A constructive study of the module structure of rings of partial differential operators

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Outline of the talk

- Notation: Let k be a field of characteristic 0 and:

$$A_n(k) := k[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle, \qquad (\partial_i\,\partial_j = \partial_j\,\partial_i,$$
$$B_n(k) := k(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle, \qquad \partial_i\, a = a\,\partial_i + \frac{\partial a}{\partial x_i})$$
$$\widehat{\mathcal{A}}(k') := k[\![t]\!]\langle \partial \rangle, \quad \mathcal{A}(k) := k'\{t\}\langle \partial \rangle, \; k' := \mathbb{R}, \mathbb{C},$$
$$\widehat{\mathcal{D}}_n(k) := k((x_1, \ldots, x_n))\langle \partial_1, \ldots, \partial_n \rangle,$$
$$\mathcal{D}_n(k') := k'\{\{x_1, \ldots, x_n\}\}\langle \partial_1, \ldots, \partial_n \rangle, \; k' := \mathbb{R}, \mathbb{C}.$$

- The goal of the talk:

Give constructive proofs of Stafford's theorems on the module structure of $A_n(k)$, $B_n(k)$, $\widehat{\mathcal{A}}(k')$, $\mathcal{A}(k)$, $\widehat{\mathcal{D}}_n(k)$ and $\mathcal{D}_n(k')$.

J. T. Stafford, "Module structure of Weyl algebras", *J. London Math. Soc.*, 18 (1978), 429–442.

$\Rightarrow$ Every left/right ideal can be generated by 2 elements.

$\Rightarrow$ Serre's splitting-off theorem, Swan's lemma, Bass' theorem, ...

Algebraic analysis

- Let D be a noetherian domain and $R \in D^{q \times p}$.
- Let us consider the left D-homomorphism:

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda := (\lambda_1 \ \ldots \ \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$M := \mathrm{coker}_D(.R) = D^{1\times p}/(D^{1\times q}\, R).$$

- Remark (Malgrange): If $\mathcal{F}$ is a left D-module, then:

$$\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

$\Rightarrow$ Algebraic analysis (Palamodov, Sato, Kashiwara, ...).

- Example: $D := A_n(\mathbb{Q})$, $\mathcal{F} := C^\infty(\mathbb{R}^n)$.

Module theory

- D is a noetherian domain $\Rightarrow$ D is a left and a right Ore domain.

$$\Rightarrow \; K := Q(D) = \{a\, b^{-1} = c^{-1}\, d \mid b,\, d \in D,\; a,\, c \in D \setminus \{0\}\}$$

$$\Rightarrow \; \mathrm{rank}_D(M) := \dim_K(K \otimes_D M).$$

- Definition: Let M be a finitely generated left D-module.

1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^{1 \times r}$.
2. M is projective if $\exists\, r \in \mathbb{Z}_+$, $\exists\, P$ such that: $M \oplus P \cong D^{1 \times r}$.
3. M is torsion-free if:

$$t(M) := \{m \in M \mid \exists\, 0 \neq d \in D : d\, m = 0\} = 0.$$

4. M is torsion if $t(M) = M$.

- Theorem (Stafford): If $D = A_n(k)$ or $D = B_n(k)$, then every projective left D-module M with $\mathrm{rank}_D(M) \geq 2$ is free.

Unimodular elements

- Definition: Let M be a left D-module. An element $m \in M$ is a unimodular element of M if there exists $\phi \in \hom_D(M, D)$ s.t.:

$$\phi(m) = 1.$$

- If m is a unimodular of M, then the short exact sequence holds:

$$\begin{array}{ccccccc} 0 \longrightarrow \ker \phi \longrightarrow M & \xrightarrow{\phi} & D \longrightarrow 0. & (\star) \\ m & \longmapsto & 1 & \\ m & \stackrel{\psi}{\longleftarrow} & 1 & \end{array}$$

$\Rightarrow \; \exists \, \psi \in \hom_D(D, M) : \; \psi(d) = d\, m, \quad \forall \, d \in D.$

$\Rightarrow \phi \circ \psi = \mathrm{id}_D \Rightarrow \; \Pi := \psi \circ \pi : \; \Pi^2 = \Pi \; \Rightarrow M = D\, m \oplus \ker \phi.$

- Remark: $m \in t(M) \; \Rightarrow \; \forall \, \varphi \in \hom_D(M, D) : \; \varphi(m) = 0$

$\Rightarrow$ if m is a unimodular element, then $m \notin t(M)$ and $D\, m \cong D$.

Characterization of unimodular elements

- Let $M := D^{1\times p}/(D^{1\times q}\, R)$, $\pi : D^{1\times p} \longrightarrow M$ be the canonical projection onto M, and $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$.
- If $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$, then Malgrange's remark yields:

$$\begin{array}{rcl} \ker_D(R.) & \cong & \hom_D(M, D) \\ \eta & \longmapsto & \phi_\eta : \; \phi_\eta(\pi(\lambda)) = \eta\, \mu \\ (\phi(y_1) \; \ldots \; \phi(y_p))^T & \longleftarrow & \phi \end{array}$$

- If $t(M) \neq M$ then $\ker_D(R.) \cong \hom_D(M, D) \neq 0$, and thus:

$$\begin{array}{rl} \exists\; Q \in D^{p\times m} : & \ker_D(R.) = Q\, D^m \\ \Rightarrow \;\; \phi \in \hom_D(M, D) : & \phi(\pi(\lambda)) = \lambda\,(Q\,\xi). \end{array}$$

- Lemma: $m = \pi(\lambda) \in M$, where $\lambda \in D^{1\times p}$, is unimodular iff:

$$\exists\; \xi \in D^m : \quad \lambda\, Q\, \xi = 1.$$

- Remark: Finding solutions of quadratic equations.

