Algorithms for Differential Systems

Linear ordinary first-order differential systems with singularities, Singularly-perturbed linear differential systems, completely integrable Pfaffian systems, Apparent Singularities

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10 Oct. 2016, INRIA Saclay, Palaiseau

miniISOLDE- Lindalg

Formal Solutions of

$$x^{p+1}\frac{dY}{dx} = A(x)Y, A \in \mathbb{C}[[x]]$$

Example (Barkatou'1997)

$$x^3\frac{dY}{dx} = \begin{bmatrix} -2x^2 & x^5+x^4+x & -x & x+x^2 \\ 1 & x-2x^2 & 0 & 0 \\ 0 & x & -x^2-x & 0 \\ -x^3 & -2x^4 & x & 0 \end{bmatrix} Y$$

Required: Compute a solution in a neighborhood of $x = 0$.

$$\Phi(x^{1/s})\, x^{C}\, exp(Q(x^{-1/s}))$$

- s is a positive integer referred to as the *ramification index*;
- Φ is a matrix of meromorphic series in $x^{1/s}$ (root-meromorphic in x) over $\mathbb{C}$;
- $Q(x^{-1/s})$ is the *exponential part*. It is a diagonal matrix whose entries are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without contant terms.
- C is a constant matrix which commutes with $Q(x^{-1/s})$.

- Existence: Turrittin, Hukuhara, Levelt, Balser-Jurkat-Lutz, ...
- Algorithms for related problems: Levelt, Hilali and Wazner (1980s), Sommeling (1993), Chen, Schaefke, van Hoeij , Barkatou,(1990s, 2004), Pfluegel (2000), Barkatou-Pfluegel (2007, 2009), ...
- Wasow (1965), Balser(2000), Hsieh and Sibuya (1999),..
- ISOLDE in MAPLE by Barkatou, E. Pfluegel (2012).
- MINIISOLDE in MAPLE and LINDALG in Mathemagix

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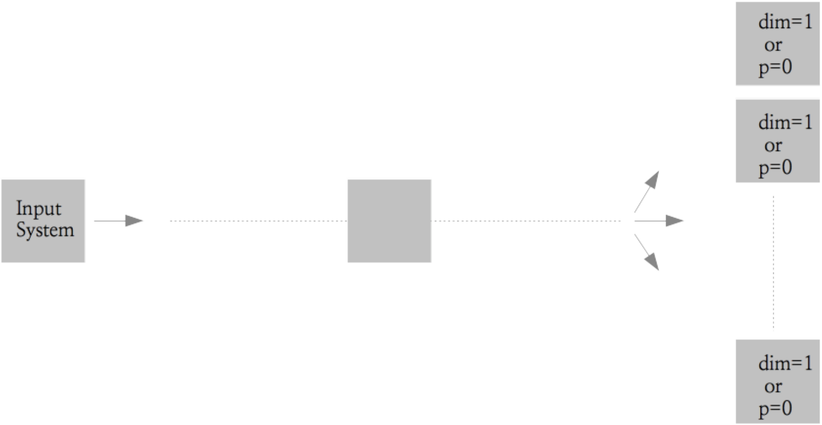

Figure: How to compute an intermediate "nicer" system(s)?

Equivalent systems

$T(x) \in GL_n(\mathbb{C}((x)))$

$$x^{p+1}\frac{dY}{dx} = A(x)Y, \quad A(x) \in \mathbb{C}[[x]]$$

$$\downarrow Y = TZ$$

$$x^{\tilde{p}+1}\frac{dZ}{dx} = \tilde{A}(x)Z, \quad \tilde{A}(x) \in \mathbb{C}[[x]]$$

$$\frac{\tilde{A}}{x^{\tilde{p}+1}} = T^{-1}\frac{A}{x^{p+1}}T - T^{-1}\frac{dT}{dx}$$

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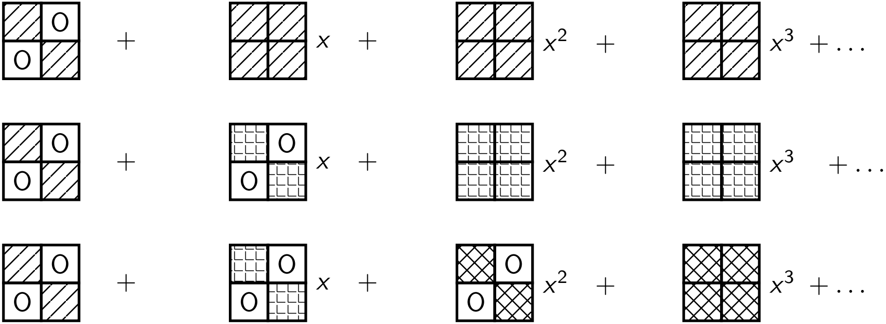
Demo: Splitting Lemma

$$x^{p+1}\frac{dY}{dx} = A(x)Y = (A_0 + A_1 x + A_2 x^2 + \dots)Y$$



MAPLE file: Splitting Lemma examples

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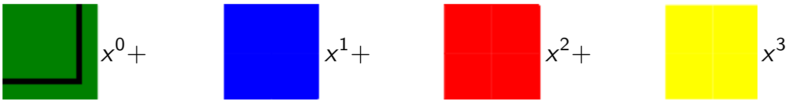
Rank Reduction

