Random Walks in the Quarter-Plane: Explicit Criterions for the Finiteness of the Associated Group in the Genus 1 Case

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Introduction

- Piecewise homogeneous random walk with sample paths in $\mathbb{Z}_+^2$, the lattice in the positive quarter plane. In the strict interior of $\mathbb{Z}_+^2$, the size of the jumps is 1, and $\{p_{ij}, |i|, |j| \leq 1\}$ will denote the generator of the process for this region. Thus a transition $(m, n) \to (m + i, n + j), m, n > 0$, can take place with probability p_{ij}, and
$$\sum_{|i|,|j|\leq 1} p_{ij} = 1.$$

- No strong assumption about the boundedness of the upward jumps on the axes, neither at $(0, 0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers.

- Original question : Find an explicit form for the invariant measure of such process.

The basic functional equation

The invariant measure $\{\pi_{i,j}, i, j \geq 0\}$ satisfies the fundamental bivariate functional equation

$$\boxed{Q(x,y)\pi(x,y) = q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + \pi_0(x,y)}, \qquad (1)$$

where in (1) the unknown functions $\pi(x,y), \pi(x), \widetilde{\pi}(y)$ are sought to be analytic in the region $\{(x,y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$, and continuous on their respective boundaries.

$$
\begin{cases}
\pi(x,y) = \sum_{i,j\geq 1} \pi_{ij} x^{i-1} y^{j-1}, \\
\pi(x) = \sum_{i\geq L} \pi_{i0} x^{i-L}, \quad \widetilde{\pi}(y) = \sum_{j\geq M} \pi_{0j} y^{j-M}, \\
Q(x,y) = xy\left[1 - \sum_{i,j\in\mathcal{S}} p_{ij} x^i y^j\right], \quad \sum_{i,j\in\mathcal{S}} p_{ij} = 1, \\
q(x,y) = x^L\left[\sum_{i\geq -L, j\geq 0} p'_{ij} x^i y^j - 1\right] \equiv x^L(P_{L0}(x,y) - 1), \\
\widetilde{q}(x,y) = y^M\left[\sum_{i\geq 0, j\geq -M} p''_{ij} x^i y^j - 1\right] \equiv y^M(P_{0M}(x,y) - 1), \\
\pi_0(x,y) = \sum_{i=1}^{L-1} \pi_{i0} x^i \left[P_{i0}(x,y) - 1\right] + \sum_{j=1}^{M-1} \pi_{0j} y^j \left[P_{0j}(x,y) - 1\right] + \pi_{00}(P_{00}(xy) - 1).
\end{cases}
$$

$\mathcal{S}$ is the set of allowed jumps, and $q, \widetilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).

Group and Genus

The function $Q(x, y)$, often referred to as the *kernel* of (1), can be rewritten in the two following equivalent forms

$$\boxed{Q(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y)}, \tag{2}$$

$$\begin{aligned} a(x) &= p_{1,1}x^2 + p_{0,1}x + p_{-1,1} & \widetilde{a}(y) &= p_{1,1}y^2 + p_{1,0}y + p_{1,-1}, \\ b(x) &= p_{1,0}x^2 + (p_{0,0} - 1)x + p_{-1,0} & \widetilde{b}(y) &= p_{0,1}y^2 + (p_{0,0} - 1)y + p_{0,-1}, \\ c(x) &= p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1} & \widetilde{c}(y) &= p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}. \end{aligned}$$

We shall also need the discriminants

$$D(x) \stackrel{\text{def}}{=} b^2(x) - 4a(x)c(x), \qquad \widetilde{D}(y) \stackrel{\text{def}}{=} \widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y). \tag{3}$$

The polynomials D and $\widetilde{D}$ are of degree 4, respectively in x and y.

Let $\mathbb{C}(x), \mathbb{C}(y)$ and $\mathbb{C}(x,y)$ denote the respective fields of rational functions of x, y and (x,y) over $\mathbb{C}$. Since in general Q is assumed to be irreducible, the quotient field $\mathbb{C}(x,y)$ with respect to Q will be denoted by $\mathbb{C}_Q(x,y)$.

Definition 1. *The* group of the random walk *is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by ξ and η given by*

$$\xi(x,y) = \left(x, \frac{c(x)}{y\,a(x)}\right), \qquad \eta(x,y) = \left(\frac{\widetilde{c}(y)}{x\,\widetilde{a}(y)}, y\right).$$

Here ξ and η are involutions satisfying $\xi^2 = \eta^2 = I$.

