Narayana polynomials and random walks in space

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Joint work with J. Borwein, A. Straub, V.H. Moll, T. Amdeberhan

Introduction

- some recent results by Lassalle
- their extension using a probabilistic approach
- about random walks

Narayana numbers and polynomials

Definitions:

- the Narayana polynomials

$$\mathcal{N}_r(z) = \sum_{k=1}^{r} N(r,k)\, z^{k-1}$$

or

$$\mathcal{N}_r(z) = \sum_{m \geq 0} \binom{r-1}{2m} C_m z^m (z+1)^{r-2m-1},$$

with

$$C_m = \frac{1}{m+1}\binom{2m}{m}, \quad \text{Catalan number}$$

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- the Narayana numbers

$$N(r,k) = \frac{1}{r}\binom{r}{k-1}\binom{r}{k}, \quad r \neq 0$$

The Narayana Triangle

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & & & & \rightarrow & 1 \\
 & & & 1 & & 1 & & & & \rightarrow & 2 \\
 & & 1 & & 3 & & 1 & & & \rightarrow & 5 \\
 & 1 & & 6 & & 6 & & 1 & & \rightarrow & 14 \\
1 & & 10 & & 20 & & 10 & & 1 & \rightarrow & 42
\end{array}
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\end{array}
$$

Each row sum is a Catalan number

$$\sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} = \frac{1}{n+2} \binom{2n+2}{n+1}.$$

Lassalle's results

M. Lassalle, "Two integer sequences related to Catalan numbers", J. Comb. Theory Ser. A, 119:923–935, 2012.

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The numbers A_n defined by

$$(z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n\geq 1} (-z)^n \binom{r-1}{2n-1} A_N \mathcal{N}_{r-2n+1}(z)$$

satisfy the recurrence

$$(-1)^{n-1} A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}.$$

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The first values are

$$A_1 = 1,\ A_2 = 1,\ A_3 = 5,\ A_4 = 56,\ A_5 = 1092,\ A_6 = 32670$$

Lassalle's results

1,1,5,56,1092 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,5,56,1092**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A180874 Lassalle's sequence connected with Catalan numbers and Narayana polynomials. +20 5

1, 1, 5, 56, 1092, 32670, 1387815, 79389310, 5882844968, 548129834616, 62720089624920, 8646340208462880, 1413380381699497200, 270316008395632253340, 59800308109377016336155, 15151722444639718679892150, 4359147487054262623576455600 (list; graph; refs; listen; history; text; internal format)

OFFSET	1,3
COMMENTS	Defined by the recurrence formula in Theorem 1, page 2 of Lasalle.
LINKS	Table of n, a(n) for n=1..17. Michel Lassalle, Catalan numbers and a new integer sequence, arXiv:1009.4225, Sep 21 2010.
FORMULA	a(n) = (-1)^(n-1) * (C(n)+sum_{j=1..n-1} (-1)^j *binomial(2n-1,2j-1) * a(j) *C(n-j)), where C() = A000108(). - R. J. Mathar, Apr 17 2011, corrected by Vaclav Kotesovec, Feb 28 2014 E.g.f.: sum_{k=0..infinity} a(k)*x^(2*k+2)/(2*k+2)!=log(x/BesselJ(1,2*x)). - Sergei N. Gladkovskii, Dec 28 2011 a(n) ~ (n!)^2 / (sqrt(Pi) * n^(3/2) * r^n), where r = BesselJZero[1, 1]^2/16 = 0.917623165132743328576236110539381686855099186384686... - Vaclav Kotesovec, added Feb 28 2014, updated Mar 01 2014
MAPLE	A000108 := proc(n) binomial(2*n, n)/(1+n) ; end proc: A180874 := proc(n) option remember; if n = 1 then 1; else A000108(n)+add((-1)^j*binomial(2*n-1, 2*j-1)*procname(j)*A000108(n-j), j=1..n-1) ; %*(-1)^(n-1) ; end if; end proc: # R. J. Mathar, Apr 16 2011
MATHEMATICA	nmax=20; a = ConstantArray[0, nmax]; a[[1]]=1; Do[a[[n]] = (-1)^(n-1)* (Binomial[2*n, n]/(n+1) + Sum[(-1)^j*Binomial[2j-1, 2j-1]*a[[j]]* Binomial[2*(n-j), n-j]/(n-j+1), {j, 1, n-1}]), {n, 2, nmax}]; a (* Vaclav Kotesovec, Feb 28 2014 *)
CROSSREFS	Cf. A000108, A001263, A188664, A115369.
KEYWORD	nonn,easy
AUTHOR	Jonathan Vos Post, Sep 22 2010

Lassalle's results

Lassalle shows that

$$\{A_n\}_{n\in\mathbb{N}}$$

is an increasing sequence of positive integers.

