When are counting sequences Stieltjes moment sequences?

Andrew Elvey Price
Joint work with
Alin Bostan, Tony Guttmann and Jean-Marie Maillard

Université de Bordeaux et Université de Tours

20/01/2020

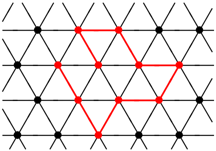
Counting sequences

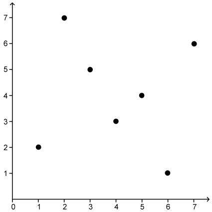
Main questions: *how many objects of size n are there?*

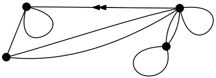
Possible objects:

- lattice paths (size = #steps)
- (pattern-avoiding) permutations (size = #elements)
- planar maps/graphs/trees (size = #edges or #vertices)
- Many more

Any such type of object defines a counting sequence $a_0, a_1, \ldots$, where a_n is the number of objects of size n.







Stieltjes moment sequences

Definition and Theorem: A sequence $a_0, a_1, a_2, \ldots$ is a *Stieltjes moment sequence* if the following (equivalent) conditions hold:

- There exists a positive measure ρ such that $a_n = \int_0^\infty x^n d\rho(x)$.

Stieltjes moment sequences

Definition and Theorem: A sequence $a_0, a_1, a_2, \ldots$ is a *Stieltjes moment sequence* if the following (equivalent) conditions hold:

- There exists a positive measure ρ such that $a_n = \int_0^\infty x^n d\rho(x)$.
- The matrices $\begin{bmatrix} a_0 & a_1 & \cdots \\ a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 & \cdots \\ a_2 & a_3 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ are positive semi-definite.

Stieltjes moment sequences

Definition and Theorem: A sequence $a_0, a_1, a_2, \ldots$ is a *Stieltjes moment sequence* if the following (equivalent) conditions hold:

- There exists a positive measure ρ such that $a_n = \int_0^\infty x^n d\rho(x)$.
- The matrices $\begin{bmatrix} a_0 & a_1 & \cdots \\ a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 & \cdots \\ a_2 & a_3 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ are positive semi-definite.
- There exist real numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ldots}}}.$$

Stieltjes moment sequences

Definition and Theorem: A sequence $a_0, a_1, a_2, \ldots$ is a *Stieltjes moment sequence* if the following (equivalent) conditions hold:

- There exists a positive measure ρ such that $a_n = \int_0^\infty x^n d\rho(x)$.
- The matrices $\begin{bmatrix} a_0 & a_1 & \cdots \\ a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 & \cdots \\ a_2 & a_3 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ are positive semi-definite.
- There exist real numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ldots}}}.$$

Question: Which counting sequences are Stieltjes moment sequences?

PROPERTIES OF STIELTJES MOMENT SEQUENCES

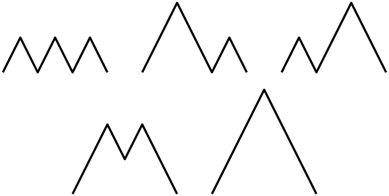
If $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence with generating function $A(t)$ then

- The ratios a_{n+1}/a_n are increasing.
- All singularities of $A(t)$ lie in $\mathbb{R}_{\geq 0}$.
- We can produce "good" lower bounds on the growth rate

$$\mu = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

EXAMPLE: CATALAN NUMBERS

The nth Catalan number c_n is the number of paths of n up steps and n down steps which are always at height ≥ 0 (Dyck paths).
The sequence starts $1, 1, 2, 5, 14, 42, \ldots$



$c_3 = 5$, as there are 5 Dyck paths of length 6.

EXAMPLE: CATALAN NUMBERS

The nth Catalan number c_n is the number of paths of n up steps and n down steps which are always at height ≥ 0 (Dyck paths).
The sequence starts $1, 1, 2, 5, 14, 42, \ldots$

Representation as moments:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \int_0^4 x^n \frac{\sqrt{4-x}}{2\pi\sqrt{x}} dx.$$

Generating function as continued fraction:

$$C(t) = \sum_{n=0}^{\infty} c_n t^n = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \ldots}}}.$$

TALK OUTLINE

- **Part 1:** Methods for proving the counting sequences are Stieltjes moments sequences
  - **Part 1a:** Combinatorial continued fractions
  - **Part 1b:** Spectral theorem for paths on graphs
  - **Part 1c:** Stieltjes inversion formula
- **Part 2:** Pattern avoiding permutations as Stieltjes moment sequences
  - **Part 2a:** Density function for 1342-avoiding permutations
  - **Part 2b:** Density function for 1234-avoiding permutations
  - **Part 2c:** Random matrices for $12\dots k$-avoiding permutations
  - **Part 2d:** Empirical analysis for 1324-avoiding permutations

Part 1: Proving that counting sequences are Stieltjes moments sequences

Part 1a: Combinatorial continued fractions

Combinatorial continued fractions

Recall $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if its generating function $A(t)$ can be written as

$$A(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \dots}}},$$

for non-negative reals α_n.