Lemma 2. *Let*

$$\delta \stackrel{def}{=} \eta\xi. \tag{4}$$

Then $\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. ■

- **The group $\mathcal{H}$ is finite of order $2n$ if, and only if,**

$$\delta^n = I. \tag{5}$$

- **The product $\delta = \eta\xi$ is non-commutative, except for $\delta^2 = I$, in which case the group is of order 4.**

- We shall write $f_\alpha = \alpha(f)$, for any automorphism $\alpha \in \mathcal{H}$ and any function $f \in \mathbb{C}_Q(x, y)$.

- The fundamental equation (1), together with ξ, η, δ, can be lifted onto the universal covering $\mathbb{C}$ (the finite complex plane).

Let $\{x_\ell\}_{1\leq\ell\leq 4}$ be the 4 roots of the discriminant $D(x)$ [see equation (3)], which are the branch points of the Riemann surface

$$\mathscr{K} = \{(x,y) \in \mathbb{C}^2 : Q(x,y) = 0\}.$$

They are always real, with $|x_1| \leq |x_2| \leq |x_3| \leq |x_4|$.

Moreover $x_1 \leq x_2$, $[x_1 x_2] \subset [-1,+1]$ and $0 \leq x_2 \leq x_3$.

Here $\mathscr{K}$ is assumed to be of genus 1 (the torus), so that the algebraic curve $Q(x,y) = 0$ admits a *uniformization given in terms of the Weierstrass $\wp$ function with periods ω_1, ω_2* and its derivatives. Indeed, letting

$$\begin{aligned} D(x) &= b^2(x) - 4a(x)c(x) \stackrel{\text{def}}{=} d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ z &\stackrel{\text{def}}{=} 2a(x)y + b(x), \end{aligned}$$

the following formulae hold (see the *Yellow Book [FIM]*).

1. If $d_4 \neq 0$ (4 finite branch points $x_1, \ldots, x_4$) then $D'(x_4) > 0$ and

$$\begin{cases} x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6}D''(x_4)\right)^2}. \end{cases} \tag{6}$$

2. If $d_4 = 0$ (3 finite branch points x_1, x_2, x_3 and $x_4 = \infty$) then

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - \dfrac{d_2}{3}}{d_3}, \\ z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases}$$

$$\omega_1 = 2i \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-D(x)}}, \qquad \omega_2 = 2 \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{D(x)}}, \qquad \omega_3 = 2 \int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{D(x)}}.$$

ω_1 is purely imaginary, while $0 < \omega_3 < \omega_2$.

- It was proved in [FIM] that the group $\mathcal{H}$ is finite of order $2n$ if and only if

$$n\omega_3 = 0 \mod (\omega_1, \omega_2),$$

or, since ω_3 is real,

$$n\omega_3 = 0 \mod (\omega_2), \tag{7}$$

where n stands for the minimal positive integer with this property.

On the *universal covering* $\mathbb{C}$ (the finite complex plane), the automorphisms ξ, η, δ become (see [FIM], Section 3.3)

$$\xi^*(\omega) = -\omega + \omega_2, \qquad \eta^*(\omega) = -\omega + \omega_2 + \omega_3, \qquad \delta^*(\omega) = \eta^*\xi^* = \omega + \omega_3. \quad (8)$$

Here $\delta = \eta\xi$ corrresponds to $\delta^* = \eta^*\xi^*$. Thus, for any $f(x, y) \in C_Q(x, y)$,

$$\delta(f(x, y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \; \omega \in \mathbb{C}.$$

In particular,

$$\begin{cases} \delta(x) = x(\delta^*(\omega)) = x(\omega + \omega_3), \\ \eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3). \end{cases} \quad (9)$$

$\mathcal{H}$ is generated by the elements ξ and η, and we can define the homomorphism

$$h(R(x,y)) \stackrel{\text{def}}{=} R(h(x),h(y)), \quad \forall h \in \mathcal{H},\ \forall R \in \mathbb{C}_Q(x,y).$$

For any $R \in \mathbb{C}_Q(x,y)$, the following equivalences hold:

$$\begin{cases} \xi(R) = R \iff R \in \mathbb{C}(x), \\ \eta(R) = R \iff R \in \mathbb{C}(y), \end{cases} \tag{10}$$

so that $\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$) is the set of elements of $\mathbb{C}_Q(x,y)$ invariant with respect to ξ (resp. η). Indeed, R has the general form

$$R(x,y) = A(x) + B(x)y \mod Q(x,y),$$

where $A(x)$ and $B(x)$ are elements of $\mathbb{C}(x)$. [Hint: $\xi(R) = R$ and $\xi(y) \neq y$, so that necessarily $B(x) \equiv 0$].

Introduce the matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix}, \tag{11}$$

and let $\vec{C}_1, \vec{C}_2, \vec{C}_3$ (resp. $\vec{D}_1, \vec{D}_2, \vec{D}_3$) denote the column vectors of $\mathbb{P}$ (resp. of $\mathbb{P}^T$, the transpose matrix of $\mathbb{P}$).