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D. Zeilberger suggested to study the sequence

$$a_n = \frac{2A_n}{C_n}$$

with first values

$$a_1 = 2,\ a_2 = 1,\ a_3 = 2,\ a_4 = 8,\ a_5 = 52,\ a_6 = 495,\ a_7 = 6470$$

and that satisfies

$$(-1)^{n-1}\, a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1}$$

Lassalle's results

Equivalently,

$$(-1)^{n-1}\, a_n = 2 + \frac{1}{2} \sum_{j=1}^{n-1} (-1)^j\, \sigma_{n,j} a_j$$

with

$$\sigma_{n,r} = \frac{2}{n} \binom{n}{r-1} \binom{n+1}{r+1}$$

which appears as OEIS A108838.

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Remark that

$$\sigma_{n,r} = \binom{n-1}{r-1}\binom{n+1}{r} - \binom{n-1}{r-2}\binom{n+1}{r+1}.$$

Lassalle's paper

A108838 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **a108838**

Displaying 1-4 of 4 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A108838 Triangle of Dyck paths counted by number of long interior inclines. +30 3

2, 3, 2, 4, 8, 2, 5, 20, 15, 2, 6, 40, 60, 24, 2, 7, 70, 175, 140, 35, 2, 8, 112, 420, 560, 280, 48, 2, 9, 168, 882, 1764, 1470, 504, 63, 2, 10, 240, 1680, 4704, 5880, 3360, 840, 80, 2, 11, 330, 2970, 11088, 19404, 16632, 6930, 1320, 99, 2 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 2,1

COMMENTS

T(n,k) is the number of Dyck n-paths (A000108) containing k long interior inclines. An incline is an ascent or a descent where an ascent (resp. descent) is a maximal sequence of contiguous upsteps (resp. downsteps). An incline is long if it consists of at least 2 steps and is interior if it does not start or end the path.

T(n,k) is the number of Dyck (n+1)-paths whose last descent has length 2 and which contain n-k peaks. For example T(3,0)=3 counts UUDUDUDD, UDUUDUDD, UDUDUUDD. - David Callan, Jul 03 2006

T(n,k) is the number of parallelogram polyominoes of semiperimeter n+1 having k corners. [Emeric Deutsch, Oct 09 2008]

T(n,k) is the number of rooted ordered trees with n non-root nodes and k leaves; see example. - Joerg Arndt, Aug 18 2014

LINKS

Table of n, a(n) for n=2..56.

Tewodros Amdeberhan, Victor H. Moll and Christophe Vignat, A probabilistic interpretation of a sequence related to Narayana Polynomials, 2012. - From N. J. A. Sloane, Sep 19 2012

David Callan, Some Identities for the Catalan and Fine Numbers

M. Delest, J. P. Dubernard and I. Dutour, Parallelogram polyominoes and corners, J. Symbolic Computation, 20(1995),503-515. [From Emeric Deutsch, Oct 09 2008]

M. P. Delest, D. Gouyou-Beauchamps and B. Vauquelin, Enumeration of parallelogram polyominoes with given bond and site parameter, Graphs and Combinatorics, 3 (1987), 325-339.

E. Deutsch, Dyck path enumeration, Discrete Math., 204, 1999, 167-202.

FORMULA

G.f. T(n, k) = 2*binom[n+1, k+2]*binom[n-2, k]/(n+1). GF G(z, t) := Sum[T(n, k)*z^n*t^k, {n>=1, k>=0}] satisfies z - ((1-z)^2 - (2*t-t^2)*z^2)*G + (t^2*z)*G^2 = 0.

