Formal Reduction of Linear Differential Systems with Singularities[1]

> *The theory of linear differential equations is so powerful that one can usually predict the local behavior of the solutions near a point x_0 without knowing how to solve the differential equation. It suffices to examine the coefficient functions of the differential equation in the neighborhood of x_0. [...] Even when the solution to a differential equation can be expressed in terms of the common higher transcendental functions, the techniques of local analysis are still very useful. For example, saying that the solutions to $\frac{d^2f}{dx^2} = x^4 f$ are expressible in terms of modified Bessel functions of order $1/6$ does not convey much qualitative information to someone who is not an expert on Bessel functions. On the other hand, an easy local analysis of the differential equation shows that solutions behave as linear combinations of $x^{-1}\ exp\ (\pm x^3/3)$ as $x \to +\infty$*[2].

We consider in this note a linear system of n ordinary differential equations ODS in the neighborhood of a singular point:

$$[A] \qquad x^{p+1}\partial F \ = \ A(x)\ F \ = \ (\sum_{k=0}^{\infty} A_k x^k)\ F \tag{1}$$

where $\partial = \frac{d}{dx}$, p is an integer, and $A(x)$ is holomorphic in some region (D) of the complex plane. Simple poles ($p = 0$) are called *first kind singularities* of $[A]$ and poles of higher order ($p > 0$) are called *second kind singularities*. Without loss of generality, we have assumed that the singularity lies at the origin. Otherwise, a simple translation of the independent variable can be performed. Moreover, the change of variable $x \mapsto 1/x$ permits to classify the point $x = \infty$.

Notations

- $\mathcal{K}[[x]]$ is the ring of formal power series in x whose coefficients lie in a computable commutative field $\mathcal{K}$ of characteristic zero ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$); $\mathcal{K}((x))$ is its fraction field, namely the field of formal meromorphic (or Laurent) series in x with coefficients in $\mathcal{K}$;
- ∂ denotes the derivation $\frac{d}{dx}$;
- We recall that a valuation of $\mathcal{K}((x))$ indicates the order in x of an element $a(x)$ of this field at zero ($val_x(0) = \infty$). It is defined by the map $val_x : \mathcal{K}((x)) \to \mathbb{Q} \cup \infty$ which satisfies the following properties for all $a(x)$, $b(x)$ in $\mathcal{K}((x))$:

[1] This note is an adaptation of Chapter 1 of my thesis, September 2015, University of Limoges——— Suzy S. Maddah, *suzy.maddah@etu.unilim.fr*

[2] [11, Quoted from Chapter 3, Introduction].

1. $val_x\ (a)\ = \infty$ if, and only if, $a = 0$;
2. $val_x\ (ab)\ = val_x\ (a) + val_x\ (b)$;
3. $val_x\ (a+b)\ \geq min\ (val_x\ (a),\ val_x\ (b))$, and equality holds if $val_x\ (a) \neq val_x\ (b)$;

- We give the blocks of a matrix M with upper indices, e.g.

$$M\ =\ \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}.$$

  The size of the different blocks is dropped unless it is unclear from the context.

- $I_{d\times n}$ (resp. I_n) stands for the identity matrix of dimesnions $d \times n$ (resp. $n \times n$); and $O_{d\times n}$ (resp. O_n) stands for the zero matrix of dimesnions $d \times n$ (resp. $n \times n$); the dimensions are dropped whenever confusion is not likely to arise.

- We say that $A \in \mathcal{M}_n(\mathrm{R})$ whenever the matrix A is a square matrix of size n whose entries lie in a ring R.

- $GL_n(\mathrm{R})$ is the general linear group of degree n over R (the set of $n \times n$ invertible matrices together with the operation of matrix multiplication).