Proposition 3. *Assume there exists a positive integer s such that*

$$\delta^s(x) = x. \tag{12}$$

Then $\delta^s = I$ and the group is of order $2s$, where s stands for the smallest integer with property (12)*.*

Sketch of proof. Each of the three following permutations

$$x \Longleftrightarrow y, \quad \delta \Longleftrightarrow \delta^{-1}, \quad \mathbb{P} \Longleftrightarrow \mathbb{P}^T,$$

implies the two other ones.

Hence, the quantity $\rho(x,y,k) \stackrel{\text{def}}{=} \delta^k(x).\delta^{-k}(y)$, for any integer $k \geq 1$, remains invariant by permuting $\mathbb{P}$ with $\mathbb{P}^T$.

• Assume first $s = 2m$. Then (12) becomes $\delta^m(x) = \delta^{-m}(x)$, and

$$\rho(x,y,m) = \delta^{-m}(x).\delta^{-m}(y) = \delta^m(x).\delta^m(y),$$

where the second equality is obtained by replacing $\mathbb{P}$ by $\mathbb{P}^T$. Then, comparing with the definition of $\rho(x,y,m)$, we get $\delta^m(y) = \delta^{-m}(y)$, which yields in turn $\delta^s(y) = y$, whence $\delta^s = I$.

• If s is odd, say $s = 2m+1$, the argument works in exactly the same way. In this case

$$\rho(x,y,m) = \delta^{-(m+1)}(x).\delta^{-m}(y) = \delta^m(x).\delta^{m+1}(y),$$

(by exchanging again $\mathbb{P}$ with $\mathbb{P}^T$), which implies $\delta^{m+1}(y) = \delta^{-m}(y)$, that is $\delta^s(y) = y$, QED. ■

Corollary 4.

1. *If there exists an integer s such that $\delta^s(x) = r(x)$, where $r(x)$ represents a rational fraction of x, then $\delta^{2s}(x) = x$ and the group is of order $4s$.*

2. *If there exists an integer s such that $\delta^s(x) = t(y)$, where $t(y)$ represents a rational fraction of y, then $\delta^{2s-1}(x) = x$ and the group is of order $4s-2$.*

In both cases, s stands for the smallest integer with the corresponding property.

Proof. Note first the identities $\xi\delta^s\xi = \delta^{-s}$ and $\eta\delta^s\xi = \delta^{-s+1}$.

So, the following chain of equalities holds.

$$\delta^s(x) = r(x) \Longrightarrow \xi\delta^s\xi(x) = \delta^s(x) \Longleftrightarrow \delta^{-s}(x) = \delta^s(x) \Longleftrightarrow \delta^{2s}(x) = x.$$

Similarly

$$\delta^s(x) = t(y) \Longrightarrow \eta\delta^s\xi(x) = \delta^s(x) \Longleftrightarrow \delta^{-s+1}(x) = \delta^s(x) \Longleftrightarrow \delta^{2s-1}(x) = x.$$

$\Longrightarrow$ In both cases, the conclusion follows from Proposition 3, in which s is replaced respectively by $2s$ and $2s-1$. ∎

Lemma 5. *On the algebraic curve $\{Q(x,y)=0\}$, the following general relations hold:*

$$\left\{\begin{array}{rcl} \eta(x) & = & \dfrac{xv(y)-u(y)}{xw(y)-v(y)}, \\ \\ \xi(y) & = & \dfrac{y\widetilde{v}(x)-\widetilde{u}(x)}{y\widetilde{w}(x)-\widetilde{v}(x)}, \end{array}\right. \tag{13}$$

where u,v,w,h (resp. $\widetilde{v},\widetilde{v},\widetilde{w},\widetilde{h}$) are polynomials of degree ≤ 2. In particular, there exist affine solutions

$$(u(y),v(y),w(y))^T = \vec{A}\,y+\vec{B},\ (\widetilde{u}(x),\widetilde{v}(x),\widetilde{w}(x))^T = \vec{E}\,x+\vec{F}, \tag{14}$$

with column vectors

$$\vec{A}=(u_0,v_0,w_0)^T,\ \vec{B}=(u_1,v_1,w_1)^T,\ \vec{E}=(\widetilde{u}_0,\widetilde{v}_0,\widetilde{w}_0)^T,\ \vec{F}=(\widetilde{u}_1,\widetilde{v}_1,\widetilde{w}_1)^T.$$