G.f.=1+z(1+r)^2, where r=r(t,z) is the Narayana function defined by (1+r)(1+tr)z=r, r(t,0)=0. - Emeric Deutsch, Jul 23 2006

For n >=0, the row polynomials sum {k = 0..n} T(n+2,k)*x^k = 2/(n+1)*(1-x)^n*P(n,2,1,(1+x)/(1-x)), where P(n,a,b,x) denotes the Jacobi polynomial.

Lassalle's results

Lassalle's main result:

Theorem

The numbers $\{A_n,\ n \geq 2\}$ *are positive, increasing integers and given by*

$$A_{n+1} = \sum_{r=1}^{n} \frac{n-r+1}{n+2} \binom{2n+1}{2r-1} A_r A_{n+1-r}.$$

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The numbers $\{a_n,\ n \geq 2\}$ are positive, increasing integers and given by

$$a_{n+1} = \frac{1}{2} \sum_{r=1}^{n} \frac{1}{n+1} \left(\begin{array}{c} n+1 \\ r+1 \end{array} \right) \left(\begin{array}{c} n+1 \\ r-1 \end{array} \right) a_r a_{n+1-r}.$$

"Both sequences seem to be new"

A Bessel function approach

The Bessel function of the first kind

$$I_\alpha(z) = \sum_{j \geq 0} \frac{1}{j!\,(j+\alpha)!} \left(\frac{z}{2}\right)^{2j+\alpha}.$$

A Bessel function approach

The Bessel function of the first kind

$$I_\alpha(z) = \sum_{j\geq 0} \frac{1}{j!\,(j+\alpha)!} \left(\frac{z}{2}\right)^{2j+\alpha}.$$

From the recurrence

$$(-1)^{n-1} a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1} \binom{n+1}{j+1} \frac{a_j}{n-j+1},$$

we deduce

Theorem

[V.H.M, T.A. , C.V.] The numbers $\{a_n\}$ satisfy

$$\sum_{j=1}^{+\infty} \frac{(-1)^{j-1} a_j}{(j+1)!} \frac{x^{j-1}}{(j-1)!} = \frac{2}{\sqrt{x}} \frac{I_2(2\sqrt{x})}{I_1(2\sqrt{x})}.$$

A Bessel function approach

Using classical contiguity properties of Bessel I functions, we recover Lassale's

Theorem

The numbers $\{a_n\}$ satisfy the recurrence, with $a_1 = 1$,

$$2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \;\; n \geq 2.$$

As a corollary, the numbers $\{a_n\}$ are positive.

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As a corollary, the numbers $\{a_n\}$ are positive.

Using the recurrence, we show moreover, by induction on n,

Theorem

The numbers $\{a_n\}$ are integers, and a_n is even if n is odd.

A Bessel function approach

Theorem

For $n \geq 3$, the sequence $\{a_n\}$ is increasing.

Proof.

Start from

$$2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1} \binom{n}{k+1} a_k a_{n-k}, \quad n \geq 2$$

so that

$$a_n \geq \frac{1}{2n} \left[\binom{n}{0} \binom{n}{2} a_1 a_{n-1} + \binom{n}{n-2} \binom{n}{2} a_{n-1} a_1 \right] = (n-1)\, a_{n-1}$$

hence for $n \geq 3$,

$$a_n - a_{n-1} \geq (n-2)\, a_{n-1} > 0.$$

□

A probabilistic approach

The symmetric beta distribution

$$f_\mu(x) = \begin{cases} \frac{1}{B\left(\mu+\frac{1}{2},\frac{1}{2}\right)} \left(1-x^2\right)^{\mu-\frac{1}{2}} & x \in [-1,1] \\ 0 & \text{else} \end{cases}$$

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The odd moments equal zero and

$$\mathbb{E}X_{\mu}^{2m} = \frac{B\left(\mu+\frac{1}{2}, 2m+\frac{1}{2}\right)}{B\left(\mu+\frac{1}{2},\frac{1}{2}\right)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+m+1)} \frac{(2m)!}{2^{2m} m!}.$$

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Remark: for $\mu = 1$,

$$\mathbb{E}\left(2X_1\right)^{2m} = C_m$$

and for $\mu = 0$,

$$\mathbb{E}\left(2X_0\right)^{2m} = \binom{2m}{m}.$$

A probabilistic approach

The moment generating function is

$$\varphi_\mu(z) = \mathbb{E}e^{zX_\mu} = \Gamma(\mu+1)\, 2^\mu \frac{I_\mu(z)}{z^\mu}.$$

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It admits the Weierstrass factorization

$$\varphi_\mu(z) = \prod_{k\geq 1}\left(1+\frac{z^2}{j_{\mu,k}^2}\right)$$

where $\{j_{\mu,k}\}$ are the zeros of the Bessel function J_μ.