Contents

1 Series representation

It is well-known that in a neighborhood of an *ordinary point* ($p \leq -1$), a solution possesses a power (Taylor) series representation locally (see, e.g., [31, The Existence Theorem, pp 3]). Moreover, the former is holomorphic in the largest disc contained in (D). A classical example is the Airy's equation which is given by

$$\partial^2 f \;=\; xf,$$

and whose first-order system representation is easily obtained by setting $F = (f, \partial f)^T$:

$$\partial F \;=\; \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} F.$$

The general solution near $x = 0$ is

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{x^{3k}}{9^k\; k!\; \Gamma(k + 2/3)} + c_1 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{9^k\; k!\; \Gamma(k + 4/3)},$$

where Γ stands for the *Gamma* function and c_0, c_1 depend on the initial conditions[3].

However, if $p \geq 0$ then such a power series representation breaks downs even in simple scalar examples.

| Example 1.1

[11, Exm 1, pp 68] Given $\partial^2 f + \frac{1}{4x^2} f = 0$ whose first-order system representation is:

$$x^2\; \partial F \;=\; \begin{bmatrix} 0 & x^2 \\ -1/4 & 0 \end{bmatrix} F.$$

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. A formal substitution in the given equation yields:

$$a_0 = a_1 = 0, \text{ and } (4k(k-1) + 1)a_k = 0, \; \forall\; k \geq 2.$$

Thus, this presentation gives the trivial solution $f(x) = 0$.

[3]One may consult The Dynamic Dictionary of Mathematical Functions, *Microsoft Research - INRIA joint Centre*

- However, considering a Frobenius series representation, $y(x) = x^{\alpha} \sum_{k=0}^{\infty} a_k x^k$ where α is a rational number and $a_0 \neq 0$, it is easy to verify that: $f(x) = a_0\sqrt{x}$.

□

Frobenius series representations may break down as well as illustrated in the following example.

| Example 1.2 ([11], Exm 2, pp 77)

Given

$$x^3\, \partial^2 f \;=\; f$$

whose first-order system representation is:

$$x^3\, \partial F \;=\; \begin{bmatrix} 0 & x^3 \\ 1 & 0 \end{bmatrix}\, F.$$

Let $f(x) = x^{\alpha} \sum_{k=0}^{\infty} a_k x^k$ where α need not be an integer and $a_0 \neq 0$. Substituting as above we have:

$$(k-1-\alpha)(k-2+\alpha)a_{k-1} - a_k = 0 \;\forall\; k \geq 1$$

and $a_0 = 0$ which is an immediate contradiction. Hence, no Frobenius representation exists for this example.

□

For a system $[A]$ with $p \geq 0$, we call p the Poincaré rank. Such systems have been studied extensively (see, e.g., [2, 31] and references therein). It is well-known that a solution is, in general, the product of not only a matrix of formal power series in a root of x (Airy's equation at $x = 0$) and a matrix power of x (Example 1.1), but also an exponential of a polynomial in a root of x^{-1} (Example 1.2).

Consider again Airy's equation but for large $|x|$ ($x \mapsto 1/x$). It is known to possess two linealry independent solutions, the Airy functions of the first and second kind, $Ai(x)$ and $Bi(x)$, having the following asymptotic representation (see, e.g. [31, Ch. VI]):

$$\begin{aligned} Ai\,(x) &= \frac{1}{2\sqrt{\pi}} x^{-1/4}\, exp\,(\frac{-2}{3x^{3/2}})\,[1 + O(|x|^{-3/2})], \\ Bi\,(x) &= \frac{1}{\sqrt{\pi}} x^{-1/4}\;\; exp\,(\frac{2}{3x^{3/2}})\,[1 + O(|x|^{-3/2})]. \end{aligned}$$

More generally, it follows from Hukuhara-Levelt-Turrittin normal form, which is an explicitly prescribed form closely resembling Jordan canonical form (JCF) of matrix presentations of linear operators, that a fundamental matrix of formal solutions FMFS of $[A]$

is given by (see, e.g., [28, 19, 31])

$$\Phi(x^{1/s})\, x^C\, exp(Q(x^{-1/s})), \tag{2}$$

where

- s is a positive integer referred to as the *ramification index*;
- Φ is a matrix of meromorphic series in $x^{1/s}$ (root-meromorphic in x) over $\mathbb{C}$;
- $Q(x^{-1/s})$ is the *exponential part*. It is a diagonal matrix whose entries are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without contant terms.
- C is a constant matrix which commutes with $Q(x^{-1/s})$.