COMBINATORIAL CONTINUED FRACTIONS

Recall $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if its generating function $A(t)$ can be written as

$$A(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \dots}}},$$

for non-negative reals α_n.

Combinatorial interpretation (Flajolet, 1980): Weighted Dyck paths where each down step from height k gets weight α_k (setting $\alpha_k = 1$ yields the Catalan numbers).

Combinatorial continued fractions

Recall $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if its generating function $A(t)$ can be written as

$$A(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \dots}}},$$

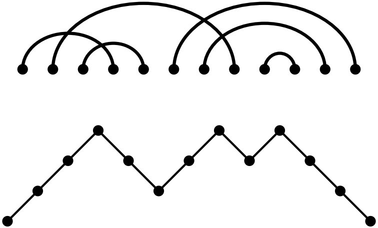
for non-negative reals α_n.

Combinatorial interpretation (Flajolet, 1980): Weighted Dyck paths where each down step from height k gets weight α_k (setting $\alpha_k = 1$ yields the Catalan numbers).

Bijections to:

- Set partitions
- Perfect matchings
- Permutations
- Phylogenetic trees

Example: Perfect matchings



- **From perfect matchings to weighted Dyck paths:** Replace each opener with an up step and each closer with a down step.
- Each down step from height k gets weight k (#arches that the closer could close).
- The continued fraction for perfect matchings has $\alpha_k = k$.

Therefore: The counting sequence for perfect matchings is Stieltjes.

Combinatorial continued fractions

More general continued fractions have also been used:

- "Jacobi-type" fractions for weighted Motzkin paths

$$A(t) = \cfrac{\delta_0}{1 - \gamma_1 t - \cfrac{\delta_1 t^2}{1 - \gamma_2 t - \cfrac{\delta_2 t^2}{1 - \ldots}}},$$

  (only Stieltjes for certain weights). (Sokal, Zeng, to appear)
- "Thron-type" fractions for weighted Schröder paths

$$A(t) = \cfrac{1}{1 - \beta_1 t - \cfrac{\alpha_1 t}{1 - \beta_2 t - \cfrac{\alpha_2 t}{1 - \ldots}}}.$$

  (Stieltjes if all $\alpha_k, \beta_k \geq 0$). (E.P., Sokal, 2020)
- "Branched continued fractions" for paths with steps $+1$ and $-k$ (Stieltjes if all weights ≥ 0). (Pétréolle, Sokal, Zhu, 2018)

Part 1b: Spectral theorem for paths on graphs

Paths on graphs

Theorem: (E.P., Guttmann 2019) On any fixed (locally-finite) undirected graph: the counting sequence of even-length excursions forms a Stieltjes moment sequence.

Paths on graphs

Theorem: (E.P., Guttmann 2019) On any fixed (locally-finite) undirected graph: the counting sequence of even-length excursions forms a Stieltjes moment sequence.

Example: On the graph below, even length excursions from the red point are counted by Catalan numbers

Paths on graphs

Theorem: (E.P., Guttmann 2019) On any fixed (locally-finite) undirected graph: the counting sequence of even-length excursions forms a Stieltjes moment sequence.
Example: On the graph below, even length excursions from the red point are counted by Catalan numbers

Proof of theorem: (Sokal)

- For finite graphs: use the adjacency matrix to count excursions $\rightarrow$ result follows from spectral theorem.
- For infinite graphs: Use sequence of finite graphs

Paths on graphs

Theorem: (E.P., Guttmann 2019) On any fixed (locally-finite) undirected graph: the counting sequence of even-length excursions forms a Stieltjes moment sequence.

Example: On the graph below, even length excursions from the red point are counted by Catalan numbers

Proof of theorem: (Sokal)

- For finite graphs: use the adjacency matrix to count excursions $\rightarrow$ result follows from spectral theorem.
- For infinite graphs: Use sequence of finite graphs

Examples:

- Walks on Cayley graphs (i.e., *cogrowth* sequences of groups).
- Lattice excursions confined to the quarter plane.
- Excursions on higher dimensional lattices.

Part 1c: Stieltjes inversion formula

Stieltjes inversion formula

Assume $a_0, a_1, \ldots$ is a Stieltjes moment sequence with

$$a_n = \int_0^\tau x^n \mu(x) dx.$$

The generating function $A(t)$ satisfies (at $z \sim \infty$)

$$\frac{1}{z} A\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n-1} = \int_0^\tau \frac{1}{z-x} \mu(x) dx =: F(z).$$

Stieltjes inversion formula

Assume $a_0, a_1, \ldots$ is a Stieltjes moment sequence with

$$a_n = \int_0^\tau x^n \mu(x) dx.$$

The generating function $A(t)$ satisfies (at $z \sim \infty$)

$$\frac{1}{z} A\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n-1} = \int_0^\tau \frac{1}{z-x} \mu(x) dx =: F(z).$$

The RHS $F(z)$ is the analytic continuation of $\frac{1}{z} A\left(\frac{1}{z}\right)$ to $\mathbb{C} \setminus [0, \tau]$.