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where $\{j_{\mu,k}\}$ are the zeros of the Bessel function J_μ.
The cumulants $\kappa_\mu(n)$ are defined as

$$\log \varphi_\mu(z) = \sum_{n\geq 1} \kappa_\mu(n) \frac{z^n}{n!}.$$

A probabilistic approach

Define the Bessel zeta function - or Rayleigh function

$$\zeta_\mu(s) = \sum_{k \geq 1} \frac{1}{j_{\mu,k}^s}.$$

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The cumulants of the symmetric beta distribution are

$$\kappa_{\mu}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2(-1)^{\frac{n}{2}+1}(n-1)!\zeta_{\mu}(n) & \text{if } n \text{ is even.} \end{cases}$$

A probabilistic approach

Define the random variable Y_μ as

$$Y_\mu = \sum_{k\geq 1} \frac{L_k}{j_{\mu,k}}$$

where $\{L_k\}$ are i.i.d. Laplace random variables. Then

$$\mathbb{E}e^{\imath z Y_\mu} = \prod_{k\geq 1}\left(1+\frac{z^2}{j_{\mu,k}^2}\right)^{-1} = \frac{1}{\mathbb{E}e^{zX_\mu}}$$

so that, with X_μ and Y_μ independent,

$$f\left(X_\mu + \imath Y_\mu + z\right) = f\left(z\right).$$

A probabilistic approach

The case $\mu = 1$ gives Lassale's sequence with

$$f_1(x) = \begin{cases} \frac{2}{\pi}\sqrt{1 - x^2}, & x \in [-1, 1] \\ 0 & \text{else} \end{cases}$$

and

$$\mathbb{E}(2X_1)^{2m} = C_m.$$

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The Narayana polynomials read

$$\mathcal{N}_r(z) = \mathbb{E}\left(1 + z + 2\sqrt{z}X_1\right)^{r-1}, \;\; r \geq 1.$$

which we rewrite as

$$\mathcal{N}_r(z) = \left(2\sqrt{z}\right)^{r-1}\mathbb{E}\left(\frac{1+z}{2\sqrt{z}} + X_1\right)^{r-1}, \;\; r \geq 1.$$

A probabilistic approach

We need the following result.

Theorem

If

$$P_n(z) = \mathbb{E}(z+X)^n,$$

then P_n satisfies the recurrence

$$P_{n+1}(z) - zP_n(z) = \sum_{m\geq 1} \binom{n}{2m-1} \kappa_1(2m) P_{n-2m+1}(z).$$

A probabilistic approach

Apply to Narayana polynomials

$$\mathcal{N}_{r+1}(z) - (1+z)\mathcal{N}_r(z) = \sum_{m \geq 1} \binom{r-1}{2m-1} \kappa_1(2m)\, 2^{2m} z^m \mathcal{N}_{r+1-2m}(z)$$

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and compare to

$$\mathcal{N}_{r+1}(z) - (1+z)\mathcal{N}_r(z) = \sum_{m \geq 1} \binom{r-1}{2m-1} (-1)^m A_m z^m \mathcal{N}_{r+1-2m}(z)$$

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so that

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and

$$a_n = \frac{A_n}{C_n} = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n).$$

A generalization

The symmetric beta distribution

$$f_\mu(x) = \begin{cases} \frac{1}{B\left(\mu+\frac{1}{2},\frac{1}{2}\right)} \left(1 - x^2\right)^{\mu-\frac{1}{2}} & x \in [-1,1] \\ 0 & \text{else} \end{cases}$$

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The cumulants are

$$\kappa_\mu(2n) = (-1)^{n+1} 2^{2n+1} (2n-1)! \zeta_\mu(2n)$$

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$$a_n = \frac{2\left(-1\right)^{n+1}\kappa_1\left(2n\right)}{\mathbb{E}\left(2X_1\right)^{2n}}$$