$$
\begin{cases}
\vec{A} & = & (\alpha\vec{C}_2 + \beta\vec{C}_1) \times \vec{C}_3, \\
\vec{B} & = & \vec{C}_1 \times (\alpha\vec{C}_3 + \beta\vec{C}_2), \\
\vec{E} & = & (\widetilde{\alpha}\vec{D}_2 + \widetilde{\beta}\vec{D}_1) \times \vec{D}_3, \\
\vec{F} & = & \vec{D}_1 \times (\widetilde{\alpha}\vec{D}_3 + \widetilde{\beta}\vec{D}_2),
\end{cases} \tag{15}
$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$ *are arbitrary complex constants, and the operator "$\times$" stands for the cross vector product. In addition, when* $\mathbb{P}$ *is of rank* 3*, none of the vectors* $\vec{A}, \vec{B}, \vec{E}, \vec{F}$ *vanish. Choosing in* (15) $\alpha = \widetilde{\alpha} = 0, \beta = \widetilde{\beta} = 1$*, gives*

$$
\begin{cases}
u(y) = y\Delta_{13} - \Delta_{12}, & \widetilde{u}(x) = x\Delta_{31} - \Delta_{21}, \\
v(y) = y\Delta_{23} - \Delta_{22}, & \widetilde{v}(x) = x\Delta_{32} - \Delta_{22}, \\
w(y) = y\Delta_{33} - \Delta_{32}, & \widetilde{w}(x) = x\Delta_{33} - \Delta_{23},
\end{cases} \tag{16}
$$

where Δ_{ij} *denotes the cofactor of the* $(i, j)^{th}$ *entry of the matrix* $\mathbb{P}$ *given in* (11).

■

Lemma 6. *Let* γ *be an endomorphism defined on the algebraic surface* $\mathscr{K}$*, which is assumed to be invariant on the field* $\mathbb{C}(x)$ *of rational functions of* x*, and such that*

$$\gamma(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)}, \tag{17}$$

where e, f, g, h *are polynomials of degree* 1 *in* x*. (Note that this is always possible, as shown in Lemma 5.) Then, for* γ *to be an involution, the condition* $f(x) + h(x) \equiv 0$ *is necessary and sufficient.*

Remark. The result of Lemma 6 does not hold if polynomials e, f, h are not of degree 1. For instance, one can check directly that, if g or e are taken to be of degree 2, then any involution γ necessarily has the form

$$\gamma(y) = \frac{(h-b)y - c}{ay + h}.$$

Groups of Order 4

Proposition 7. *The group $\mathcal{H}$ is of order 4 if, and only if,*

$$\begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00}-1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0, \tag{18}$$

and this is the only case where the matrix $\mathbb{P}$ has rank 2.

Proof. The equality $\delta^2 = I$ can be rewritten as $\xi\eta = \eta\xi$, which by Proposition 3 is, for instance, equivalent to

$$\xi\eta(x) = \eta(x),$$

where we have used $\xi(x) = x$. So, $\eta(x)$ is left invariant by ξ, which implies

$$\eta(x) \in \mathbb{C}(x).$$

Finally, η is both an involution and a conformal automorphism on $\mathbb{C}(x)$. Consequently, η is a fractional linear transform of the type

$$\eta(x) = \frac{rx+s}{tx-r},$$

where all coefficients belong to $\mathbb{C}$. The following chain of equivalences hold.

$$\begin{aligned}
\eta(x) = \frac{rx+s}{tx-r} \quad &\Leftrightarrow \quad tx.\eta(x) = r(x+\eta(x)) + s \\
&\Leftrightarrow \quad 1,\ x+\eta(x),\ x.\eta(x) \text{ are linearly dependent on } \mathbb{C} \\
&\Leftrightarrow \quad 1,\ -\frac{\widetilde{b}(y)}{\widetilde{a}(y)},\ \frac{\widetilde{c}(y)}{\widetilde{a}(y)} \text{ are linearly dependent on } \mathbb{C} \\
&\Leftrightarrow \quad \widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y) \text{ are also linearly dependent on } \mathbb{C},
\end{aligned}$$

where equation (2) has been used in the form

$$Q(x,y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y).$$

■

Groups of Order 6

Proposition 8. *$\mathcal{H}$ is of order 6 if, and only if,*

$$\begin{vmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{vmatrix} = 0, \tag{19}$$

where the Δ_{ij}'s have be given in Lemma 5.

Sketch of proof. In this case $(\xi\eta)^3 = I$, which is equivalent to

$$\eta\xi\eta = \xi\eta\xi. \tag{20}$$

Applying (20) for instance to x, we get

$$\xi\eta(x) = \eta\xi\eta(x),$$

which shows that $\xi\eta(x)$ is invariant with respect to η.