Stieltjes inversion formula

Assume $a_0, a_1, \dots$ is a Stieltjes moment sequence with

$$a_n = \int_0^\tau x^n \mu(x) dx.$$

The generating function $A(t)$ satisfies (at $z \sim \infty$)

$$\frac{1}{z} A\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n-1} = \int_0^\tau \frac{1}{z-x} \mu(x) dx =: F(z).$$

The RHS $F(z)$ is the analytic continuation of $\frac{1}{z} A\left(\frac{1}{z}\right)$ to $\mathbb{C} \setminus [0, \tau]$.

Stieltjes inversion formula:

$$\mu(x) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left(F(x + \epsilon i) - F(x - \epsilon i) \right).$$

Nature of density function

From the Stieltjes inversion formula, equations for the generating function $A(t)$ transform into equations for the density function $\mu(x)$:

Theorem: Assume that $(a_n)_{n\geq 0}$ is the moment sequence of the density μ and has generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$.

If $A(t)$ is (piecewise)

- *algebraic* (i.e., there exists a nonzero polynomial $P(x, y) \in \mathbb{R}[x, y]$ such that $P(t, A(t)) = 0$),
- *D-finite* (Batenkov, 2009) (i.e., it is the solution of a linear ODE with polynomial coefficients) or
- *D-algebraic* (i.e., it is the solution of a nonlinear ODE with polynomial coefficients)

then the same is true for μ.

Nature of density function

From the Stieltjes inversion formula, equations for the generating function $A(t)$ transform into equations for the density function $\mu(x)$:

Theorem: Assume that $(a_n)_{n\geq 0}$ is the moment sequence of the density μ and has generating function $A(t)=\sum_{n=0}^{\infty} a_n t^n$.

If $A(t)$ is (piecewise)

- *algebraic* (i.e., there exists a nonzero polynomial $P(x,y)\in\mathbb{R}[x,y]$ such that $P(t,A(t))=0$),
- *D-finite* (Batenkov, 2009) (i.e., it is the solution of a linear ODE with polynomial coefficients) or
- *D-algebraic* (i.e., it is the solution of a nonlinear ODE with polynomial coefficients)

then the same is true for μ.

For algebraicity the converse is false: eg. if $a_n=\frac{1}{n+1}$ then $\mu(x)=1$ for $x\in[0,1]$ but $A(t)=-t^{-1}\log(1-t)$.

For D-finiteness the converse is true (Bréhard, Joldes, Lasserre, 2019).

Example: Catalan numbers

Generating function

$$C(t) = \frac{1}{1 - tC(t)} = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

Then

$$F(z) := \int_0^4 \frac{1}{z - x}\mu(x)dx = \frac{1}{z}C\left(\frac{1}{z}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{z - 4}{z}}\right).$$

By the Stieltjes inversion formula, the density function is

$$\mu(x) = -\frac{1}{2\pi i}\lim_{\epsilon \to 0^+}\left(F(x + \epsilon i) - F(x - \epsilon i)\right) = \frac{1}{2\pi}\sqrt{\frac{4 - x}{x}}.$$

Example: DE for Catalan numbers

Generating function $C(t)$ satisfies

$$(1-2t)C(t)+(t-4t^2)C'(t)=1.$$

Then $F(z)=\frac{1}{z}C\left(\frac{1}{z}\right)$ satisfies

$$2F(z)-z(z-4)F'(z)=1.$$

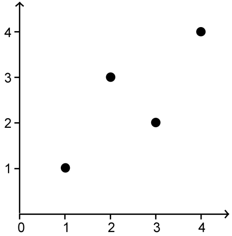
The density function $\mu(x)=-\frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\left(F(x+\epsilon i)-F(x-\epsilon i)\right)$ satisfies

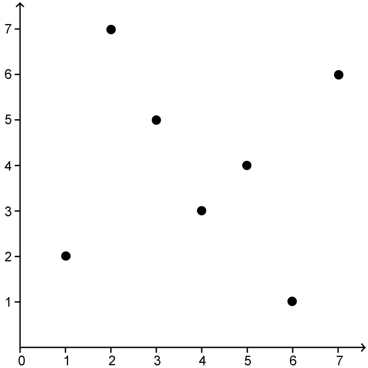
$$2\mu(x)-x(x-4)\mu'(x)=0.$$

In general: If $A(t)$ is D-finite, the density $\mu(x)$ satisfies the homogeneous part of the differential equation for $\frac{1}{x}A\left(\frac{1}{x}\right)$.

Part 2: Pattern avoiding permutations as Stietjes moment sequences

Pattern avoiding permutations

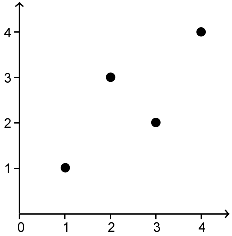


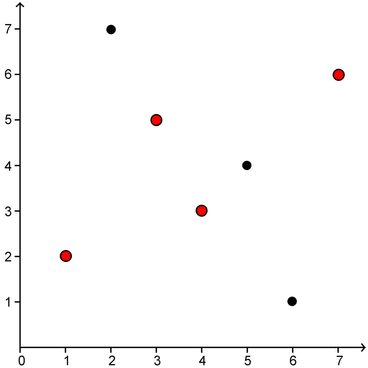


Left: 1324. **Right:** 2753416.