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$$a_n^{(\mu)} = \frac{2\left(-1\right)^{n+1}\kappa_\mu\left(2n\right)}{\mathbb{E}\left(2X_\mu\right)^{2n}}.$$

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The recurrence

$$(n+\mu)\,\zeta_\mu\,(2n) = \sum_{r=1}^{n-1} \zeta_\mu\,(2r)\,\zeta_\mu\,(2n-2r)\,,$$

translates, for $\mu = 1$, into

$$2na_n = \sum_{k=1}^{n-1} \binom{n}{k-1}\binom{n}{k+1} a_k a_{n-k},\;\; n \geq 2.$$

A generalization

$$a_n = \frac{2\left(-1\right)^{n+1}\kappa_1\left(2n\right)}{\mathbb{E}\left(2X_1\right)^{2n}}$$

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$$a_n^{(\mu)} = \frac{2\left(-1\right)^{n+1}\kappa_\mu\left(2n\right)}{\mathbb{E}\left(2X_\mu\right)^{2n}}.$$

The recurrence

$$\left(n+\mu\right)\zeta_\mu\left(2n\right) = \sum_{r=1}^{n-1}\zeta_\mu\left(2r\right)\zeta_\mu\left(2n-2r\right),$$

translates, for $\mu = 1$, into

$$2na_n = \sum_{k=1}^{n-1}\binom{n}{k-1}\binom{n}{k+1}a_k a_{n-k}, \quad n \geq 2.$$

$$a_n^{(\mu)} = \frac{1}{2\binom{n+\mu-1}{n-1}}\sum_{k=1}^{n-1}\binom{n+\mu-1}{n-k-1}\binom{n+\mu-1}{k-1}a_k^{(\mu)}a_{n-k}^{(\mu)}.$$

A generalization

We deduce

Theorem (T.A, V.H.M.,C.V.)

The coefficients $a_n^{(\mu)}$ *are positive and increasing for* $n \geq \left\lfloor \frac{\mu+3}{2} \right\rfloor$.

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Define the generalized Narayana polynomials as

$$\mathcal{N}_r^{(\mu)}(z) = \mathbb{E}\left(1 + z + 2\sqrt{z}X_\mu\right)^{r-1}, \ r \geq 1.$$

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They satisfy the recurrence

$$\mathcal{N}_{r+1}^{(\mu)}(z) - (1+z)\mathcal{N}_r^{(\mu)}(z) = -\sum_{m\geq 1} \binom{r-1}{2m-1} \kappa_\mu(2m) 2^{2m} z^m \mathcal{N}_{r+1-2m}^{(\mu)}(z)$$

Link with classical polynomials

The Gegenbauer polynomials $C_n^{(\mu)}(z)$ are defined by the horizontal generating function

$$\sum_{n\geq 0} C_n^{(\mu)}(z)\, t^n = \left(1 - 2xt + t^2\right)^{-\mu}$$

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so that

$$\mathcal{N}_{n+1}^{(\mu)}(z) = \frac{n!}{(2\mu+1)_n}(1-z)^n\, C_n^{\left(\mu+\frac{1}{2}\right)}\left(\frac{1+z}{1-z}\right).$$

Link with classical polynomials

The usual Narayana polynomials ($\mu = 1$) are given by

$$\begin{aligned}
\mathcal{N}_{n+1}(z) &= \frac{2}{(n+1)(n+2)}(1-z)^n C_n^{\left(\frac{3}{2}\right)}\left(\frac{1+z}{1-z}\right) \\
&= (1-z)^n \, {}_2F_1\left(\begin{matrix} -n, & n+3 \\ 2 & \end{matrix}; \frac{z}{z-1}\right) \\
&= \frac{(2n+2)!}{(n+2)!\,(n+1)!} z^n \, {}_2F_1\left(\begin{matrix} -n, & -n-1 \\ -2n-2 & \end{matrix}; \frac{z-1}{z}\right) \\
&= {}_2F_1\left(\begin{matrix} -n, & -n-1 \\ 2 & \end{matrix}; z\right).
\end{aligned}$$