Two kinds of transformations: Constant transformations and shearing transformations in x



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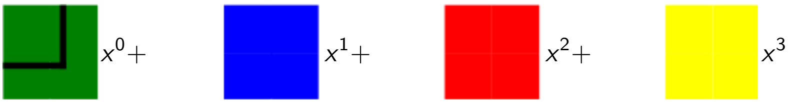

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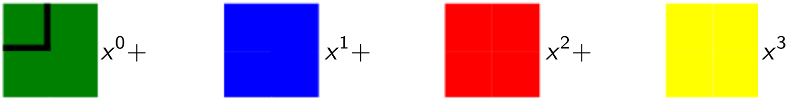

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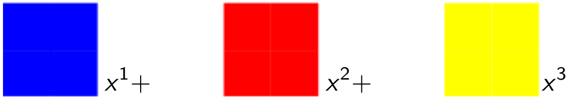
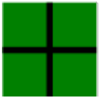
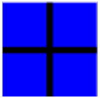
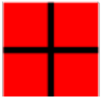
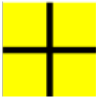


Rank Reduction

Shearing transformation?

$$T = \begin{bmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

 $x^0 +$  $x^1 +$  $x^2 +$  x^3

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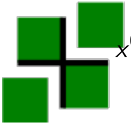
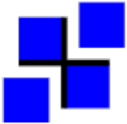
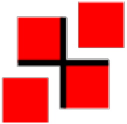
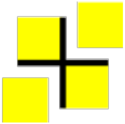
 $x^0 +$  $x^1 +$  $x^2 +$  x^3

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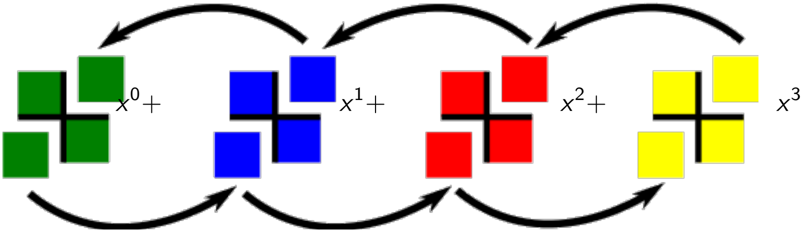

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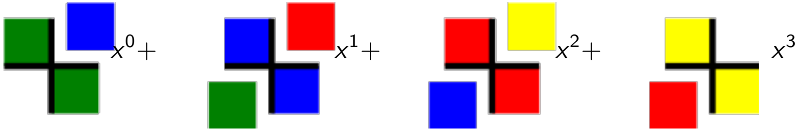

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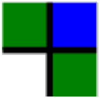
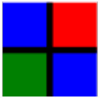
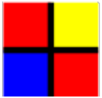
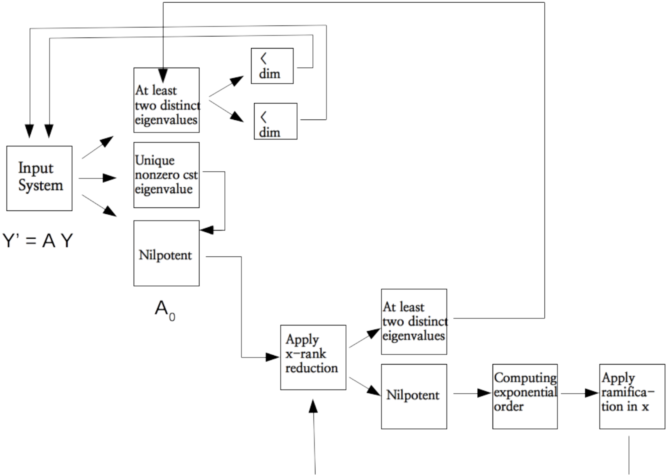
 $x^0 +$  $x^1 +$  $x^2 +$  x^3

Demo: Rank Reduction (Barkatou'1995)

MAPLE file: Rank reduction examples (Barkatou'1996)

Formal Reduction (Barkatou'1997)



$$\Phi(x^{1/s})\ x^{C}\ exp(Q(x^{-1/s}))$$

Demo: Formal Reduction (Barkatou'1997)

- MAPLE file: Examples on computing exponential parts

ParamInt

Example

$$\varepsilon^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x \end{bmatrix} Y$$

Required: Compute a solution in a full neighborhood of $x = 0$ as $\varepsilon \to 0$.

Iwano's formulation (1963)

(1) *Divide a domain [D] in* (x, ε)*-space into a finite number of subdomains so that the solution behaves quite differently as* ε *tends to zero in each of these subdomains;*

(2) *Find out a complete set of asymptotic expressions of independent solutions in each of these subdomains;*

(3) *Determine the so-called connection formula; i.e. a relation connecting two different complete sets of the asymptotic expressions obtained in* (2).

State of Art

- Existence: Turrittin, Hukuhara, Levelt, Balser-Jurkat-Lutz, Schaefke-Volkmer, ...
- Capitalized on Arnold-Wasow form (classical approach for the unperturbed counterpart): Turrittin (1952), Iwano-Sibuya (60's), Wasow (1979), ...
- Algorithmic Treatment excluding *turning points*: G. Chen (1990), ...
- Scalar n^{th}-order: Iwano-Sibuya (1963) , Macutan (1999), ...
- Analytic Reduction : Fruchard - Schaefke (2013) , Hulek, ...