Pattern avoiding permutations

The permutation 2753416 *contains* the pattern 1324.



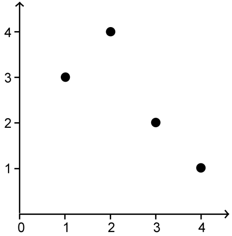


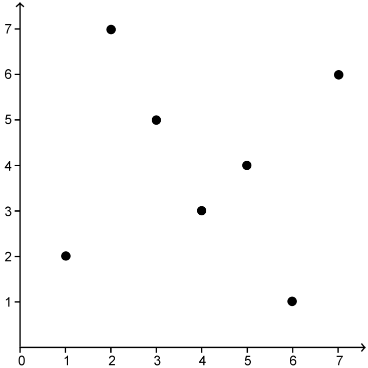
Left: 1324.

Right: 2753416.

PATTERN AVOIDING PERMUTATIONS

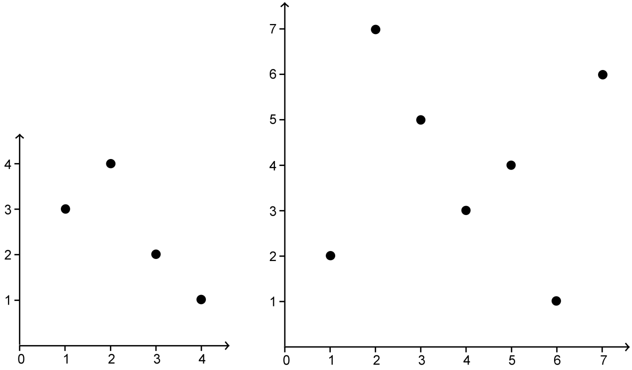
The permutation 2753416 *contains* the pattern 1324.
The permutation 2753416 *avoids* the pattern 3421.





Left: 3421. **Right:** 2753416.

The permutation 2753416 *contains* the pattern 1324.
The permutation 2753416 *avoids* the pattern 3421.
Problem: Count (by size) permutations avoiding a given pattern τ



Left: 3421. **Right:** 2753416.

Pattern avoiding permutations

The permutation 2753416 *contains* the pattern 1324.
The permutation 2753416 *avoids* the pattern 3421.
Problem: Count (by size) permutations avoiding a given pattern τ

- For each n, the number of permutations of length n avoiding π is denoted $Av_n(\pi)$.
- Two patterns π and τ are said to be "Wilf equivalent" if $Av_n(\pi) = Av_n(\tau)$ for every $n \in \mathbb{Z}_{\geq 0}$.

LENGTH 3 PATTERNS

Length 3 patterns

- For length 3 patterns, there would seem to be two Wilf classes: 132, 213, 231 and 312 are all equivalent to each other due to reflective symmetry, and similarly 123 and 321 are equivalent.

Length 3 patterns

- For length 3 patterns, there would seem to be two Wilf classes: 132, 213, 231 and 312 are all equivalent to each other due to reflective symmetry, and similarly 123 and 321 are equivalent.
- Remarkably, these are all Wilf equivalent (MacMahon 1915, Knuth 1968) as

$$Av_n(123) = Av_n(132) = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number.

LENGTH 4 PATTERNS

LENGTH 4 PATTERNS

- For length 4 patterns, there are only three Wilf classes, which have representatives 1342, 1234 and 1324.

Length 4 patterns

- For length 4 patterns, there are only three Wilf classes, which have representatives 1342, 1234 and 1324.
- (Bóna, 1997) The sequence $Av_n(1342)$ is given by the coefficients of the algebraic generating function

$$\frac{32t}{1+12t-8t^2-(1-8t)^{3/2}}.$$

LENGTH 4 PATTERNS

- For length 4 patterns, there are only three Wilf classes, which have representatives 1342, 1234 and 1324.
- (Bóna, 1997) The sequence $Av_n(1342)$ is given by the coefficients of the algebraic generating function

$$\frac{32t}{1+12t-8t^2-(1-8t)^{3/2}}.$$

- (Gessel, 1990) The sequence $Av_n(1234)$ is not algebraic, but has the D-finite generating function

$$\frac{1+5t}{6t^2}-\frac{(1-t)^{\frac{1}{4}}(1-9t)^{\frac{3}{4}}}{6t^2}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64t}{(1-t)(1-9t)^3}\right).$$

LENGTH 4 PATTERNS

- For length 4 patterns, there are only three Wilf classes, which have representatives 1342, 1234 and 1324.
- (Bóna, 1997) The sequence $Av_n(1342)$ is given by the coefficients of the algebraic generating function