Example

- Let $D := A_3(\mathbb{Q})$ and $M := D^{1\times 3}/(D^{1\times 3}\, R)$, where:

$$R := \begin{pmatrix} \frac{1}{2}\, x_2\, \partial_1 & x_2\, \partial_2 + 1 & x_2\, \partial_3 + \frac{1}{2}\, \partial_1 \\ -\frac{1}{2}\, x_2\, \partial_2 - \frac{3}{2} & 0 & \frac{1}{2}\, \partial_2 \\ -\partial_1 - \frac{1}{2}\, x_2\, \partial_3 & -\partial_2 & -\frac{1}{2}\, \partial_3 \end{pmatrix}.$$

- $\ker_D(R.) = Q\, D$, where $Q := (-\partial_2 \quad \partial_1 + x_2\, \partial_3 \quad - x_2\, \partial_2 - 2)^T$.
- Q admits a left inverse $T := \frac{1}{2}\, (x_2 \quad 0 \quad -1)$, i.e., $T\, Q = 1$

$\Rightarrow$ $m := \pi(T)$ is a unimodular element of M.

- Let $\phi \in \hom_D(M, D)$ be defined by $\phi(\pi(\lambda)) = \lambda\, Q$, $\forall\, \lambda \in D^{1\times 3}$:

$$\phi(m) = \phi(\pi(T)) = T\, Q = 1.$$

- $\ker \phi = \ker_D(.Q)/(D^{1\times 3}\, R) = 0$ since $\ker_D(.Q) = D^{1\times 3}\, R$

$$\Rightarrow \quad M = D\, m \cong D.$$

Very simple domain

- Definition: A noetherian domain D is very simple if:

$$\forall\, a,\, b,\, c \in D, \quad \forall\, d \in D \setminus \{0\}, \quad \exists\, u,\, v \in D :$$
$$D\, a + D\, b + D\, c = D\,(a + (d\, u)\, c) + D\,(b + (d\, v)\, c).$$

- Remark: Every f.g. left ideal of D can be generated by 2 elts and the stable rank of D is 2 ($\mathrm{sr}(D) = 2$).

- Since D satisfies the right Ore condition, namely

$$\forall\, d_1,\, d_2 \in D \setminus \{0\},\, \exists\, e_1,\, e_2 \in D \setminus \{0\} : \; d = d_1\, e_1 = d_2\, e_2,$$
$$\Rightarrow \; D\, a + D\, b + D\, c = D\,(a + d_1\,(e_1\, u)\, c) + D\,(b + d_2\,(e_2\, v)\, c).$$

- If D is a very simple domain, then:

$$\forall\, a,\, b,\, c \in D, \quad \forall\, d_1,\, d_2 \in D \setminus \{0\}, \quad \exists\, u,\, v \in D :$$
$$D\, a + D\, b + D\, c = D\,(a + d_1\, u\, c) + D\,(b + d_2\, v\, c).$$

Very simple domain

- If D is a very simple domain, then:

$$\forall\, a,\, b,\, c \in D, \quad \forall\, d_1,\, d_2 \in D \setminus \{0\}, \quad \exists\, u,\, v \in D :$$
$$D\,a + D\,b + D\,c = D\,(a + d_1\,u\,c) + D\,(b + d_2\,v\,c).$$

- Application: If $a = b = 0$, $c = 1$, then

$$\forall\, d_1,\, d_2 \in D \setminus \{0\}, \quad \exists\, u,\, v \in D : \quad D = D\,d_1\,u + D\,d_2\,v,$$
$$\Leftrightarrow\ \exists\, x,\, y \in D : \quad x\,d_1\,u + y\,d_2\,v = 1.$$

- Remark: $d := d_1 = d_2 \in D \setminus \{0\}$, $\exists\, u,\, v \in D$: $x\,d\,u + y\,d\,v = 1$

$$\Rightarrow\ D\,d\,D = D\ \Rightarrow\ D \text{ is simple.}$$

Stafford's main theorem

- Theorem (Stafford 78): If k is a field of characteristic 0, then $A_n(k)$ and $B_n(k)$ are very simple domains.

- Constructive proofs: Hillebrand-Schmale (JSC 01), Leykin (JSC 04, `Dmodules`), STAFFORD package (Q.-Robertz).

- Theorem (Caro-Levcovitz 10): $\widehat{\mathcal{D}}_n(k)$ and $\mathcal{D}_n(k)$ ($k = \mathbb{R}, \mathbb{C}$) are very simple domains.

- Theorem (Q.-Robertz 10): $\widehat{\mathcal{A}}(k)$ and $\mathcal{A}(k)$ ($k = \mathbb{R}, \mathbb{C}$) are very simple domains.