The case $\mu = \frac{1}{2}$

The density $f_{\frac{1}{2}}$ is the uniform density on $[-1, 1]$ with

$$\mathbb{E}\,(2X_1)^{2n} = \frac{2^{2n}}{2n+1}$$

and

$$\kappa_{\frac{1}{2}}(2n) = 2^{2n}\frac{B_{2n}}{2n}$$

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Theorem

The sequence

$$a_n^{\left(\frac{1}{2}\right)} = 2^{2n}\frac{2n+1}{n}|B_{2n}|$$

with first terms

$$a_1^{\left(\frac{1}{2}\right)} = 2,\ a_2^{\left(\frac{1}{2}\right)} = \frac{4}{3},\ a_1^{\left(\frac{1}{2}\right)} = \frac{32}{9},\ a_4^{\left(\frac{1}{2}\right)} = \frac{96}{5},\ a_5^{\left(\frac{1}{2}\right)} = \frac{512}{3}$$

is positive and increasing.

The case $\mu = \frac{1}{2}$

The convolution identity

$$(n+\mu)\,\zeta_\mu(2n) = \sum_{r=1}^{n-1} \zeta_\mu(2r)\,\zeta_\mu(2n-2r)$$

with $\mu = \frac{1}{2}$ gives the well-known

$$\sum_{k=1}^{n-2} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1)\,B_{2n},\ \ n \geq 1$$

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The moments-cumulants identity

$$\kappa_{\frac{1}{2}}(n) = \mathbb{E}X_{\frac{1}{2}}^{n} - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_{\frac{1}{2}}(j)\,\mathbb{E}X_{\frac{1}{2}}^{n-j}$$

gives

$$\sum_{j=1}^{n} \binom{2n+1}{2j} 2^{2j} B_{2j} = 2n,\ \ n \geq 1.$$

The limite case $\mu = -\frac{1}{2}$

The density $f_{-\frac{1}{2}}$ is the discrete uniform density on $\{-1, 1\}$ with

$$\mathbb{E}\left(2X_{-\frac{1}{2}}\right)^{2n} = 2^{2n}$$

and the cumulants are expressed in terms of Euler numbers

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$$a_n^{\left(-\frac{1}{2}\right)} = (-1)^n 2^{2n} E_{2n-1}$$

with first terms

$$a_1^{\left(\frac{1}{2}\right)} = 2,\ a_2^{\left(\frac{1}{2}\right)} = 4,\ a_1^{\left(\frac{1}{2}\right)} = 32,\ a_4^{\left(\frac{1}{2}\right)} = 544,\ a_5^{\left(\frac{1}{2}\right)} = 15872$$

is positive and increasing.

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gives

$$\sum_{k=1}^{n} \binom{2n-1}{2k-1} 2^{2k-1} E_{2k-1} = 1, \;\; n \geq 1.$$

Random Walks: densities

The probability density $p_n(\nu; x)$ of the distance to the origin in $d \geq 2$ dimensions after $n \geq 2$ steps is, for $x > 0$,

$$p_n(\nu; x) = \frac{1}{2^\nu \nu!} \int_0^\infty (tx)^{\nu+1} J_\nu(tx) j_\nu^n(t)\, dt$$

with

$$\nu = \frac{d}{2} - 1.$$

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with

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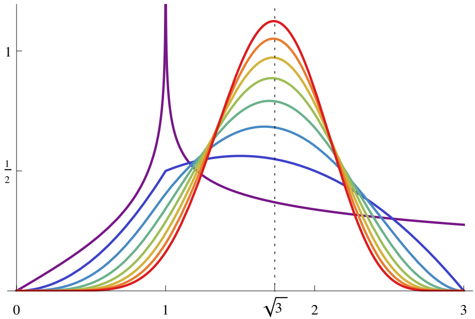
For $x > 0$ and $n = 1, 2, \ldots$ the function

$$\psi_n(\nu; x) = \frac{\nu!}{2\pi^{\nu+1}} \frac{p_n(\nu; x)}{x^{2\nu+1}}$$

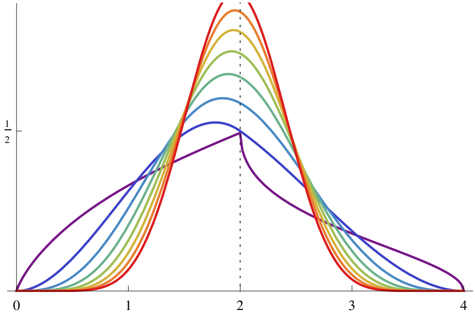
satisfies

$$\psi_n(\nu; x) = \frac{\nu!^2 2^{2\nu}}{(2\nu)!\pi} \int_{-1}^{+1} \psi_{n-1}\left(\nu; \sqrt{1 + 2\lambda x + x^2}\right) \left(1 - \lambda^2\right)^{\nu - \frac{1}{2}} d\lambda.$$