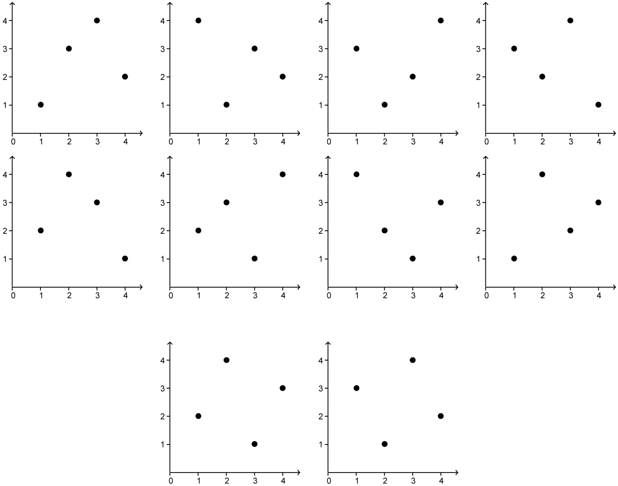
$$\frac{32t}{1+12t-8t^2-(1-8t)^{3/2}}.$$

- (Gessel, 1990) The sequence $Av_n(1234)$ is not algebraic, but has the D-finite generating function

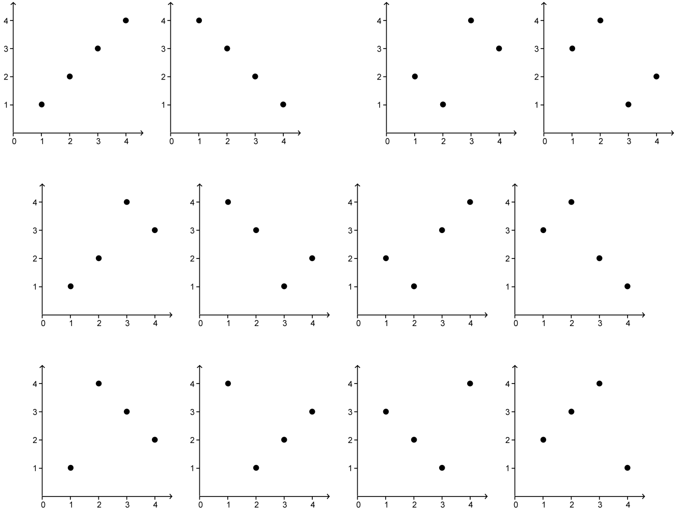
$$\frac{1+5t}{6t^2}-\frac{(1-t)^{\frac{1}{4}}(1-9t)^{\frac{3}{4}}}{6t^2}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64t}{(1-t)(1-9t)^3}\right).$$

- The sequence $Av_n(1324)$ is a mystery.

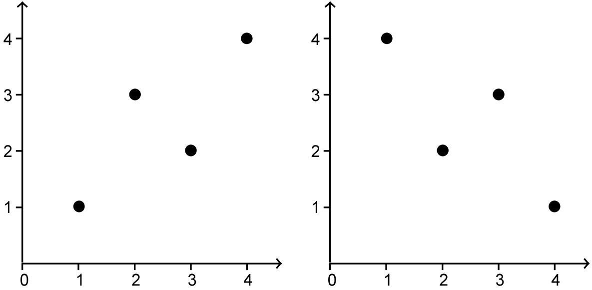
Length 4 patterns: Algebraic cases



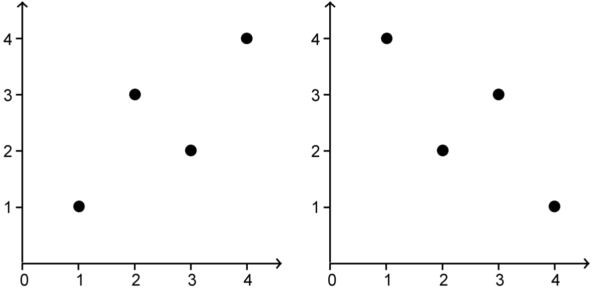
LENGTH 4 PATTERNS: D-FINITE CASES



Length 4 patterns: unsolved cases



LENGTH 4 PATTERNS: UNSOLVED CASES



- Conway, Guttmann and Zinn-Justin (2018) computed $Av_n(1324)$ for $n \leq 50$. Their analysis suggests the sequence is not D-finite.
- Bevan, Brignall, E.P. and Pantone (2018) showed that the exponential growth rate μ_{1324} lies in the interval $(10.37, 13.5)$.

Increasing patterns

The only other solved cases are $Av_n(12\dots k)$

- (Gessel 1990): For any fixed k, the generating function $\sum_{n=0}^{\infty} Av_n(12\dots k)t^n$ is D-finite
- (Bergeron, Gascon 2000): computed explicit differential equations for $k \leq 11$.
- (Rains 1998): Connection to random matrices

Pattern avoiding permutations

Theorem: For any pattern τ of length at most 4 except (possibly) 1324 and 4231, the sequence $Av_n(\tau)$ forms a Stieltjes moment sequence.

Pattern avoiding permutations

Theorem: For any pattern τ of length at most 4 except (possibly) 1324 and 4231, the sequence $Av_n(\tau)$ forms a Stieltjes moment sequence.

Conjecture: The sequence $Av_n(1324)$ forms a Stieltjes moment sequence.

Theorem: For any pattern τ of length at most 4 except (possibly) 1324 and 4231, the sequence $Av_n(\tau)$ forms a Stieltjes moment sequence.