Proufound advancement in the last two decades witihn the research line of unperturbed singular linear differential systems in contrast to the perturbed ones (Wasow' 1985).

$$\varepsilon^h \frac{dY}{dx} = A(x,\varepsilon)Y = \sum_{k=0}^{\infty} A_k(x)\varepsilon^k Y.$$

BUT we need to consider the more general systems:

$$\mathcal{K}_\varepsilon = \{f = \sum_{k\in\mathbb{Z}} f_k(x)\varepsilon^k \in \mathbb{C}((x))((\varepsilon)) \quad \text{s.t.}$$
$$val_x(f_k) \geq \sigma k + p \quad \text{for some } \sigma \in \mathbb{Q}^-, p \in \mathbb{Q}\}$$

$$x^p \xi^h \frac{dY}{dx} = A(x,\xi)Y = \sum_{k=0}^{\infty} A_k(x)\xi^k Y.$$

- $\xi = x^\sigma \varepsilon$, $\sigma \in \mathbb{Q}^-$, $p \in \mathbb{Q}$;
- For all $k \geq 0$, $A_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$;
- σ is called *restraining index*;
- $h > 0$, (and $A_0(x) \neq 0$);
- At the starting point, $\sigma = 0$ and $p = 0$.

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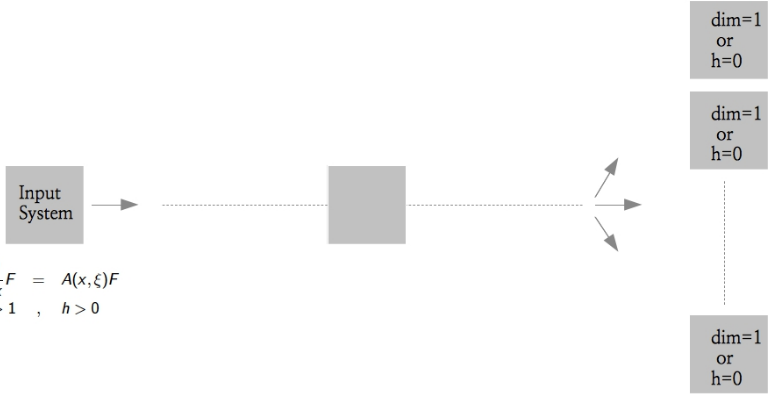
Construct formal solutions:
Abbas-Barkatou-Maddah'ISSAC2014,
Barkatou-Maddah'2016
(MAPLE package ParamInt)

(3) *Determine the so-called connection formula; i.e. a relation connecting two different complete sets of the asymptotic expressions obtained in (2).*



$$x^p\xi^h\frac{d}{dx}F = A(x,\xi)F$$
$$n>1 \quad , \quad h>0$$

Figure: How to compute an intermediate “nicer” system(s)?

Equivalent systems

$$[A_{\sigma_A}] \qquad x^p \xi^h \frac{dF}{dx} = A(x,\xi)F, \quad \xi = x^{\sigma_A}\varepsilon$$

$$\downarrow \quad F = TG, \quad T \in GL_n(K_\varepsilon) \qquad ?$$

$$[\tilde{A}_{\sigma_{\tilde{A}}}] \qquad \tilde{\xi}^{\tilde{h}} x^{\tilde{p}} \frac{dG}{dx} = \tilde{A}(x,\tilde{\xi})\, G, \quad \tilde{\xi} = x^{\sigma_{\tilde{A}}}\varepsilon$$

We have:

$$\frac{\tilde{A}(x,\tilde{\xi})}{\tilde{\xi}^{\tilde{h}} x^{\tilde{p}}} = T^{-1} \frac{A(x,\xi)}{\xi^h x^p}\, T - T^{-1}\frac{dT}{dx}.$$

Example

$$\varepsilon^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x \end{bmatrix} Y$$

With $\sigma_A = 0$ we can write:

$$\xi^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \xi & 0 & x \end{bmatrix} Y \quad \text{where} \quad A_0(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{bmatrix}.$$

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Let $T = diag(1, x, x^2)$. Then $Y = TG$ yields:

$$\xi^2 \frac{dG}{dx} = x\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x^{-3}\xi + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x^{-2}\xi^2 \} G.$$

Setting $\tilde{\xi} = x^{-3}\xi = x^{-3}\varepsilon$, the former can be rewritten equivaently as:

$$\tilde{\xi}^2 x^5 \frac{dG}{dx} = \{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^4 & 0 \\ 0 & 0 & -2x^4 \end{bmatrix} \tilde{\xi}^2 \} G.$$

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Let $G = TW$ where $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -2 \\ -1 & -1 & 0 \end{bmatrix} \tilde{\xi} + O(\tilde{\xi}^2)$.