- Conclusion: Let $D = A_n(k), B_n(k), \widehat{\mathcal{A}}(k), \mathcal{A}(k), \widehat{\mathcal{D}}_n(k), \mathcal{D}_n(k)$.

$$\forall\, d_1, d_2 \in D \setminus \{0\}, \quad \exists\, u, v, x, y \in D: \quad x\, d_1\, u + y\, d_2\, v = 1.$$

- Example: $D := A_1(\mathbb{Q})$, $d_1 := \partial = \frac{d}{dt}$, $d_2 := t$:

$$x = -t\,(t+1)\,(2\,t+1),\ u = 1,\ y = (2\,t+1)\,\partial - 4,\ v = t+1.$$

Stafford's theorem 2

- Proposition: Let M be a finitely generated left D-module with $\mathrm{rank}_D(M) \geq 2$. Then, there exists a unimodular element $m \in M$:

$$\Rightarrow \quad M = D\, m \oplus M' \cong D \oplus M'.$$

- **Main idea:** Find $m_1,\, m_2 \in M$ and $\phi_1,\, \phi_2 \in \hom_D(M, D)$:

$$\phi_1(m_1) \neq 0, \quad \phi_2(m_2) \neq 0, \quad \phi_1(m_2) = 0, \quad \phi_2(m_1) = 0.$$

D is a very simple domain $\Rightarrow y_1\, \phi_1(m_1)\, z_1 + y_2\, \phi_2(m_2)\, z_2 = 1$.

Let us consider:

$$\begin{cases} \phi := \phi_1\, z_1 + \phi_2\, z_2 \in \hom_D(M, D), \\ m := y_1\, m_1 + y_2\, m_2. \end{cases}$$

Then, m is unimodular since:

$$\begin{aligned} \phi(m) &= (y_1\, \phi_1(m_1) + y_2\, \phi_1(m_2))\, z_1 + (y_1\, \phi_2(m_1) + y_2\, \phi_2(m_2))\, z_2 \\ &= y_1\, \phi_1(m_1)\, z_1 + y_2\, \phi_2(m_2)\, z_2 = 1. \end{aligned}$$

Stafford's theorem 2

• Proposition: Let M be a finitely generated left D-module with $\mathrm{rank}_D(M) \geq 2$. Then, there exists a unimodular element $m \in M$:

$$\Rightarrow \; M = D\,m \oplus M' \cong D \oplus M'.$$

Proof. ($\sim$ Serre's splitting-off theorem) Let $M := D^{1 \times p}/(D^{1 \times q}\,R)$.

$\mathrm{rank}_D(M) \geq 2 \;\Rightarrow\; t(M) \neq M \;\Rightarrow\; \ker_D(R.) \cong \hom_D(M, D) \neq 0$.

Let $Q \in D^{p \times m}$ be such that $\ker_D(R.) = Q\,D^m$.

Applying the functor $\hom_D(\,\cdot\,, D)$ to the following exact sequence

$$D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m,$$

we get the complex $D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{.Q} D^{1 \times m}$.

$$t(M) = \ker_D(.Q)/(D^{1 \times q}\,R) \cong \mathrm{ext}^1_D(M, D).$$

• Pick $m_1 \notin t(M) \Rightarrow$ take $m_1 = \pi(\lambda_1)$ such that $\lambda_1\,Q \neq 0$.

Stafford's theorem 2

- $\phi_1 \in \hom_D(M, D)$ is defined by $\phi_1(\pi(\lambda)) = \lambda\, Q\, \xi_1$, $\forall\, \lambda \in D^{1\times p}$.
- Take $\phi_1 \in \hom_D(M, D)$ such that $\phi_1(m_1) \neq 0$ by considering:

$$\xi_1 \in D^m : \quad \lambda_1\, Q\, \xi_1 \neq 0.$$

- Let $\mu_1 = Q\, \xi_1$. The following commutative exact diagram holds:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow .\mu_1 & & \downarrow \phi_1 & & \\ 0 & \longrightarrow & D & \xrightarrow{\mathrm{id}_D} & D & \longrightarrow & 0. \end{array}$$

- $\ker \phi_1 = \ker_D(.\mu_1)/(D^{1\times q} R)$.
- $0 \neq D\, \phi_1(m_1) \subseteq \mathrm{im}\, \phi_1 \subseteq D \quad \Rightarrow \quad \mathrm{rank}_D(\mathrm{im}\, \phi_1) = 1$.
- The short exact sequence $0 \longrightarrow \ker \phi_1 \longrightarrow M \longrightarrow \mathrm{im}\, \phi_1 \longrightarrow 0$

yields $\quad \mathrm{rank}_D(\ker \phi_1) = \mathrm{rank}_D(M) - \mathrm{rank}_D(\mathrm{im}\, \phi_1) \geq 1$.

Stafford's theorem 2

- $\mathrm{rank}_D(\ker \phi_1) \geq 1 \;\Rightarrow\; \ker \phi_1$ is not a torsion left D-module.
- Pick $m_2 \in \ker \phi_1 = \ker_D(.\mu_1)/(D^{1\times q}\, R)$ such that $m_2 \notin t(M)$:

Let $S \in D^{r\times p}$ such that $\ker_D(.\mu_1) = D^{1\times r}\, S$.