Conjecture: The sequence $Av_n(1324)$ forms a Stieltjes moment sequence.

Conjollary: The growth rate $\mu_{1324} \geq 10.607$.

Pattern avoiding permutations

Theorem: For any pattern τ of length at most 4 except (possibly) 1324 and 4231, the sequence $Av_n(\tau)$ forms a Stieltjes moment sequence.

Conjecture: The sequence $Av_n(1324)$ forms a Stieltjes moment sequence.

Conjollary: The growth rate $\mu_{1324} \geq 10.607$.

Theorem: For any k, the sequence $Av_n(12 \ldots k)$ forms a Stieltjes moment sequence.

Pattern avoiding permutations

Theorem: For any pattern τ of length at most 4 except (possibly) 1324 and 4231, the sequence $Av_n(\tau)$ forms a Stieltjes moment sequence.

Conjecture: The sequence $Av_n(1324)$ forms a Stieltjes moment sequence.

Conjollary: The growth rate $\mu_{1324} \geq 10.607$.

Theorem: For any k, the sequence $Av_n(12\ldots k)$ forms a Stieltjes moment sequence.

Question: For any permutation pattern τ, does $Av_n(\tau)$ form a Stieltjes moment sequence?

Part 2a: Density function for 1342-avoiding permutations

Density for 1342-avoiding permutations

(Bóna, 1997): The generating function

$$A(t) := \sum_{n=0}^{\infty} Av_n(1342)t^n = \frac{1 + 20t - 8t^2 + (1-8t)^{3/2}}{2(1+t)^3}$$

Then the transformation

$$F(z) := \int_0^8 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right) = \frac{z^2 + 20z - 8 + (z-8)^{3/2}\sqrt{z}}{2(z+1)^3}.$$

By the Stieltjes inversion formula, the density function is

$$\mu(x) = -\frac{1}{2\pi i}\lim_{\epsilon \to 0^+}\left(F(x+\epsilon i) - F(x-\epsilon i)\right) = \frac{(8-x)^{3/2}\sqrt{x}}{2\pi(1+x)^3}.$$

Density for 1342-avoiding permutations

(Bóna, 1997): The generating function

$$A(t) := \sum_{n=0}^{\infty} Av_n(1342)t^n = \frac{1 + 20t - 8t^2 + (1-8t)^{3/2}}{2(1+t)^3}$$

Then the transformation

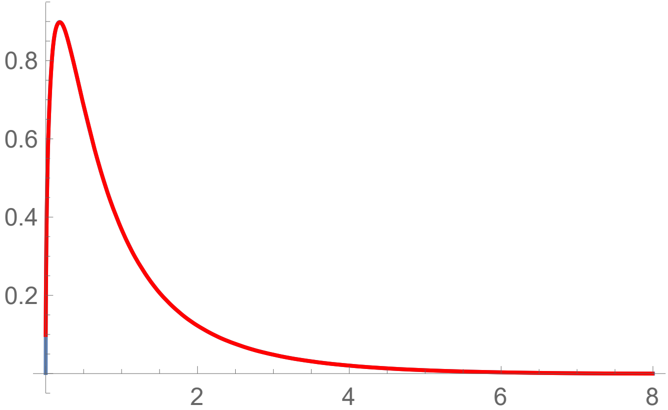
$$F(z) := \int_0^8 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right) = \frac{z^2 + 20z - 8 + (z-8)^{3/2}\sqrt{z}}{2(z+1)^3}.$$

By the Stieltjes inversion formula, the density function is

$$\mu(x) = -\frac{1}{2\pi i}\lim_{\epsilon \to 0^+}\left(F(x+\epsilon i) - F(x-\epsilon i)\right) = \frac{(8-x)^{3/2}\sqrt{x}}{2\pi(1+x)^3}.$$

Non-negative for $x \in [0,8]$! Therefore $(Av_n(1342))_{n\geq 0}$ is a Stieltjes moment sequence.

Density for 1342-avoiding permutations



Density function for $Av_n(1324)$.

Part 2b: density function for 1234-avoiding permutations

DENSITY FOR 1234-AVOIDING PERMUTATIONS

The generating function $A(t) := \sum_{n=0}^{\infty} Av_n(1234)t^n$ is given by

$$A(t) = \frac{1+5t}{6t^2} - \frac{(1-t)^{\frac{1}{4}}(1-9t)^{\frac{3}{4}}}{6t^2}\, {}_2F_1\left(\left[-\frac{1}{4}, \frac{3}{4}\right], [1], \frac{-64t}{(1-t)(1-9t)^3}\right).$$

Density for 1234-avoiding permutations

The transformed generating function

$F(z) := \int_0^9 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right)$ is given by

$$F(z) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64z^3}{(z-1)(z-9)^3}\right).$$

By the Stieltjes inversion formula, the density function is

$\mu(x) = -\frac{1}{\pi}\Im(F(z)).$

DENSITY FOR 1234-AVOIDING PERMUTATIONS

The transformed generating function
$F(z) := \int_0^9 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right)$ is given by

$$F(z) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64z^3}{(z-1)(z-9)^3}\right).$$

By the Stieltjes inversion formula, the density function is
$\mu(x) = -\frac{1}{\pi}\Im(F(z)).$
This is wrong!! Problem: $F(z)$ is not analytic on $\mathbb{C}\setminus\mathbb{R}$.