Then with $W = [W_1, W_2]^T$ we have:

$$\tilde{\xi}^2 x^5 \frac{dW_1}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 1 & -1 \\ 1 & -1 + x^4 \end{bmatrix} \tilde{\xi}^2 + O(\tilde{\xi}^3) \right\} W_1.$$

$$\tilde{\xi}^2 x^5 \frac{dW_2}{dx} = \{ 1 + \tilde{\xi} + (1 + 2x^4)\tilde{\xi}^2 + O(\tilde{\xi}^3) \} W_2.$$

We have:

$$\tilde{\xi}^2 x^5 \frac{dW_1}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 1 & -1 \\ 1 & -1 + x^4 \end{bmatrix} \tilde{\xi}^2 + O(\tilde{\xi}^3) \right\} W_1.$$

$$\tilde{\xi}^2 x^5 \frac{dW_2}{dx} = \{1 + \tilde{\xi} + (1 + 2x^4)\tilde{\xi}^2 + O(\tilde{\xi}^3) \} W_2.$$

- The second subsystem is scalar and the exponential part is $exp(\int \tilde{\xi}^{-2} x^{-5} (1 + O(\tilde{\xi}))\, dx) = exp(\frac{1}{2} \varepsilon^{-2} x^2 (1 + O(\varepsilon^{-2} x^2)))$.
- The first subsystem has a nilpotent leading matrix and requires further reduction.

We have:

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- The second subsystem is scalar and the exponential part is $exp(\int \tilde{\xi}^{-2} x^{-5}(1 + O(\tilde{\xi}))\, dx) = exp(\frac{1}{2}\varepsilon^{-2} x^2 (1 + O(\varepsilon^{-2} x^2)))$.
- The first subsystem has a nilpotent leading matrix and requires further reduction.

$$
\begin{aligned}
\xi^2 x^5 \frac{dW_1}{dx} &= B(x,\xi) W_1 \\
&= \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \xi + \begin{bmatrix} 1 & -1 \\ 1 & -1 + x^4 \end{bmatrix} \xi^2 + O(\xi^3) \right\} W_1 .
\end{aligned}
$$

- Set $\xi = \tilde{\xi}^2 x^3$.

- Apply ε-rank reduction via $Diag(1, \tilde{\xi})$, we get the following $\tilde{\varepsilon}$-irreducible system

$$\tilde{\xi}^3 x^{11} \frac{dS}{dx} = \tilde{B}(x, \tilde{\xi}) S \quad \text{where}$$

$$\begin{aligned} \tilde{B}(x,\tilde{\xi}) \quad &= \quad \begin{bmatrix} 0 & 1 \\ -x^3 & 0 \end{bmatrix} + \begin{bmatrix} -x^3 & 0 \\ 0 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 0 & 0 \\ x^6 & 0 \end{bmatrix} \tilde{\xi}^2 \\ &+ \quad \begin{bmatrix} x^6 & 0 \\ 0 & x^6(-1+x^4) \end{bmatrix} \tilde{\xi}^3 + \begin{bmatrix} 0 & -x^6 \\ 0 & 0 \end{bmatrix} \tilde{\xi}^4 + O(\tilde{\xi}^5). \end{aligned}$$

- Apply turning point algorithm:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & x^{3/2} \end{bmatrix} U$$

which yields

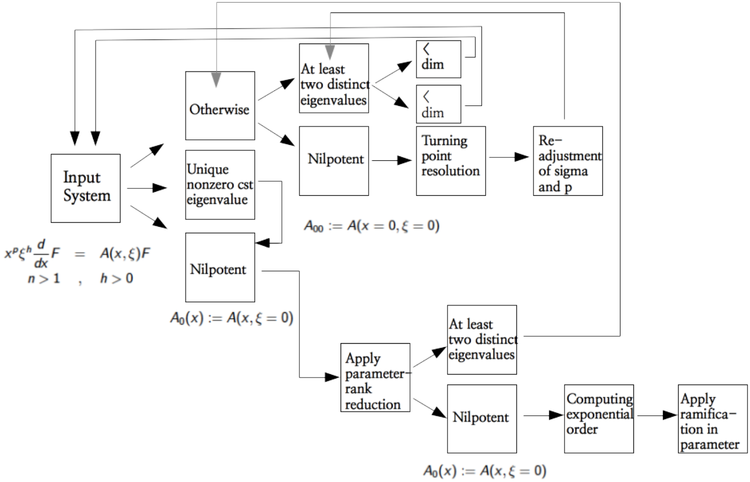
$$\tilde{\xi}^3 x^{19/2} \frac{dU}{dx} = \tilde{B}(x, \tilde{\xi}) U \quad \text{where}$$

$$\begin{aligned} \tilde{B}(x, \tilde{\xi}) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -x^{3/2} & 0 \\ 0 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 0 & 0 \\ x^3 & 0 \end{bmatrix} \tilde{\xi}^2 \\ &+ \begin{bmatrix} x^{9/2} & 0 \\ 0 & x^{9/2}(-1 + x + x^4) \end{bmatrix} \tilde{\xi}^3 + \begin{bmatrix} 0 & -x^3 \\ 0 & 0 \end{bmatrix} \tilde{\xi}^4 + O(\tilde{\xi}^5). \end{aligned}$$

- Applying Splitting Lemma and re-substituting for $\tilde{\xi}^2 x^3 = \xi = x^{-3}\varepsilon$, we get

$$\varepsilon^{3/2} x^{1/2} \frac{dR}{dx} = \tilde{\tilde{B}}(x,\varepsilon) R \quad \text{where}$$

$$\tilde{\tilde{B}}(x,\varepsilon) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} x^{-3/2} \varepsilon^{1/2} + O(x^{-3}\varepsilon).$$