Similarly, $\eta\xi(y)$ is invariant with respect to ξ. Hence (20) is plainly equivalent to

$$\begin{cases} \xi\eta(x) = P(y), \\ \eta\xi(y) = R(x), \end{cases}$$

where P and R are rational. Then

$$y = R(\xi\eta(x)) = R \circ P(y),$$

or, equivalently,

$$R \circ P = I, \tag{21}$$

so that P and R are fractional linear transforms.

Thus (21) yields the relation

$$\xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s}. \tag{22}$$

Hence, there is a linear dependence on $\mathbb{C}$ between the 4 elements 1, $\xi(y)$, $\eta(x)$, $\xi(y)\eta(x)$, with 4 unknown constants (in fact three by homogeneity). Starting from equation (13), we choose $\eta(x)$ by means of (16),

$$\eta(x) = \frac{y(x\Delta_{23} - \Delta_{13}) - x\Delta_{22} + \Delta_{12}}{y(x\Delta_{33} - \Delta_{23}) - x\Delta_{32} + \Delta_{22}}. \tag{23}$$

Instantiating now (23) in (22), we obtain

$$\xi(y) = \frac{y[p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23})] + p(\Delta_{12} - x\Delta_{22}) + q(\Delta_{22} - x\Delta_{32})}{y[r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23})] + r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32})}. \tag{24}$$

Then, according to (17),

$$\xi(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)},$$

where

$$\left\{\begin{array}{lcl} e(x) & = & p(x\Delta_{22} - \Delta_{12}) + q(x\Delta_{32} - \Delta_{22}), \\ f(x) & = & p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23}), \\ g(x) & = & r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23}), \\ h(x) & = & r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32}), \end{array}\right. \tag{25}$$

and we can compare system (25) with the solution presented in equation (15) of Lemma 5.

The final step is to analyze the feasibility of a global linear system formed of 8 equations with 6 unknown variables. . . ■

Criterion for Groups of Order $4m$

Proposition 9. *The group $\mathcal{H}$ is of order $4m$ if and only if the Weierstrass $\wp$ function with periods (ω_1, ω_2) satisfies the equation*

$$\wp(m\omega_3) = \wp(\omega_2/2). \tag{26}$$

Proof. Recalling that $\delta = \eta\xi$, we have here $\delta^{2m} = I$, that is

$$(\xi\eta)^m = (\eta\xi)^m. \tag{27}$$

By applying equation (27) at x (or even at an arbitrary element of $\mathbb{C}(x)$), and replacing $\xi(x)$ by x in the right-hand side, we obtain

$$\xi\delta^m(x) = \delta^m(x),$$

showing that the involution $\delta^m(x)$ is invariant with respect to ξ.

Hence $\delta^m(x)$ is an element of $\mathbb{C}(x)$, so that

$$\delta^m(x) = F(x) = \frac{xf - e}{xg - f}, \tag{28}$$

where $F(x)$ is a simple fractional linear transform, with constants e, f, g, to be determined. Hence, equation (28) implies the existence of a linear dependence between the functions

$$x.\delta^m(x),\ x + \delta^m(x),\ \mathbf{1}. \tag{29}$$

Lemma 10. *For the group to be of order $4m$, a necessary and sufficient condition is that the three functions*

$$x(\omega - m\omega_3/2).x(\omega + m\omega_3/2),\ x(\omega - m\omega_3/2) + x(\omega + m\omega_3/2),\ \mathbf{1}, \tag{30}$$

be linearly dependent, $\forall \omega \in \mathbb{C}$. ■

Recall that $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, and let, for arbitrary u, v,

$$A(u,v) \stackrel{\text{def}}{=} \wp(u+v) + \wp(u-v), \quad B(u,v) \stackrel{\text{def}}{=} \wp(u+v)\wp(u-v)$$

Then, setting for now $X \stackrel{\text{def}}{=} \wp(u)$, $Y \stackrel{\text{def}}{=} \wp(v)$, we have

Lemma 11.

$$A(u,v) = \frac{(X+Y)(4XY-g_2)-2g_3}{2(X-Y)^2}, \tag{31}$$

$$B(u,v) = \frac{(XY)^2 + \frac{g_2}{2}XY + g_3(X+Y) + \frac{g_2^2}{16}}{(X-Y)^2}. \tag{32}$$

Let

$$S(u,v) \stackrel{\text{def}}{=} x(u+v) + x(u-v), \quad P(u,v) \stackrel{\text{def}}{=} x(u+v)x(u-v). \tag{33}$$

Since

$$x(\omega) = p + \frac{q}{\wp(\omega) - r}, \tag{34}$$

where p, q, r are known constants [see equation (6)], we have

$$\begin{cases} S = \dfrac{2pB + (q-2pr)A + 2r(pr-q)}{B - rA + r^2}, \\ P = \dfrac{p^2 B + p(q-pr)A + (pr-q)^2}{B - rA + r^2}. \end{cases} \tag{35}$$

Taking $u = \omega, v = m\omega_3/2$, the claims involving (29) and (30) are merely equivalent to the existence of a non-trivial linear relation

$$eS + fP + g = 0, \quad \forall X \in \mathbb{C}. \tag{36}$$

In other words $S, P, \mathbf{1}$, considered as functions of X, are linearly dependent.