Density for 1234-avoiding permutations

The transformed generating function

$F(z) := \int_0^9 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right)$ is given by

$$F(z) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64z^3}{(z-1)(z-9)^3}\right).$$

By the Stieltjes inversion formula, the density function is

$\mu(x) = -\frac{1}{\pi}\Im(F(z))$.

This is wrong!! Problem: $F(z)$ is not analytic on $\mathbb{C}\setminus\mathbb{R}$.

Correct Formula:

$$\mu(x) = -\frac{3}{\pi}\Im(F(z)) \qquad \text{for } x\in[1,9],$$

$$\mu(x) = \frac{3}{\pi}\Re(F(z)) \qquad \text{for } x\in[0,1].$$

Density for 1234-avoiding permutations

Problem: $F(z)$ is not analytic on $\mathbb{C} \setminus \mathbb{R}$.

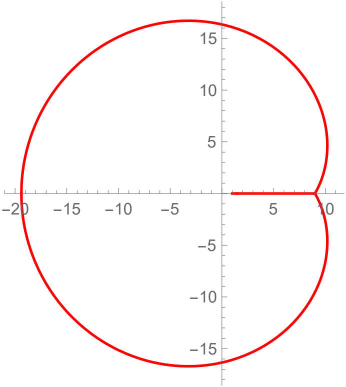


Diagram of non-analytic points of $F(z)$. So $F(z)$ is only the "correct" function in the outer region.

Density for 1234-avoiding permutations

The transformed generating function

$F(z) := \int_0^9 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right)$ is given by

$$F(z) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64z^3}{(z-1)(z-9)^3}\right).$$

By the Stieltjes inversion formula, the density function is

$\mu(x) = -\frac{1}{\pi}\Im(F(z)).$

This is wrong!! Problem: $F(z)$ is not analytic on $\mathbb{C}\setminus\mathbb{R}$.

Correct Formula:

$$\mu(x) = -\frac{3}{\pi}\Im(F(z)) \qquad \text{for } x \in [1,9],$$

$$\mu(x) = \frac{3}{\pi}\Re(F(z)) \qquad \text{for } x \in [0,1].$$

Density for 1234-avoiding permutations

The transformed generating function

$F(z) := \int_0^9 \frac{1}{z-x}\mu(x)dx = \frac{1}{z}A\left(\frac{1}{z}\right)$ is given by

$$F(z) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4},\frac{3}{4}\right],[1],\frac{-64z^3}{(z-1)(z-9)^3}\right).$$

By the Stieltjes inversion formula, the density function is

$\mu(x) = -\frac{1}{\pi}\Im(F(z))$.

This is wrong!! Problem: $F(z)$ is not analytic on $\mathbb{C}\setminus\mathbb{R}$.

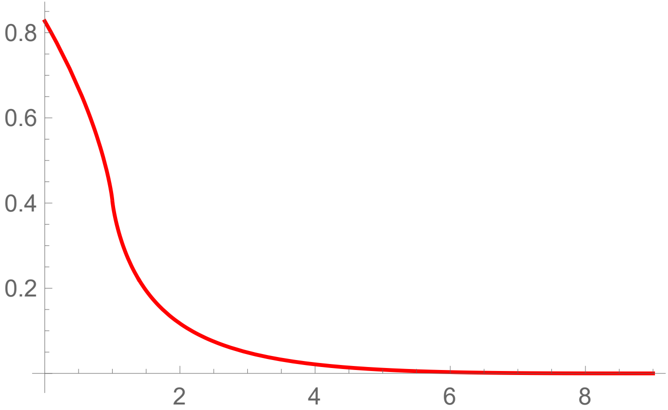
Correct Formula:

$$\mu(x) = -\frac{3}{\pi}\Im(F(z)) \qquad \text{for } x \in [1,9],$$

$$\mu(x) = \frac{3}{\pi}\Re(F(z)) \qquad \text{for } x \in [0,1].$$

Very interesting modular properties behind the factor of 3.

Density for 1234-avoiding permutations



Density function for $Av_n(1234)$.

Part 2c: Random matrices for $12\ldots k$-avoiding permutations

RAINS' RANDOM MATRIX RESULT

Rains (1998) showed the following theorem:

Theorem: For X a random variable defined as the norm squared of the trace of a Haar-random $(k-1) \times (k-1)$ unitary matrix, $E(X^n) = Av_n(12 \cdots k)$.

Rains' random matrix result

Rains (1998) showed the following theorem:
Theorem: For X a random variable defined as the norm squared of the trace of a Haar-random $(k-1)\times(k-1)$ unitary matrix, $E(X^n) = Av_n(12\cdots k)$. i.e., $A_n(12\ldots(k+1))$ is equal to

$$\frac{1}{(2\pi)^k \cdot k!}\cdot\int_{[0,2\,\pi]^k} |e^{i\theta_1}+\cdots+e^{i\theta_k}|^{2n}\cdot\prod_{1\le p<q\le k}|e^{i\theta_p}-e^{i\theta_q}|^2\,d\theta_1\ldots d\theta_k.$$

Consequences:

- $Av_n(12\cdots k)$ is a Stieltjes moment sequence.
- The density function of X is piecewise D-finite.
- Our exact results for density functions for $Av_n(12\cdots k)$ also describe the density of X.