Find $\nu \in D^{1\times r}$ such that $\lambda_2 = \nu\, S$ satisfies $\lambda_2\, Q = \nu\,(S\, Q) \neq 0$.

- Find $\xi_2 \in D^m$ such that $\lambda_2\, Q\, \xi_2 \neq 0$ and define $\mu_2 = Q\, \xi_2$ and:

$$\phi_2 \in \hom_D(M, D) : \; \phi_2(\pi(\lambda)) = \lambda\, \mu_2, \; \forall\, \lambda \in D^{1\times p}.$$

Then, $\phi_2(m_2) = \phi_2(\pi(\lambda_2)) = \lambda_2\, Q\, \xi_2 \neq 0$.

- Since $\lambda_2 \in \ker_D(.\mu_1)$, $\phi_1(m_2) = \phi_1(\pi(\lambda_2)) = \lambda_2\, \mu_1 = 0$.
- We get $m_1,\, m_2 \in M$ and $\phi_1,\, \phi_2 \in \hom_D(M, D)$ such that:

$$\phi_1(m_1) \neq 0, \quad \phi_2(m_2) \neq 0, \quad \phi_1(m_2) = 0.$$

Stafford's theorem 2

- If $\phi_2(m_1) \neq 0$, then by the right Ore property of D:

$$\exists\ r_1,\ r_2 \in D \setminus \{0\} : \quad \phi_1(m_1)\, r_1 + \phi_2(m_1)\, r_2 = 0.$$

Then, $\phi_2' = \phi_1\, r_1 + \phi_2\, r_2 \in \hom_D(M, D)$ satisfies:

$$\begin{cases} \phi_2'(m_1) = \phi_1(m_1)\, r_1 + \phi_2(m_1)\, r_2 = 0, \\ \phi_2'(m_2) = \phi_1(m_2)\, r_1 + \phi_2(m_2)\, r_2 = \phi_2(m_2)\, r_2 \neq 0. \end{cases}$$

$\Rightarrow$ one can suppose w.l.o.g. that $\phi_2(m_1) = 0$.

- Since D is strongly simple, there exist $y_1, y_2, z_1, z_2 \in D$ such that:

$$y_1\, \phi_1(m_1)\, z_1 + y_2\, \phi_2(m_2)\, z_2 = 1.$$

Stafford's theorem 2

- Let us consider $\mu^\star = \mu_1\, z_1 + \mu_2\, z_2 \in \ker_D(R.)$ and:

$$\phi = \phi_1\, z_1 + \phi_2\, z_2 \in \hom_D(M, D) : \quad \phi(\pi(\lambda)) = \lambda\, \mu^\star.$$

- If $\lambda^\star = y_1\, \lambda_1 + y_2\, \lambda_2 \in D^{1\times p}$, then

$$m = \pi(\lambda^\star) = y_1\, m_1 + y_2\, m_2 \in M$$

is unimodular since $(\phi_1(m_2) = 0,\ \phi_2(m_1) = 0)$:

$$\begin{aligned}\phi(m) &= (y_1\, \phi_1(m_1) + y_2\, \phi_1(m_2))\, z_1 + (y_1\, \phi_2(m_1) + y_2\, \phi_2(m_2))\, z_2 \\ &= y_1\, \phi_1(m_1)\, z_1 + y_2\, \phi_2(m_2)\, z_2 = 1.\end{aligned}$$

Stafford's theorem 2

- $M = D\, m \oplus \ker \phi$, where $\ker \phi = \ker_D(.\mu^\star)/(D^{1\times q}\, R)$.
- Let $S \in D^{s\times p}$ be such that $\ker_D(.\mu^\star) = D^{1\times s}\, S$.
- Let $S_2 \in D^{t\times s}$ be such that $\ker_D(.T) = D^{1\times t}\, S_2$.
- Let $F \in D^{q\times s}$ be such that $R = F\, S$.

$$\Rightarrow \ \ker \phi \cong L := D^{1\times s}/(D^{1\times (q+t)}\, (F^T \quad S_2^T)^T),$$

where the isomorphism is defined by

$$\pi(\gamma\, S) \longmapsto \sigma(\gamma),$$

and $\sigma : D^{1\times (q+t)} \longrightarrow L$ is the canonical projection onto L.

- Theorem (Stafford): Let D be a very simple domain and M a finitely generated left D-module. Then:

$$M \cong D^{1\times r} \oplus M', \quad \mathrm{rank}_D(M') \leq 1.$$

If $t(M) = 0$, then $\mathrm{rank}_D(M') = 1$ and $M' \cong D\, d_1 + D\, d_2$.