Here $Y = \wp(m\omega_3/2)$, and the independence condition reads

$$\Delta(Y) \stackrel{\text{def}}{=} \begin{vmatrix} 4Y & 4Y^2 - g_2 & -(g_2Y + 2g_3) \\ 2Y^2 & g_2Y + 2g_3 & 2g_3Y + g_2^2/8 \\ 1 & -2Y & Y^2 \end{vmatrix} = 0, \qquad (37)$$

which yields exactly (26), by using the factorization of $\Delta(Y)$ as the product of 3 polynomials of degree 2 in Y.

However, the computation of $\wp(m\omega_3/2)$, via the recursive relationship

$$\wp((l+1)\omega_3/2) + \wp((l-1)\omega_3/2) = \frac{(\wp(l\omega_3/2) + \wp(\omega_3/2))(4\wp(l\omega_3/2)\wp(\omega_3/2) - g_2) - 2g_3}{2(\wp(l\omega_3/2) - \wp(\omega_3/2))^2},$$

is hardly exploitable. . .

• **Case $m = 2k$**. Applying the operator δ^{-k} in (29) amounts to saying that

$$\delta^k(x).\xi\delta^k(x),\ \ \delta^k(x) + \xi\delta^k(x),\ \ \mathbf{1}, \tag{38}$$

are linearly dependent. But $\delta^k(x).\xi\delta^k(x)$ and $\delta^k(x) + \xi\delta^k(x)$ are elements of $\mathbb{C}(x)$, and by (34), (31), (32), they are in fact ratios of polynomials of degree 2 in x with the same denominator.

In addition, letting $\zeta_j(x) \stackrel{\text{def}}{=} \delta^j(x) + \xi\delta^j(x)$, the following recursive scheme holds.

$$\begin{cases} \zeta_0(x) & = 2x,\ \ \zeta_1(x) = \delta^{-1}(x) + \delta(x), \\ \zeta_j(x) & = \zeta_{j-1}(\zeta_1(x)) - \zeta_{j-2}(x),\ \ \forall j \geq 2. \end{cases} \tag{39}$$

• **Case $m = 2k - 1$**. Upon applying here the operator δ^{-k+1} in (29) and using the identity $\delta^{-k+1}(x) = \eta\delta^k(x)$, we obtain that

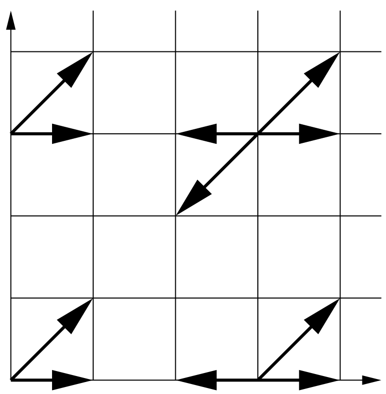
$$\delta^k(x).\eta\delta^k(x),\ \ \delta^k(x) + \eta\delta^k(x),\ \ \mathbf{1} \tag{40}$$

are linearly dependent. Moreover, $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are elements of $\mathbb{C}(y)$, namely ratios of polynomials of degree 2 in y with the same denominator.

Proposition 12. *The group $\mathcal{H}$ is of order 8 if, and only if, the third order determinant*

$$\begin{vmatrix} \begin{array}{r} 2\,\Delta_{22}\Delta_{32} \\ -(\Delta_{21}\Delta_{33}+\Delta_{31}\Delta_{23}) \end{array} & \begin{array}{r} 2\,(\Delta_{22}^2-\Delta_{12}\Delta_{31}+\Delta_{21}\Delta_{23}) \\ +\Delta_{11}\Delta_{33}+\Delta_{31}\Delta_{13} \end{array} & \begin{array}{r} 2\,\Delta_{12}\Delta_{22} \\ -(\Delta_{11}\Delta_{23}+\Delta_{21}\Delta_{13}) \end{array} \\ \Delta_{32}^2-\Delta_{31}\Delta_{33} & -2\,\Delta_{32}\Delta_{22}+\Delta_{31}\Delta_{23}+\Delta_{21}\Delta_{33} & \Delta_{22}^2-\Delta_{21}\Delta_{23} \\ \Delta_{22}^2-\Delta_{21}\Delta_{23} & -2\,\Delta_{22}\Delta_{12}+\Delta_{11}\Delta_{23}+\Delta_{13}\Delta_{21} & \Delta_{12}^2-\Delta_{11}\Delta_{13} \end{vmatrix}$$

is equal to zero, where Δ_{ij} denotes the cofactor of the $(i,j)^{th}$ entry of the matrix $\mathbb{P}$ given in (11). ■



Example: Gessel's walk.