Input System
$x^p \xi^h \frac{d}{dx} F = A(x,\xi) F$
$n > 1$, $h > 0$
Otherwise
Unique nonzero cst eigenvalue
Nilpotent
$A_0(x) := A(x, \xi = 0)$
At least two distinct eigenvalues
Nilpotent
$A_{00} := A(x=0, \xi=0)$
< dim
< dim
Turning point resolution
Re−adjustment of sigma and p
Apply parameter−rank reduction
At least two distinct eigenvalues
Nilpotent
$A_0(x) := A(x, \xi = 0)$
Computing exponential order
Apply ramifica−tion in parameter

Example

$$\xi^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \xi & x & 0 \end{bmatrix} Y \quad \text{where} \quad A_0(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x & 0 \end{bmatrix}, \sigma_A = 0.$$

Here, $s = 2$. So we can set $x = t^2$ and let $Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & x^{1/2} & 0 \\ 0 & 0 & x \end{bmatrix} G$ then:

$$\xi^2 \frac{dG}{dx} = x^{1/2} \{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x^{-3/2}\xi + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x^{-1}\xi^2 \} G,$$

or with the readjustment $\tilde{\xi} = x^{-3/2}\xi = x^{-3/2}\varepsilon$:

$$\tilde{\xi}^2 x^{5/2} \frac{dG}{dx} = \{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^2 & 0 \\ 0 & 0 & -2x^2 \end{bmatrix} \tilde{\xi}^2 \} G,$$

Example

$\xi^2 \frac{dY}{dx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \xi & x & 0 \end{bmatrix} Y$ where $A_0(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x & 0 \end{bmatrix}, \sigma_A = 0$. Here,

$s = 2$. So we can set $x = t^2$ and let $Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & x^{1/2} & 0 \\ 0 & 0 & x \end{bmatrix} G$ then:

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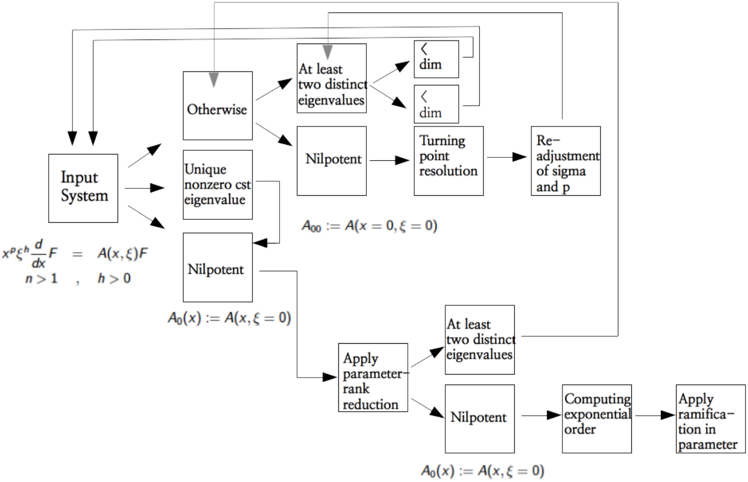
or with the readjustment $\tilde{\xi} = x^{-3/2}\xi = x^{-3/2}\varepsilon$:

$$\tilde{\xi}^2 x^{5/2} \frac{dG}{dx} = \{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^2 & 0 \\ 0 & 0 & -2x^2 \end{bmatrix} \tilde{\xi}^2 \} G,$$


Input System
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$A_0(x) := A(x, \xi = 0)$
Computing exponential order
Apply ramifica−tion in parameter

Demo: Formal Reduction

Construction of a fundamental matrix of formal solutions by a recursive algorithm:

$$Y = (\sum_{i=0}^{\infty} \Phi_i(x^{1/s})\, \varepsilon^{i/d})\ exp(\int Q(x^{1/s}, \varepsilon^{-1/d})),$$

where s, d are positive integers; $\int Q$ is the diagonal matrix whose entries are polynomials in $\varepsilon^{-1/d}$ with coefficients in $\mathbb{C}((x^{1/s}))$

MAPLE file: Examples on formal reduction

Demo: Formal Reduction

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Maple file: Examples on formal reduction

PfaffInt

Example

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ x_2^3 \frac{\partial}{\partial x_2} Y = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases} .$$

Required: Compute a solution in a neighborhood of $(x, y) = (0, 0)$.

General form

$$[\underline{A}] \quad \begin{cases} x_1^{p_1+1} \frac{\partial}{\partial x_1} Y = A_{(1)}(x_1, x_2, \dots, x_m)\, Y \\ x_2^{p_2+1} \frac{\partial}{\partial x_2} Y = A_{(2)}(x_1, x_2, \dots, x_m)\, Y \\ \vdots \\ x_m^{p_m+1} \frac{\partial}{\partial x_m} Y = A_{(m)}(x_1, x_2, \dots, x_m)\, Y \end{cases}$$

For $i, j \in \{1, \dots, m\}$,

- p_i is an integer and $\underline{p} = (p_1, \dots, p_m)$ is called Poincaré rank
- $A_{(i)} \in \mathrm{R} = \mathbb{C}[[x_1, \dots, x_m]]$ (i^{th}-component), and
- Integrability conditions:

$$x_i^{p_i+1} \frac{\partial}{\partial x_i} A_{(j)}(x) - x_j^{p_j+1} \frac{\partial}{\partial x_j} A_{(i)}(x) = A_{(i)}(x)\, A_{(j)}(x) - A_{(j)}(x)\, A_{(i)}(x).$$

