Démonstration --- des fractions continues dans gfun.

En bref

```
> with(gfun:-cfrac);
```

$$[diffeq2cfrac, holexpr2cfrac, ppcfrac] \qquad (1.1)$$

```
> f:=arctan(x);
```

$$f := \arctan(x) \qquad (1.2)$$

```
> ppcfrac(holexpr2cfrac(f,y,x=0,z,t,simregular));
```

$$z(t) = \cfrac{a_0\, t^{\alpha_0}}{1 + \cfrac{a_1\, t^{\alpha_1}}{1 + \cfrac{\cdots}{1 + \cfrac{a_n\, t^{\alpha_n}}{1 + \ldots}}}},\ a_n\, t^{\alpha_n} = \begin{cases} t & n = 0 \\ \dfrac{n^2 t^2}{4 n^2 - 1} & \mathit{otherwise} \end{cases} \qquad (1.3)$$

En détails

```
> infolevel[gfuncfrac]:=3;
```

$$\mathit{infolevel}_{\mathit{gfuncfrac}} := 3 \qquad (2.1)$$

```
> diffeq:=holexprtodiffeq(f,y(x));
```

$$\mathit{diffeq} := \left\{ (x^2+1)\left(\frac{d}{dx} y(x)\right) - 1,\ y(0) = 0 \right\} \qquad (2.2)$$

```
> diffeq2cfrac(diffeq,y,x=0,simregular,time):
diffeq2cfrac: Expanding  Z(x) = y(x) as an infinite continued
fraction.
diffeq2cfrac:  Guess: with  16 terms.
diffeq2cfrac:
```

$$Z(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} - \frac{1}{15}x^{15} + \frac{1}{17}x^{17} - \frac{1}{19}x^{19} + \frac{1}{21}x^{21} - \frac{1}{23}x^{23} + \frac{1}{25}x^{25} - \frac{1}{27}x^{27} + \frac{1}{29}x^{29} - \frac{1}{31}x^{31} + O(x^{33})$$

```
diffeq2cfrac:
```

$$Z(x) = x \Big/ \left(1 + \frac{1}{3} x^2 \Big/ \left(1 \right.\right.$$

$$+ \frac{4}{15} x^2 \Big/ \left(1 + \frac{9}{35} x^2 \Big/ \left(1 \right.\right.$$

$$+ \frac{16}{63} x^2 \bigg/ \bigg(1 + \frac{25}{99} x^2 \bigg/ \bigg(1$$

$$+ \frac{36}{143} x^2 \bigg/ \bigg(1 + \frac{49}{195} x^2 \bigg/ \bigg(1$$

$$+ \frac{64}{255} x^2 \bigg/ \bigg(1$$

$$+ \frac{81}{323} \frac{x^2}{1 + \frac{100}{399} \frac{x^2}{1 + \frac{121}{483} \frac{x^2}{1 + \frac{144}{575} \frac{x^2}{1 + \frac{169}{675} \frac{x^2}{1 + \frac{196}{783} \frac{x^2}{1 + \frac{225}{899} x^2}}}}}} \bigg) \bigg) \bigg)$$

$$\left.\left.\left.\left.\left.\left.\right)\right)\right)\right)\right)\right) + O\left(x^{32}\right)$$

diffeq2cfrac:

$$y(x) = \frac{a_0 x^{\alpha_0}}{1 + \frac{a_1 x^{\alpha_1}}{1 + \frac{\dots}{1 + \frac{a_n x^{\alpha_n}}{1 + \dots}}}}, \ a_n x^{\alpha_n} = \begin{cases} x & n = 0 \\ \frac{n^2 x^2}{4 n^2 - 1} & \text{otherwise} \end{cases}$$

diffeq2cfrac: Lemma: formula is true iff val(H_n) -> infinity

diffeq2cfrac:

$$H(n, x) = \left(\frac{\partial}{\partial x} P(n, x)\right) x^2 Q(n, x) - P(n, x) \left(\frac{\partial}{\partial x} Q(n, x)\right) x^2 + \left(\frac{\partial}{\partial x} P(n, x)\right) Q(n, x) - Q(n, x)^2 - P(n, x) \left(\frac{\partial}{\partial x} Q(n, x)\right)$$

diffeq2cfrac: Proof: computing a first recurrence for (H_n). (which does not conclude)

diffeq2cfrac:

$$\dots x^8 H(n) + \dots x^4 H(n+1) + \left(\dots x^4 + \dots x^2\right) H(n+2) + \dots H(n+3) + \dots H(n+4) = 0$$

tick: ellapsed CPU time: 1.532 seconds.

