downarrow$ Sigma

CLOSED FORM

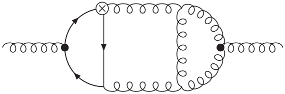
Automatization

Example: All n-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

(Nuclear Physics B, 2012; arXiv:1206.2252v1)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\Vert$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\text{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$ Example.

Mathematica Session:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}\Big[\dfrac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!},$

$\{\{s,0,n\text{-}j+r\text{-}2\},\{r,0,j+1\},\{j,0,n\text{-}2\}\}\Big]$

Out[4]= $\dfrac{-n^2-n-1}{n^2(n+1)^3}+\dfrac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\dfrac{2S_{-2}(n)}{n+1}+\dfrac{S_1(n)}{(n+1)^2}+\dfrac{S_2(n)}{-n-1}$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

A typical sum

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$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(n)=\sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\cdots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;n)+S_{2,1,1,1}(1,\frac{1}{2},1,2;n)$$

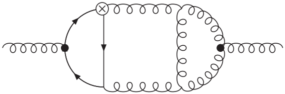
$$+\dots$$

where, e.g.,

145 S–sums occur

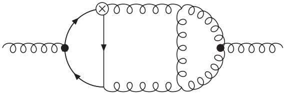
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;n)=\sum_{i=1}^{n}\frac{\displaystyle\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\displaystyle\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

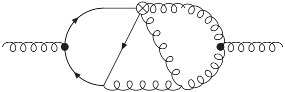
Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

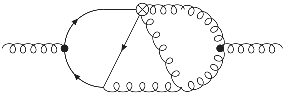
$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_1(n), S_2(n), S_3(n), S_4(n), S_{-3,1}(n),$$
$$S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+\big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$-2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

New Strategies

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$

$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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multivariate
Almquist/Zeilberger
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

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Holonomic/difference field Approach
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$$a_0(\varepsilon, n)D_\varepsilon(n) + \dots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

given

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
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+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

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+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
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\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Apply Sigma's recurrence solver

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products: **game over**

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]$$
$$+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]$$
$$+$$
$$\vdots$$
$$+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]$$
$$=h_0(n)+h_1(n)\varepsilon+h_2(n)\varepsilon^2+\dots$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0'(n)+h_1'(n)\varepsilon+h_2'(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big]\\
&\qquad = \underbrace{h'_0(n)}_{=\mathbf{0}} + h'_1(n)\varepsilon + h'_2(n)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

Remark: Works the same for Laurent series.

(see J. Blümlein, S. Klein, CS, F. Stan. J. Symbolic Comput. 47, 2012; arXiv:1011.2656v2)

Appendix

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

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- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
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$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

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(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) = g + f}$$

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$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) = g + f}$$

Such a difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is called Σ^*-extension

Construction of Σ^*-extensions

- Let $(\mathbb{F},\sigma)$ be a difference field with constant field

$$\text{const}_\sigma\mathbb{F} := \{k\in\mathbb{F}|\sigma(k)=k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

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(Karr 1981) Then $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$ iff

$$\nexists g\in\mathbb{F}: \quad \boxed{\sigma(g)=g+f}$$

There are 2 cases:

1. $\boxed{\nexists g\in\mathbb{F}:\ \sigma(g)=g+f}$ $(\mathbb{F}(t),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

$$\text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} | \sigma(k) = k\}.$$

- Adjoin a new variable t to $\mathbb{F}$ (i.e., $\mathbb{F}(t)$ is a rational function field).
- Extend the shift operator s.t.

$$\sigma(t) = t + f \qquad \text{for some } f \in \mathbb{F}.$$

(Karr 1981) Then $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$ iff

$$\nexists g \in \mathbb{F} : \quad \boxed{\sigma(g) = g + f}$$

There are 2 cases:

1. $\boxed{\nexists g \in \mathbb{F} : \ \sigma(g) = g + f}$ $(\mathbb{F}(t), \sigma)$ is a Σ^*-extension of $(\mathbb{F}, \sigma)$
2. $\boxed{\exists g \in \mathbb{F} : \sigma(g) = g + f}$ No need for a Σ^*-extension!