General form

$$[\underline{A}] \quad \begin{cases} x_1^{p_1+1} \frac{\partial}{\partial x_1} Y = A_{(1)}(x_1, x_2, \dots, x_m)\, Y \\ x_2^{p_2+1} \frac{\partial}{\partial x_2} Y = A_{(2)}(x_1, x_2, \dots, x_m)\, Y \\ \vdots \\ x_m^{p_m+1} \frac{\partial}{\partial x_m} Y = A_{(m)}(x_1, x_2, \dots, x_m)\, Y \end{cases}$$

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General form

$$[\underline{A}] \quad \left\{ \begin{array}{lcl} x_1^{p_1+1} \frac{\partial}{\partial x_1} Y & = & A_{(1)}(x_1, x_2, \ldots, x_m)\, Y \\ x_2^{p_2+1} \frac{\partial}{\partial x_2} Y & = & A_{(2)}(x_1, x_2, \ldots, x_m)\, Y \\ \vdots & & \\ x_m^{p_m+1} \frac{\partial}{\partial x_m} Y & = & A_{(m)}(x_1, x_2, \ldots, x_m)\, Y \end{array} \right.$$

For $i, j \in \{1, \ldots, m\}$,

- p_i is an integer and $\underline{p} = (p_1, \ldots, p_m)$ is called Poincaré rank
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Fundamental matrix of formal solutions

$$\Phi(x_1^{1/s_1}, \ldots, x_m^{1/s_m}) \prod_{i=1}^{m} x_i^{C_i} \prod_{i=1}^{m} \exp(Q_i(x_i^{-1/s_i}))$$

- Φ is an invertible matrix whose entries are meromorphic series in $(x_1^{1/s_1}, \ldots, x_m^{1/s_m})$ over $\mathbb{C}$;
- $Q_i(x_i^{-1/s_i})$ is a diagonal matrix of polynomials in x_i^{-1/s_i} over $\mathbb{C}$ without contant terms.
- C_i is a constant matrix which commutes with $Q_i(x_i^{-1/s_i})$.

- H. Charrière, P. Deligne, R. Gérard, A. H. M. Levelt, Y. Sibuya, A. van den Essen, . . . (70's and 80's)
- Algorithms: Reduction of Poincaré rank, Constructing Solutions of regular systems - Barkatou and LeRoux (2006), Closed Form Solutions of Integrable Connections: Barkatou-Cluzeau-ElBacha-Weil'2012

Example

Poincaré Rank Reduction, Barkatou-LeRoux'2006

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = A_{(1)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & -x_2 + x_1^3 \end{bmatrix} \right) Y \\ x_2^2 \frac{\partial}{\partial x_2} Y = A_{(2)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} \right) Y \end{cases}$$

$$\downarrow \quad Y = \left(\begin{bmatrix} x_1^3 & -x_2^2 \\ 0 & x_2 \end{bmatrix} \right) G$$

$$\begin{cases} x_1 x_2 \frac{\partial}{\partial x_1} G = \tilde{A}_{(1)}(x_1, x_2)\ G = \left(\begin{bmatrix} -2x_2 & 0 \\ -1 & x_2 \end{bmatrix} \right) G \\ x_2^3 \frac{\partial}{\partial x_2} G = \tilde{A}_{(2)}(x_1, x_2)\ G = \left(\begin{bmatrix} -x_2^2 & 0 \\ -2x_1^3 & -2x_2^2 \end{bmatrix} \right) G. \end{cases}$$

Example

Poincaré Rank Reduction, Barkatou-LeRoux'2006

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = A_{(1)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & -x_2 + x_1^3 \end{bmatrix} \right) Y \\ x_2^2 \frac{\partial}{\partial x_2} Y = A_{(2)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} \right) Y \end{cases}$$

$$\downarrow \quad Y = \left(\begin{bmatrix} x_1^3 & -x_2^2 \\ 0 & x_2 \end{bmatrix} \right) G$$

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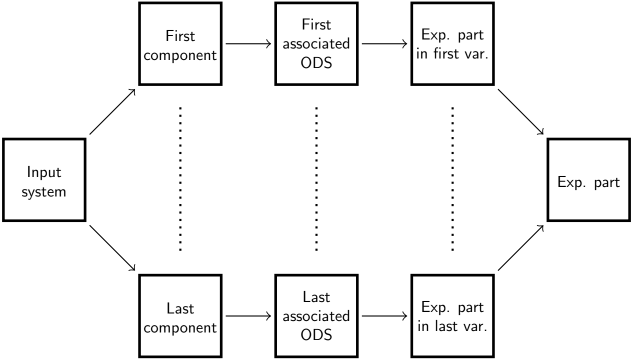


Figure: Computing the exponential part from associated ODS's

Example

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ \\ x_2^3 \frac{\partial}{\partial x_2} Y = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases} .$$

Associated system:

$$\begin{cases} x_1^4 \frac{d}{dx_1} \mathcal{Y} = \begin{bmatrix} x_1^3 + x_1^2 & 0 \\ -1 & x_1^3 + x_1^2 \end{bmatrix} \mathcal{Y} \\ \\ x_2^3 \frac{d}{dx_2} \mathcal{Y} = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} \mathcal{Y} \end{cases} .$$

Example

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ x_2^3 \frac{\partial}{\partial x_2} Y = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases}.$$