Criterion for Groups of Order $4m - 2$

Here

$$\delta^{2m-1} = I, \tag{41}$$

which by Proposition 3 and Corollary 4 is equivalent to

$$\eta(\delta^m(x)) = \delta^m(x),$$

that is

$$\delta^m(x) = G(y) \in \mathbb{C}(y). \tag{42}$$

Similarly, upon applying (41) to y, we get

$$\delta^{-m}(y) = \delta^{m+1}(y) = \xi(\delta^{-m}(y),$$

whence

$$\delta^{-m}(y) = F(x) \in \mathbb{C}(x).$$

Applying now δ^{-m} to both members of (42) yields

$$x = \delta^{-m}(G(y)) = G(\delta^{-m}(y)) = G \circ F(x),$$

which shows that $G \circ F = I$, and hence G and F are simple fractional linear transforms.

Setting for instance

$$G(y) = -\frac{py+q}{ry+s},$$

where p, q, r, s are arbitrary complex constants, the problem is to achieve the linear relation

$$r\, y\delta^m(x) + s\, \delta^m(x) + py + q = 0 \mod Q(x,y), \tag{43}$$

which is necessary and sufficient for the group to be of order $4m - 2$.

Final results

For the group to be finite, there is a unique condition tantamount to the cancellation of a determinant, the elements of which are intricate functions of the coefficients of the transition matrix $\mathbb{P}$, but nonetheless recursively computable.

- The determinant is of order 3, for groups of order $4m, \; m \geq 1$.
- The determinant is of order 4, for groups of order $4m-2, \; m \geq 1$.
- The condition depends on the entries of the matrix $\mathbb{P}$ in a polynomial way, as shown in the next three theorems.

Theorem 13. *For any integer $s \geq 1$, we have*

$$\delta^s(x) = \frac{y\,U_s(x) + V_s(x)}{W_s(x)} \mod Q(x,y), \tag{44}$$

where U_s, V_s, W_s are second degree polynomials.

Theorem 14. *The finiteness of the group is always equivalent to the cancellation of a* single constant, *which depends on the entries of $\mathbb{P}$ in a polynomial way. In other words, the group is finite if and only if the non-negative (p_{ij})'s belong to the intersection of some algebraic hypersurface with the hyperplane $\sum p_{ij} = 1$.*

Theorem 15. *For the group $\mathcal{H}$ to be finite, the necessary and sufficient condition is $\det(\Omega) = 0$, where Ω is a matrix of order 3 (resp. 4) when the group is of order $4m$ (resp. 4m+2).*

Sketch of proofs.

Let

$$\delta^s(x) - \xi\delta^s(x) \overset{\text{def}}{=} 2H, \quad X = \wp(\omega),\ Y = \wp(s\omega_3).$$

Then

$$\delta^s(x) = \frac{S(\omega, s\omega_3)}{2} + H,$$

with $S(\omega, s\omega_3)$ given by (35).

$$H = \frac{q[\wp(s\omega_3 - \omega) - \wp(s\omega_3 + \omega)]}{2[\wp(s\omega_3 + \omega) - r][\wp(s\omega_3 - \omega) - r]} = \frac{q\,\wp'(\omega)\wp'(s\omega_3)}{D(X,Y)} = \frac{2q^2\wp'(s\omega_3)[2a(x)y + b(x)]}{(x-p)^2 D(X,Y)}$$

where, by using (31), (32), (35),

$$D(X,Y) = 2(X-Y)^2(B - rA + r^2)$$

is a polynomial of second degree in X and Y.

On the other hand, by construction, we can a priori write

$$\delta^s(x) = M_s(x)y + N_s(x) \mod Q(x,y), \tag{45}$$

where M_s and N_s are rational fractions whose numerators and denominators are polynomials of (a priori) unknown degrees, but with coefficients given in terms of polynomials of the entries of $\mathbb{P}$. The decomposition (45) is unique, so that, comparing with (44), we have

$$\begin{cases} M_s(x) = \dfrac{4q^2\wp'(s\omega_3)a(x)}{W_s(x)}, \\ N_s(x) = \dfrac{V_s(x)}{W_s(x)}, \end{cases}$$

where V_s, W_s are the second degree polynomials given by Theorem 13.