diffeq2cfrac: Guess: P-recurrence for (H_n), defining a new sequence (h_n).

diffeq2cfrac:

$$\dots x^2 h(n) + \dots h(n+1) = 0$$

diffeq2cfrac: proof: h_n=H_n, by induction on n.

diffeq2cfrac: - initial conditions: ok (were guessed with).

diffeq2cfrac: - recurrence step.

diffeq2cfrac: defining (z_n) with z_n:=defH[h/H].

diffeq2cfrac: deciding (z_n)=0: P-recurrence computation.

diffeq2cfrac: QED.

diffeq2cfrac: QED.

diffeq2cfrac: CPU time : 2.540 seconds.

En plus?

```
> infolevel[gfuncfrac]:=0;
```

(3.1)

$$infolevel_{gfuncfrac} := 0 \tag{3.1}$$

```
> for f in [ln(1+x),exp(3*x^2),cos(x),BesselJ(2,x),erf(x)] do
    print(y(x)=f);
    diffeq:=holexprtodiffeq(f,y(x));
    print(ppcfrac(diffeq2cfrac(diffeq,y,x=0,'simregular')));
  od;
```

$$y(x) = \ln(1+x)$$

$$diffeq := \left\{(1+x)\left(\frac{d}{dx}y(x)\right) - 1, y(0) = 0\right\}$$

$$y(x) = \cfrac{a_0 x^{\alpha_0}}{1 + \cfrac{a_1 x^{\alpha_1}}{1 + \cfrac{\cdots}{1 + \cfrac{a_n x^{\alpha_n}}{1 + \ldots}}}}, a_n x^{\alpha_n} =$$

$$\begin{cases} x & n = 0 \\ \frac{1}{8}\,\frac{\left(2\,(-1)^{n+1}\,n + 2\,n^2 + (-1)^{n+1} + 2\,n + 1\right)x}{(n+1)\,n} & otherwise \end{cases}$$

$$y(x) = e^{3x^2}$$

$$diffeq := \left\{-6\,x\,y(x) + \frac{d}{dx}y(x), y(0) = 1\right\}$$

$$y(x) = 1 + \cfrac{a_0 x^{\alpha_0}}{1 + \cfrac{a_1 x^{\alpha_1}}{1 + \cfrac{\cdots}{1 + \cfrac{a_n x^{\alpha_n}}{1 + \ldots}}}}, a_n x^{\alpha_n} =$$

$$\begin{cases} 3\,x^2 & n = 0 \\ \frac{3}{4}\,\frac{\left(2\,(-1)^n\,n + (-1)^n - 1\right)x^2}{(n+1)\,n} & otherwise \end{cases}$$

$$y(x) = \cos(x)$$

$$diffeq := \left\{\frac{d^2}{dx^2}y(x) + y(x), y(0) = 1, D(y)(0) = 0\right\}$$

$$\frac{\frac{d}{dx}\,y(x)}{y(x)} = \frac{a_0\,x^{\alpha_0}}{1+\frac{a_1\,x^{\alpha_1}}{1+\frac{\ldots}{1+\frac{a_n\,x^{\alpha_n}}{1+\ldots}}}},\ a_n\,x^{\alpha_n} = \begin{cases} -x & n=0 \\ -\frac{x^2}{4\,n^2-1} & \textit{otherwise} \end{cases}$$

$$y(x) = \mathrm{BesselJ}(2, x)$$

$$\textit{diffeq} := \left\{x^2\left(\frac{d^2}{dx^2}\,y(x)\right) + x\left(\frac{d}{dx}\,y(x)\right) + \left(x^2-4\right)y(x),\ D^{(2)}(y)(0) = \frac{1}{4}\right\}$$