Random Walks: 3 steps



Random Walks: 4 steps



Random Walks: moments

The moments

$$W_n(\nu; s) = \int_0^\infty x^s f_n(\nu; x)\, dx$$

satisfy

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+\nu n)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \ldots, k_n} \binom{k+n\nu}{k_1+\nu, \ldots, k_n+\nu}$$

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and the recursion

$$W_{n_1+n_2}(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!\,(j+\nu)!} W_{n_1}(\nu; 2j)\, W_{n_2}(\nu; 2k-2j)$$

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and in particular

$$W_n(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!\,(j+\nu)!} W_{n-1}(\nu; 2j)$$

Random Walks: 2 dimensions

In two dimensions:

$$W_2(0;2k) : 1;\ 2;\ 6;\ 20;\ 70;\ 252;\ 924;\ 3432;\ 12870;$$

$$W_3(0;2k) : 1;\ 3;\ 15;\ 93;\ 639;\ 4653;\ 35169;\ 272835;\ 2157759;$$

$$W_4(0;2k) : 1;\ 4;\ 28;\ 256;\ 2716;\ 31504;\ 387136;\ 4951552;$$

$$W_5(0;2k) : 1;\ 5;\ 45;\ 545;\ 7885;\ 127905;\ 2241225;\ 41467725;$$

$$W_6(0;2k) : 1;\ 6;\ 66;\ 996;\ 18306;\ 384156;\ 8848236;\ 218040696;$$

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In fact

$$W_n(0;2k) = \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\ldots,k_n}^2$$

count abelian squares.

Random Walks: 4 dimensions

In four dimensions:

$$W_2(1;2k): 1;\ 2;\ 5;\ 14;\ 42;\ 132;\ 429;\ 1430;\ 4862;\ \ \text{(Catalan)}$$

$$W_3(1;2k): 1;\ 3;\ 12;\ 57;\ 303;\ 1743;\ 10629;\ 67791;$$

$$W_4(1;2k): 1;\ 4;\ 22;\ 148;\ 1144;\ 9784;\ 90346;$$

$$W_5(1;2k): 1;\ 5;\ 35;\ 305;\ 3105;\ 35505;\ 444225;$$

$$W_6(1;2k): 1;\ 6;\ 51;\ 546;\ 6906;\ 99156;\ 1573011$$

Random Walks: Narayana polynomials again

The distance to the origin R_n verifies

$$R_{n+1} \sim \sqrt{1 + 2\Lambda R_n + R_n^2}$$

with

$$\Lambda = \cos\theta \sim \frac{\nu!}{\sqrt{\pi}\left(\nu - \frac{1}{2}\right)!}\left(1 - \lambda^2\right)^{\nu - \frac{1}{2}}, \ -1 \leq \lambda \leq +1$$

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As a consequence

$$\mathbb{E} R_{n+1}^{2k} = \mathbb{E}\left(1 + 2\Lambda R_n + R_n^2\right)^k$$

and since

$$\mathcal{N}_k^{(\nu)}(z) = \mathbb{E}\left(1 + 2\Lambda\sqrt{z} + z\right)^{k-1},$$

we deduce (μ is now ν)

$$\mathbb{E}\left(R_{n+1}^{2k}\right) = \mathbb{E}\mathcal{N}_{k+1}^{(\nu)}\left(R_n^2\right).$$

Random Walks: 4 steps

The recursion

$$W_n(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!\,(j+\nu)!} W_{n-1}(\nu; 2j)$$

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Denote

$$A_{k,j}(\nu) = \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!\,(j+\nu)!}$$

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and build the Narayana triangle (or Catalan triangle A001263)

$$A(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 1 & 3 & 1 & 0 & \\ 1 & 6 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad A^3(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \\ 12 & 9 & 1 & 0 & \\ 57 & 72 & 18 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}$$

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For example,

$$W_3(1;2k) : 1;\ 3;\ 12;\ 57;$$

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