Associated system:

$$\begin{cases} x_1^4 \frac{d}{dx_1} \mathcal{Y} = \begin{bmatrix} x_1^3 + x_1^2 & 0 \\ -1 & x_1^3 + x_1^2 \end{bmatrix} \mathcal{Y} \\ x_2^3 \frac{d}{dx_2} \mathcal{Y} = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} \mathcal{Y} \end{cases}.$$

Example

$$\begin{cases} x_1^4 \frac{dY}{dx_1} = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ \\ x_2^3 \frac{dY}{dx_2} = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases} .$$

With MINIISOLDE or LINDALG we compute from the asscoiated system

$$\Phi(x_1, x_2)\; x_1^{C_1}\; x_2^{C_2} \exp\left(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix}\right) \exp\left(\begin{bmatrix} \frac{3}{{x_2}^2} + \frac{2}{x_2} & 0 \\ 0 & \frac{3}{{x_2}^2} + \frac{2}{x_2} \end{bmatrix}\right).$$

Upon applying

$$Y = \exp(\frac{-1}{x_1}) \exp(\frac{3}{x_2^2} + \frac{2}{x_2}) \; G,$$

we have

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} G = \begin{bmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & x_1^3 - x_2 \end{bmatrix} G \\ x_2^2 \frac{\partial}{\partial x_2} G = \begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} G \end{cases}$$

And so, it is left to obtain:

$$G(x_1, x_2) = \Phi(x_1, x_2) \; x_1^{C_1} \; x_2^{C_2}.$$

For rank-reduction, we apply $G = T_1 H$ where

$$T_1 = \begin{bmatrix} x_2 x_1^3 & -x_2 \\ 0 & 1 \end{bmatrix}$$

which yields:

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} H = \begin{bmatrix} -2 & 0 \\ -x_2 & 1 \end{bmatrix} H, \\ x_2 \frac{\partial}{\partial x_2} H = \begin{bmatrix} -2 & 0 \\ -2x_1^3 & -1 \end{bmatrix} H. \end{cases}$$

Finally, we compute

$$T_2 = \begin{bmatrix} 1 & 0 \\ \frac{x_2}{3} + 2x_1^3 & -1 \end{bmatrix}.$$

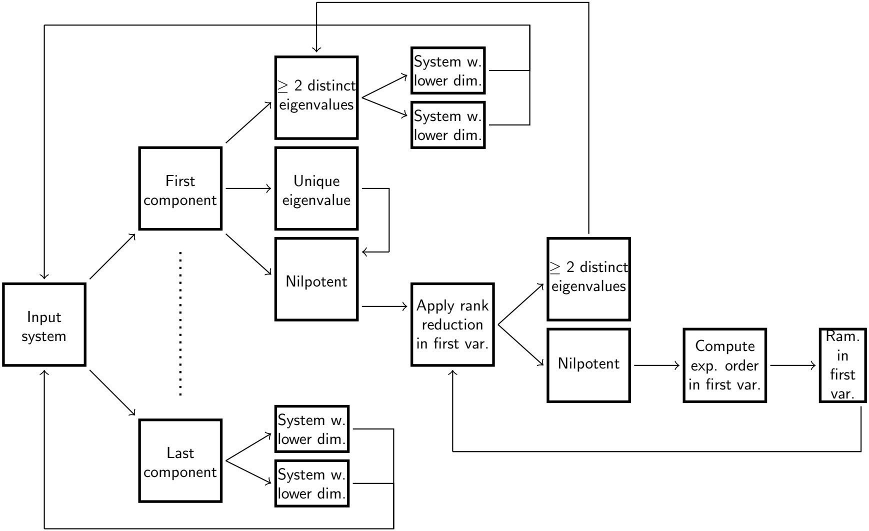
Then $H = T_2\, U$ yields

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} U = C_1\, U = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} U, \\ x_2 \frac{\partial}{\partial x_2} U = C_2\, U = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} U. \end{cases}$$

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} F = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} F \\ \\ x_2^3 \frac{\partial}{\partial x_2} F = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} F \end{cases}.$$

A fundamental matrix of formal solutions is given by

$$T_1 \; T_2 \; x_1^{C_1} \; x_2^{C_2} \; \exp(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix}) \; \exp(\begin{bmatrix} \frac{3}{{x_2}^2} + \frac{2}{x_2} & 0 \\ 0 & \frac{3}{{x_2}^2} + \frac{2}{x_2} \end{bmatrix}).$$


Input system
First component
Last component
≥ 2 distinct eigenvalues
Unique eigenvalue
Nilpotent
System w. lower dim.
System w. lower dim.
System w. lower dim.
System w. lower dim.
Apply rank reduction in first var.
≥ 2 distinct eigenvalues
Nilpotent
Compute exp. order in first var.
Ram. in first var.