Example

- Let $D := A_3(\mathbb{Q})$, $R := (\partial_1 \quad \partial_2 \quad \partial_3)$, and $M := D^{1\times 3}/(D\,R)$.
- We have $\mathrm{rank}_D(M) = 2$ and $\ker_D(R.) = Q\,D^3$, where:

$$Q := \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}.$$

- Let $\lambda_1 := (0 \quad -1 \quad 0)$. Then, $\lambda_1\,Q = (-\partial_3 \quad 0 \quad \partial_1) \neq 0$.
- Taking $\xi_1 := (0 \quad 0 \quad 1)^T$, $\mu_1 = Q\,\xi_1 = (\partial_2 \quad -\partial_1 \quad 0)^T$ and:

$$m_1 := \pi(\lambda_1), \quad \phi_1(\pi(\lambda)) = \lambda\,\mu_1, \quad d_1 := \phi_1(m_1) = \lambda_1\,\mu_1 = \partial_1.$$

- $\ker_D(.\mu_1) = D^{1\times 2}\,S$, where S and thus $S\,Q$ are defined by:

$$S := \begin{pmatrix} \partial_1 & \partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S\,Q = \begin{pmatrix} \partial_2\,\partial_3 & -\partial_1\,\partial_3 & 0 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}.$$

Example

- Let $\nu := (0 \quad 1)$. Then, $\nu\,(S\,Q) \neq 0$ and $\lambda_2 := \nu\,S = (0 \quad 0 \quad 1)$.
- Let $\xi_2 := (0 \quad 1 \quad 0)^T$. Then, $\mu_2 := Q\,\xi_2 = (-\partial_3 \quad 0 \quad \partial_1)^T$,

$$m_2 = \pi(\lambda_2), \quad \phi_2(\pi(\lambda)) = \lambda\,\mu_2,$$

$$d_2 := \phi_2(m_2) = \lambda_2\,\mu_2 = \partial_1, \quad \phi_2(m_1) = \lambda_1\,\mu_2 = 0.$$

- Computing a solution of $y_1\,d_1\,z_1 + y_2\,d_2\,z_2 = 1$, we get:

$$z_1 = 1, \quad z_2 = x_1 + 1, \quad y_1 = -x_1 - 1, \quad y_2 = 1.$$

$\Rightarrow$ $m = \pi(\lambda^\star)$ is a unimodular element of M, where:

$$\lambda^\star := y_1\,\lambda_1 + y_2\,\lambda_2 = (0 \quad x_1 + 1 \quad 1).$$

- $\mu^\star = \mu_1\,z_1 + \mu_2\,z_2 = (\partial_2 - (x_1 + 1)\,\partial_3 \quad -\partial_1 \quad (x_1 + 1)\,\partial_1 + 1)^T$.
- Let $\phi \in \hom_D(M, D)$ be defined by $\phi(\pi(\lambda)) = \lambda\,\mu^\star$.
- Then, we have $\phi(m) = \lambda^\star\,\mu^\star = 1$

$$\Rightarrow \quad M = D\,m \oplus \ker\phi, \quad \ker\phi = \ker_D(.\mu^\star)/(D\,R).$$

Example

- We have $\ker_D(.\mu^\star) = D^{1\times 2}\, S$, where:

$$S := \begin{pmatrix} 1 & -(x_1+1)\,(\partial_2 - (x_1+1)\,\partial_3) & (x_1+1)\,\partial_3 - \partial_2 \\ 0 & (x_1+1)\,\partial_1 + 2 & \partial_1 \end{pmatrix}.$$

- $\ker_D(.S) = 0$.
- Let $F := (\partial_1 \quad \partial_2 - (x_1+1)\,\partial_3)$ be such that $R = F\, S$.

$$\Rightarrow \;\; M \cong D \oplus \ker \phi \cong D \oplus D^{1\times 2}/(D\, F).$$

- $\partial_1\, u_1 + \partial_2\, u_2 + \partial_3\, u_3 = 0 \;\Leftrightarrow\; \partial_1\, v_2 + (\partial_2 - (x_1+1)\,\partial_3)\, v_3 = 0,$

$$\left\{\begin{array}{l} v_1 = (x_1+1)\, u_2 + u_3, \\ v_2 = u_1 - (x_1+1)\,(\partial_2 - (x_1+1)\,\partial_3)\, u_2 + ((x_1+1)\,\partial_3 - \partial_2)\, u_3, \\ v_3 = ((x_1+1)\,\partial_1 + 2)\, u_2 + \partial_1\, u_3, \end{array}\right.$$

$$\Leftrightarrow \quad \left\{\begin{array}{l} u_1 = (\partial_2 - (x_1+1)\,\partial_3)\, v_1 + v_2, \\ u_2 = -\partial_1\, v_1 + v_3, \\ u_3 = ((x_1+1)\,\partial_1 + 1)\, v_1 - (x_1+1)\, v_3. \end{array}\right.$$

Stafford's theorem 3

- Theorem (Stafford): Let M and N be finitely generated left D-modules satisfying $M \subseteq N$ and $\mathrm{rank}_D(M) \geq 2$. Then, there exists $m \in M$ which is a unimodular element of N. Hence, we get:

$$M = D\, m \oplus M' \;\subseteq\; N = D\, m \oplus N', \quad M' = M \cap N'.$$

- Remark: This result ressembles the properties of vector spaces over a division ring where $\mathrm{rank}_D(M) \geq 1$.

- The result is a relative version of Stafford's theorem 2:

$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & L & \longrightarrow & 0\\
\downarrow .P' & & \downarrow .P & & \downarrow \iota & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & N & \longrightarrow & 0,
\end{array}$$

where ι is an injective left D-homomorphism and $M := \iota(L) \subseteq N$.

- Idea: Replace $\lambda \in D^{1\times p}$ by $\lambda\, P \in D^{1\times p'}$!