By homogeneity, we can always rewrite

$$M_s(x) = \frac{A_s a(x)}{F_s(x)},$$

where A_s, K_s are real constants with

$$A_s = K_s[4q^2\wp'(s\omega_3)], \quad F_s(x) = K_s W_s(x).$$

• By Corollary 4, the group is of order $4s$ if and only if $M_s \equiv 0$, that is $A_s = 0$, where now A_s *depends only on the entries of* $\mathbb{P}$ in a complicated polynomial form. It is also equivalent to $\wp'(s\omega_3) = 0$.

• When the group is of order $4s+2$, exchange the role of x and y by uniformizing $y(\omega)$. Then, *mutatis mutandis*, this yields

$$\delta^s(x) = \frac{\widetilde{A}_s \widetilde{a}(y)x + \widetilde{V}_s(y)}{\widetilde{F_s}(y)} \mod Q(x,y),$$

where $\widetilde{F}_s, \widetilde{V}_s$ are second degree polynomials. . . ■

As for combinatorics..?

Let $f(i,j,k)$ denote the number of paths starting from $(0,0)$ and ending at (i,j) at time k (or after k steps). Then the corresponding CGF

$$F(x,y,z) = \sum_{i,j,k\geq 0} f(i,j,k) x^i y^j z^k \tag{46}$$

satisfies the functional equation

$$K(x,y,z)F(x,y,z) = c(x)F(x,0,z) + \widetilde{c}(y)F(0,y,z) + c_0(x,y,z), \tag{47}$$

where

$$K(x,y;z) = xy\left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right].$$

Note that here the group depends on z. . .

About the genus 0 case

Here the Riemann surface $\mathscr{K} = \{(x,y) \in \mathbb{C}^2 : Q(x,y) = 0\}$ is of genus 0 (the Riemann Sphere) and admits a uniformization in terms of simple rational functions.

For all non-singular random walks, $\mathbf{S}$ has genus 0 if, and only if, one of the following relations holds:

$$M_x = M_y = 0, \tag{48}$$

$$p_{10} = p_{11} = p_{01} = 0, \tag{49}$$

$$p_{10} = p_{1,-1} = p_{0,-1} = 0, \tag{50}$$

$$p_{-1,0} = p_{-1,-1} = p_{0,-1} = 0, \tag{51}$$

$$p_{01} = p_{-1,0} = p_{-1,1} = 0. \tag{52}$$

Define the drift $\overrightarrow{\mathbf{M}} = (\sum i p_{ij}, \sum j p_{ij})$ and $\theta = \arccos(-r)$, where r denotes the *correlation coefficient*.

Theorem. *[Fayolle-Raschel, MPRF 2011]*

(a) When $\overrightarrow{\mathbf{M}} = 0$, the group $\mathcal{H}$ is finite if and only if θ/π is rational, in which case its order is equal to

$$2\inf\{\ell \in \mathbb{Z}_+^* : \ell\theta/\pi \in \mathbb{Z}\}.$$

(b) When $\overrightarrow{\mathbf{M}} \neq 0$, the order of $\mathcal{H}$ is always infinite in the four remaining cases ■

Sketch of proof of part (a). The main idea consists in working by continuity from the genus 1 case ! Now letting $\overrightarrow{\mathbf{M}} \to 0$, so that $x_2, x_3 \to 1$, we have

$$\begin{cases} \omega_1 \to i\infty, \\ \omega_2 \to \alpha_2 = \dfrac{\pi}{[C(x_4-1)(1-x_1)]^{1/2}}, \\ \omega_3 \to \alpha_3 = \displaystyle\int_{X_0(y_1)}^{x_1} \frac{\mathrm{d}x}{(1-x)[C(x-x_1)(x-x_4)]^{1/2}}. \\ \dfrac{\theta}{\pi} = \displaystyle\lim_{\overrightarrow{\mathbf{M}}\to 0} \frac{\omega_2}{\omega_3} = \frac{\alpha_2}{\alpha_3}. \end{cases} \tag{53}$$

Sketch of proof of part (b).

By symmetry, it suffices to consider the case $p_{10} = p_{1,-1} = p_{0,-1} = 0$. Then

$$\begin{cases} \omega_1 \to i\alpha_1, \text{ with } \alpha_1 \in (0,\infty), \\ \omega_2 \to \infty, \\ \omega_3 \to \alpha_3 \in (0,\infty). \end{cases}$$

Hence, the limit group can be interpreted as the group of transformations

$$\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \alpha_3 \rangle$$

on $\mathbb{C}/(\alpha_1\mathbb{Z})$. This group is obviously infinite, and so is $\mathcal{H}$.

Thank you for your attention!

But, what to do now ?

The trick will be to avoid the pitfalls, seize the opportunities, and get back home by six o'clock.

[Woody Allen, *My Speech to the Graduates*, Side Effects, 1980].