$$x\left(\frac{\frac{d}{dx}\,y(x)}{y(x)} - \frac{2}{x}\right) = \frac{a_0\,x^{\alpha_0}}{1+\frac{a_1\,x^{\alpha_1}}{1+\frac{\ldots}{1+\frac{a_n\,x^{\alpha_n}}{1+\ldots}}}},\ a_n\,x^{\alpha_n} =$$

$$\begin{cases} -\frac{1}{6}\,x^2 & n=0 \\ -\frac{1}{4}\,\frac{x^2}{(n+3)\,(n+2)} & \textit{otherwise} \end{cases}$$

$$y(x) = \mathrm{erf}(x)$$

$$\textit{diffeq} := \left\{2\,x\left(\frac{d}{dx}\,y(x)\right) + \frac{d^2}{dx^2}\,y(x),\ y(0) = 0,\ D(y)(0) = \frac{2}{\sqrt{\pi}}\right\}$$

$$x\left(\frac{\frac{d}{dx}\,y(x)}{y(x)} - \frac{1}{x}\right) = \frac{a_0\,x^{\alpha_0}}{1+\frac{a_1\,x^{\alpha_1}}{1+\frac{\ldots}{1+\frac{a_n\,x^{\alpha_n}}{1+\ldots}}}},\ a_n\,x^{\alpha_n} = \qquad \textbf{(3.2)}$$

$$\begin{cases} -\frac{2}{3}\,x^2 & n=0 \\ -\frac{2\,(-1)^n\,(n+1)\,x^2}{(2\,n+1)\,(2\,n+3)} & \textit{otherwise} \end{cases}$$

(Encore?)

Ici, un exemple de réduction d'ordre de récurrences,
sur une liste obtenue au cours du développement de exp(x) en fraction continue,
qui permet de faire la preuve en 3 secondes (au lieu de plusieurs minutes).

```
> bigrecsC := [{(-n^6-17*n^5-119*n^4-439*n^3-900*n^2-972*n-432)*u[1](n)+(n^6+26*n^5+277*n^4+1548*n^3+4788*n^2+7776*n+5184)*u[1](n+1)+(2*n^6+60*n^5+744*n^4+4876*n^3+17794*n^2+34244*n+27120)*u[1](n+2)+(-2*n^6-78*n^5-1262*n^4-10846*n^3-52240*n^2-133752*n-142272)*u[1](n+3)+(-n^6-43*n^5-765*n^4-7205*n^3-37874*n^2-105312*n-120960)*u[1](n+4)+(n^6+52*n^5+1125*n^4+12962*n^3+83888*n^2+289152*n+414720)*u[1](n+5), u[1](0) = -1/2160, u[1](1) = -1/3600, u[1](2) = -1/14000, u[1](3) = -1/19600, u[1](4) = -1/49392}, {(n^3+11*n^2+40*n+48)*u[2](n)+(-n^3-14*n^2-65*n-100)*u[2](n+1)+(-n^3-19*n^2-120*n-252)*u[2](n+2)+(n^3+22*n^2+161*n+392)*u[2](n+3), u[2](0) = 1/60, u[2](1) = 1/100, u[2](2) = 1/140}, {(n^3+12*n^2+47*n+60)*u[3](n)+(-n^3-17*n^2-94*n-168)*u[3](n+1)+(-n^3-18*n^2-105*n-196)*u[3](n+2)+(n^3+23*n^2+176*n+448)*u[3](n+3), u[3](0) = 1/150, u[3](1) = 0, u[3](2) = 1/490}, {(-n-4)*u[4](n)+u[4](n+1)+(n+7)*u[4](n+2), u[4](0) = -1/5, u[4](1) = 1/5}];
```