Part 2d: Empirical analysis for 1324-avoiding permutations

Continued fraction coefficients

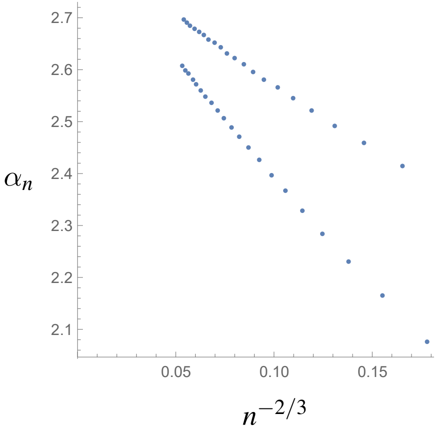
Using the exact numbers $Av_n(1324)$ for $n \leq 50$, we computed the first 50 continued fraction coefficients α_n

$$\sum_{n=0}^{\infty} Av_n(1324)t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \dots}}}.$$

They are all positive $\Rightarrow$ the sequence might be Stieltjes.

Lower bounds assuming Stieltjes: Set $\alpha_k = 0$ for $k > 50$. This yields the lower bound 10.6 for the growth rate. (Method first used by Haagerup, Haagerup and Ramirez-Solano 2015).

CONTINUED FRACTION COEFFICIENTS



Plot of α_n vs. $n^{-2/3}$ for the sequence $a_n = |\operatorname{Av}_n(1324)|$, using $n \in [10, 50]$.

Estimating the density function

We estimate the density function by a polynomial $P(x)$ of degree 52 satisfying:

- $Av_n(1324) = \int_0^{\tau} x^n P(x)dx$ for $x \leq 50$
- $P(\tau) = P'(\tau) = 0$,

where we set $\tau = 12$.

ESTIMATING THE DENSITY FUNCTION

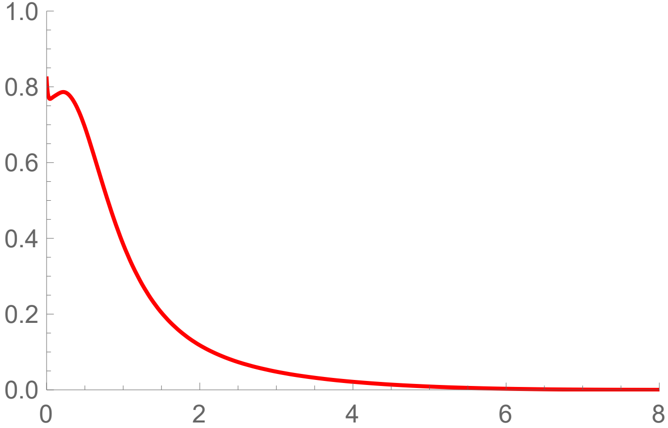
We estimate the density function by a polynomial $P(x)$ of degree 52 satisfying:

- $Av_n(1324) = \int_0^\tau x^n P(x) dx$ for $x \leq 50$
- $P(\tau) = P'(\tau) = 0$,

where we set $\tau = 12$.

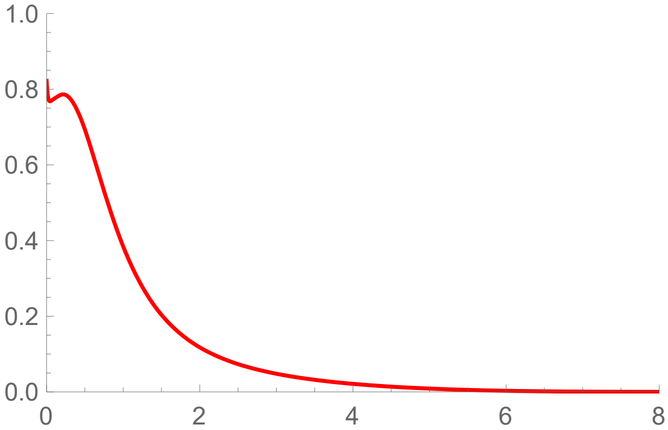
Using the same method for 1342 and 1234 (with $\tau = 8$ or 9) yields an approximation visually indistinguishable from the exact formula.

Estimating the density function



Polynomial approximation to the density function for $Av_n(1324)$.

Estimating the density function



Polynomial approximation to the density function for $Av_n(1324)$.

Open questions

- Is $(Av_n(1324))_{n\geq 0}$ a Stieltjes moment sequence?
- Is $(Av_n(\tau))_{n\geq 0}$ a Stieltjes moment sequence for any τ?
- Is there a combinatorial interpretation to being a Stieltjes moment sequence?
- What else does being a Stieltjes moment sequence gain?

Thank You!