Algorithm

1. Compute $Q' \in D^{p' \times m'}$ such that $\ker_D(R'.) = Q' \, D^{m'}$.
2. Pick $\lambda_1 \in D^{1 \times p}$ such that $\lambda_1 \, (P \, Q') \neq 0$.
3. Find $\xi_1 \in D^{m'}$ such that $(\lambda_1 \, P \, Q') \, \xi_1 \neq 0$ and $\mu_1 := Q' \, \xi_1$.
4. Compute $S \in D^{r \times p}$ such that $\ker_D(.(P \, \mu_1)) = D^{1 \times r} \, S$.
5. Pick $\nu \in D^{1 \times r}$ such that $\nu \, (S \, P \, Q') \neq 0$ and $\lambda_2 := \nu \, S$.
6. Find $\xi_2 \in D^{m'}$ such that $(\lambda_2 \, P \, Q') \, \xi_2 \neq 0$ and $\mu_2 := Q' \, \xi_2$.
7. If $\lambda_1 \, P \, \mu_2 \neq 0$, then compute $r_1, \, r_2 \in D \setminus \{0\}$ such that

$$(\lambda_1 \, P \, \mu_1) \, r_1 + (\lambda_1 \, P \, \mu_2) \, r_2 = 0,$$

   and $\mu_2 \longleftarrow \mu_1 \, r_1 + \mu_2 \, r_2$.
8. Compute $y_1, \, y_2, \, z_1, \, z_2 \in D$ such that:

$$y_1 \, (\lambda_1 \, P \, \mu_1) \, z_1 + y_2 \, (\lambda_2 \, P \, \mu_2) \, z_2 = 1.$$

9. Return $\lambda^\star = y_1 \, \lambda_1 + y_2 \, \lambda_2$ and $\mu^\star = \mu_1 \, z_1 + \mu_2 \, z_2$.

- Let $P \in D^{p \times p'}$ and $L = D^{1 \times p'}/(D^{1 \times p} P)$.
- $M = D^{1 \times p} P \subseteq N = D^{1 \times p'}$.
- If $\mathrm{rank}_D(M) \geq 2$, then Stafford's theorem 2 shows that there exists a unimodular element λP which is a unimodular of $D^{1 \times p'}$

$$\Rightarrow \; D^{1 \times p} P = D\,(\lambda P) \oplus M' \; \subseteq \; D^{1 \times p'} = D\,(\lambda P) \oplus N'$$

$$\Rightarrow \; L = D^{1 \times p'}/(D^{1 \times p'} P) \cong N'/M'.$$

- Let $R \in D^{q \times p}$ be such that $\ker_D(.P) = D^{1 \times q} R$.
- The following commutative exact diagram holds:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ \downarrow & & \downarrow .P & & \downarrow \iota & & \\ 0 & \longrightarrow & D^{1 \times p'} & \xrightarrow{\mathrm{id}} & D^{1 \times p'} & \longrightarrow & 0. \end{array}$$

$\Rightarrow$ We can apply the above algorithm with $(R, P, 0)$.

Main result

- Theorem: Let $P \in D^{p \times p'}$ and $L := D^{1 \times p'}/(D^{1 \times p}\, P)$.
If $\mathrm{rank}_D(D^{1 \times p}\, P) \geq 2$ and $p' \geq 3$, then $\exists\, \overline{P} \in D^{s \times (m-1)}$ s.t.:

$$L \cong \overline{L} := D^{1 \times (p'-1)}/(D^{1 \times s}\, \overline{P}).$$

Moreover, if $p \geq 3$, then $\overline{P}$ can be chosen so that $s = p - 1$, i.e.:

$$L \cong \overline{L} := D^{1 \times (p'-1)}/(D^{1 \times (p-1)}\, \overline{P}).$$

- Corollary: Let $P \in D^{p \times p'}$, $p' \geq 2$, and $L := D^{1 \times p'}/(D^{1 \times p}\, P)$ be a torsion left D-module. Then, there exists $\overline{P} \in D^{2 \times s}$ such that:

$$L \cong \overline{L} := D^{1 \times 2}/(D^{1 \times s}\, \overline{P}).$$

Moreover, $L \cong I/J$, where $I = D\, d_1 + D\, d_2$ is a projective ideal.

Example

- Let $D := A_3(\mathbb{Q})$ and $L := D^{1\times 3}/(D^{1\times 3}\,P)$, where:

$$P := \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}.$$

- $\ker_D(.P) = D\,(\partial_1 \quad \partial_2 \quad \partial_3) \;\Rightarrow\; \mathrm{rank}_D(D^{1\times 3}\,P) = 2.$
- $p' = p = 3$. Algorithm with $(R, P, 0)$ + basis computations yield

$$L = D^{1\times 3}/(D^{1\times 3}\,P) \cong \overline{L} = D^{1\times 2}/(D^{1\times 2}\,\overline{P}),$$

$$\overline{P} := \begin{pmatrix} -(\partial_3 + (x_1+1)\,\partial_2)^2 & ((x_1+1)\,\partial_2 + \partial_3)\,\partial_1 - \partial_2 \\ -((x_1+1)\,\partial_2 + \partial_3)\,\partial_1 - 2\,\partial_2 & \partial_1^2 \end{pmatrix}.$$

- $\ker_D(.\overline{P}) = D\,(-\partial_1 \quad (x_1+1)\,\partial_2 + \partial_3) \;\Rightarrow\; \mathrm{rank}_D(D^{1\times 2}\,\overline{P}) = 1.$

More results

- Theorem (Stafford). Let P be a finitely generated left D-module that is not a torsion module, i.e., $\mathrm{rank}_D(P) \geq 1$. Then, 1 ⇔ 2:

  1. $\mathrm{rank}_D(P) = r \geq 1$.
  2. a. Either $P \cong D^{1\times r}$, i.e., P can be generated by r elements,
     b. or P can be generated by $r+1$ elements but no fewer.