$$bigrecsC := \Big[\Big\{ \left(-n^6 - 17 n^5 - 119 n^4 - 439 n^3 - 900 n^2 - 972 n - 432\right) u_1(n) + \left(n^6 + 26 n^5 + 277 n^4 + 1548 n^3 + 4788 n^2 + 7776 n + 5184\right) u_1(n+1) + \left(2 n^6 + 60 n^5 + 744 n^4 + 4876 n^3 + 17794 n^2 + 34244 n + 27120\right) u_1(n+2) + \left(-2 n^6 - 78 n^5 - 1262 n^4 - 10846 n^3 - 52240 n^2 - 133752 n - 142272\right) u_1(n+3) + \left(-n^6 - 43 n^5 - 765 n^4 - 7205 n^3 - 37874 n^2 - 105312 n - 120960\right) u_1(n+4) + \left(n^6 + 52 n^5 + 1125 n^4 + 12962 n^3 + 83888 n^2 + 289152 n + 414720\right) u_1(n+5), u_1(0) = -\frac{1}{2160}, u_1(1) = -\frac{1}{3600}, u_1(2) = -\frac{1}{14000}, u_1(3) = -\frac{1}{19600}, u_1(4) = -\frac{1}{49392} \Big\}, \Big\{ \left(n^3 + 11 n^2 + 40 n + 48\right) u_2(n) + \left(-n^3 - 14 n^2 - 65 n - 100\right) u_2(n+1) + \left(-n^3 - 19 n^2 - 120 n - 252\right) u_2(n+2) + \left(n^3 + 22 n^2 + 161 n + 392\right) u_2(n+3), u_2(0) = \frac{1}{60}, u_2(1) = \frac{1}{100}, u_2(2) = \frac{1}{140} \Big\}, \Big\{ \left(n^3 + 12 n^2 + 47 n + 60\right) u_3(n) + \left(-n^3 - 17 n^2 - 94 n - 168\right) u_3(n+1) + \left(-n^3 - 18 n^2 - 105 n - 196\right) u_3(n+2) + \left(n^3 + 23 n^2 + 176 n + 448\right) u_3(n+3), u_3(0) = \frac{1}{150}, u_3(1) = 0, u_3(2) = \frac{1}{490} \Big\}, \Big\{ (-n - 4) u_4(n) + u_4(n+1) + (n+7) u_4(n+2), u_4(0) = -\frac{1}{5}, u_4(1) = \frac{1}{5} \Big\} \Big] \quad \textbf{(4.1)}$$

```
> recsC := [seq(reducerecorder(bigrecsC[i],u[i](n)), i=1..4)];
```

$$recsC := \Big[\Big\{ \left(-2 n^5 - 33 n^4 - 208 n^3 - 629 n^2 - 916 n - 516\right) u_1(n) + \left(-2 n^4 - 34 n^3\right. \quad \textbf{(4.2)}$$

$$-212\,n^2-568\,n-544\big)\,u_1(n+1)+\big(2\,n^5+51\,n^4+510\,n^3+2493\,n^2+5932\,n$$
$$+5460\big)\,u_1(n+2),\,u_1(0)=-\frac{1}{2160},\,u_1(1)=-\frac{1}{3600}\Big\},\Big\{\big(-n^2-7\,n-12\big)\,u_2(n)$$
$$+(-2\,n-10)\,u_2(n+1)+\big(n^2+13\,n+42\big)\,u_2(n+2),\,u_2(0)=\frac{1}{60},\,u_2(1)=\frac{1}{100}\Big\}$$
$$,\Big\{\big(-n^2-8\,n-15\big)\,u_3(n)+\big(n^2+14\,n+49\big)\,u_3(n+2),\,u_3(0)=\frac{1}{150},\,u_3(1)=0\Big\},$$
$$\Big\{(-n-4)\,u_4(n)+u_4(n+1)+(n+7)\,u_4(n+2),\,u_4(0)=-\frac{1}{5},\,u_4(1)=\frac{1}{5}\Big\}\Big]$$

```
> ord:=L->map(rec->nops(rec)-1,L);
```

$$ord := L \rightarrow map(rec \rightarrow nops(rec)-1, L) \qquad \textbf{(4.3)}$$

```
> ord(bigrecsC);
```

$$[5, 3, 3, 2] \qquad \textbf{(4.4)}$$

```
> ord(recsC);
```

$$[2, 2, 2, 2] \qquad \textbf{(4.5)}$$