Demo

MAPLE file: Examples on formal reduction (Abbas-M. Barkatou-S.S. Maddah-ISSAC'14 and M. Barkatou- M. Jaroschek-S.S.Maddah-Submitted'2015)

Perturbed Pfaffian System in Quantum Chromodynamics

- Communicated by Clemens Raab, Group of Elementary Particle Theory;
- DESY Research Center of Experimental Physics, Hamburg.

$$\begin{cases} \dfrac{\partial}{\partial x} Y = A(x,y,\varepsilon) Y = \frac{1}{\varepsilon\,(-1+x)\,x\,(xy-y+1)\,(xy+\varepsilon-y+1)} \begin{bmatrix} O_{4\times 4} & O_{4\times 3} \\ M(x,y) & N(x,y) \end{bmatrix} Y \\ \dfrac{\partial Y}{\partial y} = B(x,y,\varepsilon) Y = \frac{1}{y\,(y-1)\,(xy-y+1)\,(xy+\varepsilon-y+1)} \begin{bmatrix} E(x,y) & O_{4\times 3} \\ H(x,y) & L(x,y) \end{bmatrix} Y \end{cases}$$

Objective

Construct a solution in a neighborhood of $(x, y = 1)$

- $Y = T\,Z$ where $T = Diag(\varepsilon, \varepsilon, \varepsilon, \varepsilon, 1, 1, 1)$ yields an equivalent system non-singular in ε;
- Apply a translation of independent variable $z = y - 1$.
- Let $Z = \begin{bmatrix} G \\ R \end{bmatrix}$ be a FMFS. We have

$$\begin{cases} \dfrac{\partial G}{\partial x} = O_{4\times 1} \\ z\dfrac{\partial G}{\partial z} = E(z, \varepsilon)G \end{cases}$$

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$$\begin{cases} \dfrac{\partial R}{\partial x} = \tilde{N}(x, z, \varepsilon)R + \tilde{M}(x, z, \varepsilon)R \\ \dfrac{\partial R}{\partial z} = \tilde{L}(x, z, \varepsilon)R + \tilde{H}(x, z, \varepsilon)R \end{cases}$$

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Application in Statistics: Muirhead system

Communicated by N. Takayama, Kobe University, Japan

$$\begin{cases} \frac{\partial}{\partial x} F = \frac{1}{x^2(y-x)^2} A(x,y)\ F \\ \\ \frac{\partial}{\partial y} F = \frac{1}{y^2(x-y)^2} B(x,y)\ F \end{cases}$$

Objective

Construct a Fundamental Matrix of Formal Solutions in nbhd of $(0,0)$

- No Normal Crossings
- $A(x,y) = P\ B(y,x)$ where P is a permutation matrix.

AppSing

$$\frac{dY}{dx} = A(x)Y, \quad A(x) \in \mathcal{M}_n(\mathbb{C}(x))$$

Apparent singularities

If x_0 is a pole of $A(x)$ but there exists a fundamental matrix of formal solutions whose entries are holomorphic in some neighborhood of x_0, then x_0 is an apparent singularity.

Detecting and removing apparent singularities (M. Barkatou, S.S. Maddah, ISSAC'15): MAPLE file: Examples on removing apparent singularities

- LINDALG: MATHEMAGIX package for symbolic resolution of linear systems of differential equations with singularities.
- MINIISOLDE : MAPLE package for symbolic resolution of linear Systems of Differential Equations with singularities.
- PARAMINT: MAPLE package for symbolic resolution of singularly-perturbed linear systems of differential equations, prototype implementaton.
- PFAFFINT: MAPLE package for symbolic resolution of completely integrable pfaffian systems with normal crossings, prototype implementation.
- APPSING: MAPLE package for removing apparent singularities of systems of linear differential equations with rational function coefficients.
- PARAMALG: MAPLE package for differential-like reduction of matrices perturbed by a parameter.

Nov. 14th

Generalized Hypergeometric Solutions of Linear Differential Systems : Test equivalence between an input system and a hypergeometric system

Given system

$$[A] \qquad \partial W = A(x) W \quad \text{where}$$

$$A(x) = \begin{bmatrix} -\frac{3x^4+15x^3-x^2-86x-85}{(x^2-x-3)(x+3)^2(x+4)(x+1)} & \frac{(x+4)(2x^3+11x^2+12x-8)}{(x^2-x-3)(x+3)^2} & \frac{(x^2+1)(x+4)^2(x+2)}{(x^2-x-3)(x+3)^2x} \\ \frac{-1}{(x^2-x-3)(x+3)^2(x+4)} & \frac{-(3x^4+18x^3+23x^2-23x-35)}{(x^2-x-3)(x+3)^2(x+1)} & \frac{-(x+2)(x+4)^2}{(x^2-x-3)(x+3)^2x} \\ \frac{x(30x^2+79x+12)}{30(x+2)^3(x+3)} & \frac{x(15x^3-169x-147)(x+4)}{-30(x+2)^3(x+3)} & \frac{15x^5+121x^4+323x^3+394x^2+387x+270}{15(x+2)(x+3)^2x(x+1)} \end{bmatrix}.$$

$$[H_{2,0}] \qquad \partial Y = H_{2,0}(x)Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2x^2} & \frac{45x+4}{30x^2} & 1 - \frac{-14}{15x} \end{bmatrix}$$

- The change of variable $x \to \frac{(x+2)^2}{x+3}$
- The gauge transformation $Y = T(x)Z$ where
$$T(x) = \begin{bmatrix} 1 & x^2+1 & 0 \\ \frac{1}{x+4} & 1 & 0 \\ 0 & 0 & \frac{1}{x} \end{bmatrix}.$$
- The exp-product transformation $Z = We^{\int \frac{1}{x+1} dx}$

Result

Solutions of input system $[A]$ can be expressed in terms of generalized hypergeometric functions.

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