- Corollary (Stafford): A finitely generated left D-module P can be generated by 2 elements iff $\mathrm{rank}_D(P) \leq 1$.

- Theorem (cancellation): If M is such that $\mathrm{rank}_D(M) \geq 2$, then:

$$M \oplus D \cong N \oplus D \;\Rightarrow\; M \cong N.$$

- Corollary: If M is such that $\mathrm{rank}_D(M) \geq 2$ and:

$$M \oplus D^{1\times q} \cong D^{1\times p} \;\Rightarrow\; M \cong D^{1\times (p-q)}.$$

Swan's lemma

• Lemma: Let M be s.t. $\mathrm{rank}_D(M) \geq 2$, $(d^\star,\, m^\star) \in \mathrm{U}(D \oplus M)$. Then, there exists $\phi \in \hom_D(D, M)$ such that:

$$m^\star + \phi(d^\star) \in \mathrm{U}(M).$$

• Proof: Let $\omega = \omega_1 \oplus \omega_2 \in \hom_D(D \oplus M, D)$ be such that:

$$\omega((d^\star,\, m^\star)) = \omega_1(d^\star) + \omega_2(m^\star) = 1.$$

• $\mathrm{rank}_D(M) \geq 2 \Rightarrow \exists\, m_1,\, m_2 \in M$, $\exists\, \varphi_1,\, \varphi_2 \in \hom_D(M, D)$:

$$\varphi_1(m_1) \neq 0, \quad \varphi_1(m_2) = 0, \quad \varphi_2(m_1) = 0, \quad \varphi_2(m_2) \neq 0.$$

$$\Rightarrow \; r,\, s \in D : \quad \varphi_1(m^\star)\, D + \varphi_2(m^\star)\, D + d^\star\, D$$

$$= (\varphi_1(m^\star) + d^\star\, r\, \varphi_1(m_1))\, D + (\varphi_2(m^\star) + d^\star\, s\, \varphi_2(m_2))\, D$$

$$\Rightarrow \; \exists\, \alpha,\, \beta \in D :$$

$$d^\star = (\varphi_1(m^\star) + d^\star\, r\, \varphi_1(m_1))\, \alpha + (\varphi_2(m^\star) + d^\star\, s\, \varphi_2(m_2))\, \beta.$$

Swan's lemma

- Let $\chi = \varphi_1\,\alpha + \varphi_2\,\beta \in \hom_D(M, D)$ and $\phi \in \hom_D(D, M)$:

$$\forall\, d \in D, \quad \phi(d) = d\,(r\,m_1 + s\,m_2).$$

$$\begin{aligned}\chi(\phi(d^\star)) &= \chi(d^\star\,\phi(1)) = d^\star\,\chi(r\,m_1 + s\,m_2)\\ &= d^\star\,(\varphi_1(r\,m_1 + s\,m_2)\,\alpha + \varphi_2(r\,m_1 + s\,m_2)\,\beta)\\ &= d^\star\,(r\,\varphi_1(m_1)\,\alpha + s\,\varphi_2(m_2)\,\beta),\end{aligned}$$

$$\begin{aligned}\chi(m^\star + \phi(d^\star)) &= \varphi_1(m^\star)\,\alpha + \varphi_2(m^\star)\,\beta + d^\star\,(r\,\varphi_1(m_1)\,\alpha + s\,\varphi_2(m_2)\,\beta)\\ &= (\varphi_1(m^\star) + d^\star\,r\,\varphi_1(m_1))\,\alpha + (\varphi_2(m^\star) + d^\star\,s\,\varphi_2(m_2))\,\beta\\ &= d^\star.\end{aligned}$$

Swan's lemma

- Let $t = \omega_1(1) - \omega_2(\phi(1)) \in D$ and $\varphi = \omega_2 + \chi\, t \in \hom_D(M, D)$.

$$\begin{aligned} \Rightarrow\ \varphi(m^\star + \phi(d^\star)) &= (\omega_2 + \chi\, t)(m^\star + \phi(d^\star)) \\ &= \omega_2(m^\star) + \omega_2(\phi(d^\star)) + \chi(m^\star + \phi(d^\star))\, t \\ &= 1 - \omega_1(d^\star) + \omega_2(d^\star\, \phi(1)) + d^\star\, t \\ &= 1 - d^\star\, \omega_1(1) + d^\star\, \omega_2(\phi(1)) + d^\star\, t \\ &= 1 - d^\star\, t + d^\star\, t = 1 \end{aligned}$$

$\Rightarrow\ m^\star + \phi(d^\star) \in \mathrm{U}(M)$.

- Equivalent formulation: Let $d^\star \in D$ and $m^\star = \pi(\lambda^\star)$ be such that there exist $e \in D$ and $\mu \in \ker_D(R.)$:

$$\omega((d^\star,\, m^\star)) = d^\star\, e + \lambda^\star\, \mu = 1.$$

Then, there exist $\overline{\lambda} \in D^{1\times p}$ and $\overline{\mu} \in \ker_D(R.)$ such that

$$\varphi(m^\star + \phi(d^\star)) = \varphi(\pi(\lambda^\star + d^\star\, \overline{\lambda})) = (\lambda^\star + d^\star\, \overline{\lambda})\, \overline{\mu} = 1,$$

where $\phi \in \hom_D(D, M)$ defined by $\phi(d) = d\, \pi(\overline{\lambda})$ and $\varphi = \varphi_{\overline{\mu}}$.

- Theorem (cancellation): If M is such that $\mathrm{rank}_D(M) \geq 2$, then:

$$M \oplus D \cong N \oplus D \;\Rightarrow\; M \cong N.$$

$$\left\{\begin{array}{l} M = D^{1\times p}/(D^{1\times q}\, R), \\ N = D^{1\times p'}/(D^{1\times q'}\, R'), \\ P = (0 \quad R) \in D^{q\times(1+p)}, \\ P' = (0 \quad R') \in D^{q'\times(1+p')}, \\ L = D^{1\times(1+p)}/(D^{1\times q}\, P) \cong D \oplus M, \\ L' = D^{1\times(1+p')}/(D^{1\times q'}\, P') \cong D \oplus N, \end{array}\right. \quad \left\{\begin{array}{l} X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \\ X_{11} \in D, \; X_{12} \in D^{1\times p'}, \\ X_{21} \in D^{p}, \; X_{22} \in D^{p\times p'}, \\ \tau = \mathrm{id} \oplus \pi, \\ \tau' = \mathrm{id} \oplus \pi'. \end{array}\right.$$

- Let f be a left D-isomorphism defined by:

$$\begin{array}{rcl} f : L & \longrightarrow & L' \\ \tau(\zeta) & \longmapsto & \tau'(\zeta\, X). \end{array}$$

Bass' theorem

$$f((1 \quad 0)) = f(\tau(1 \quad 0)) = \tau'((X_{11} \quad X_{12})) = (X_{11},\ \pi'(X_{12})) \in \mathrm{U}(L').$$

- Applying Swan's lemma to $d^\star = X_{11}$ and $n^\star = \pi'(X_{12})$, there exist $\overline{\lambda} \in D^{1\times p'}$ such that $\phi \in \hom_D(D, N)$ defined by

$$\forall\, d \in D, \quad \phi(d) = d\,\pi'(\overline{\lambda})$$

satisfies $\pi'(X_{12}) + \phi(X_{11}) = \pi'(X_{12} + X_{11}\,\overline{\lambda}) \in \mathrm{U}(M')$, i.e., there exists $\overline{\mu} \in \ker_D(R'.)$ such that $(X_{12} + X_{11}\,\overline{\lambda})\,\overline{\mu} = 1.$

$$\begin{pmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix} := \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} 1 & \overline{\lambda} \\ 0 & I_{p'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\overline{\mu}\,X_{11} & I_{p'} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & X_{11}\,\overline{\lambda} + X_{12} \\ X_{21} - (X_{21}\,\overline{\lambda} + X_{22})\,\overline{\mu}\,X_{11} & X_{21}\,\overline{\lambda} + X_{22} \end{pmatrix}.$$

- The left D-homomorphism from M to N is then defined by:

$$\begin{array}{rcl} \varpi : M & \longrightarrow & N \\ \pi(\lambda) & \longmapsto & \pi'\left(\lambda\,(X'_{22} + (X'_{21} - X'_{22}\,\overline{\mu})\,X'_{12})\right). \end{array}$$

Stafford's theorem

- Corollary: If M is such that $\mathrm{rank}_D(M) \geq 2$ and

$$M \oplus D^{1\times q} \cong D^{1\times p} \;\Rightarrow\; M \cong D^{1\times (p-q)}.$$

- Proof: M is stably free $\Rightarrow$ split exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \quad \text{i.e.,} \quad R\,S = I_q.$$

$$\begin{array}{rcl} g : D^{1\times q} \oplus M & \longrightarrow & D^{1\times p} \\ (\theta, \pi(\lambda)) & \longmapsto & (\theta \quad \lambda) \left(\begin{array}{c} R \\ I_p - S\,R \end{array} \right), \end{array} \quad \begin{array}{rcl} g^{-1} : D^{1\times p} & \longrightarrow & D^{1\times q} \oplus M \\ \lambda & \longmapsto & (\lambda\,S,\ \pi(\lambda)). \end{array}$$

- $P = (0 \quad R) \in D^{q\times (q+p)}$, $\quad P' = 0$, $\quad L = D^{1\times (q+p)}/(D^{1\times q}\,P)$.
- We apply Bass' theorem on the left D-isomorphism:

$$\begin{array}{rcl} f : L & \longrightarrow & L' = D^{1\times p} \\ \tau((\theta \quad \lambda)) & \longmapsto & (\theta \quad \lambda) \left(\begin{array}{c} R \\ I_p - S\,R \end{array} \right). \end{array}$$