This gives another classification of the singularity: If $Q(x^{-1/s})$ is a zero matrix then $x = 0$ is a *regular singular point*. In this case, $s = 1$ and the formal series $\Phi(x^{1/s})$ converges whenever the series of $A(x)$ does: All solutions grow at most like a finite power of $|x|$. Otherwise, $x = 0$ is an *irregular singular point* (see, e.g. [31, Ch 4 Sec 2 pp 111]), and the elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (see, e.g. [31, Theorem 19.1, pg 110]). This classification however, based upon the knowledge of a FMFS, is not immediately apparent for a given differential system and will be discussed further in Section 7.1.

Algorithms to related problems leading to the construction of formal solutions of system $[A]$ and n^{th}-order scalar differential equations have been developed by various authors (see, e.g., [3, 7, 10, 9, 26, 29, 27, 32, 12, 23] and references therein).

2 Formal Reduction

Formal reduction is the algorithmic procedure that constructs a fundamental matrix of formal solutions (2) of $[A]$ (see, e.g., [3, 25], and references therein). A recursive algorithmic procedure attaining formal reduction was developed by Barkatou in [3]. As in the classical approach, it consists of computing at every step, depending on the nature of the eigenvalues of the leading matrix coefficient A_0, a *transformation* (change of basis). Accordingly, either a block-diagonalized *equivalent* system, an *irreducible* equivalent system, or the *formal exponential order* are computed. These operations will be described individually in the following sections. One can also refer to this tutorial by Barkatou or to cited articles. Upon performing such a reduction, the resulting (uncoupled) system(s) is (are) of either dimension(s) or Poincaré rank(s) lower than that of $[A]$. Based on the former operations, the packages ISOLDE [8] (also MINIISOLDE) and LINDALG [22] written respectively in MAPLE and MATHEMAGIX, are dedicated to the symbolic resolution of such systems.

3 Gauge transformations

Given system $[A]$, let $T(x) \in GL_n(\bar{\mathcal{K}}((x)))$. A transformation (change of basis) $F = T\,G$ yields

$$[\tilde{A}] \qquad x^{\tilde{p}+1}\,\partial G = \tilde{A}(x)\,G, \tag{3}$$

where $\tilde{A} \in \mathcal{M}_n(\bar{\mathcal{K}}[[x]])$, $\tilde{p}$ is an integer, and

$$\frac{\tilde{A}}{x^{\tilde{p}+1}} = T^{-1}\,\frac{A}{x^{p+1}}\,T \;-\; T^{-1}\,\partial T. \tag{4}$$

We say that system $[\tilde{A}]$ is *equivalent* to system $[A]$ via $T(x)$ and we write $[\tilde{A}] := T[A]$. One can observe that this transformation deviates from smilarity with the term $T^{-1}\,\partial T$. The theory of differential operators henceforth deviates from that of linear operators. We refer to $T^{-1}AT$ as the *similarity term* of (4). Two special types are the constant transformation and the shearing transformation. We remark that in literature, T is referred to sometimes as *gauge transformation* or *coordinate transformation.*

3.1 Constant Transformations

Let $T \in GL_n(\bar{\mathcal{K}})$. Clearly, $F = T\,G$ yields system $[\tilde{A}]$ such that $\tilde{p} = p$ and $\tilde{A} \;=\; T^{-1}\,A\,T$.

3.2 Shearing Transformation

The shearing transformation is a polynomial transformation of the general form:

$$S \;=\; Diag\,(x^{\alpha_1}, \ldots, x^{\alpha_n}), \quad \text{where} \quad \alpha_1, \ldots, \alpha_n \in \mathbb{Z}.$$

Such transformations can change the leading matrix coefficient radically upon altering its eigenvalues. Given system $[A]$, it follows from (4) that the transformation $F = T\,G$ yields system $[\tilde{A}]$ such that for $p = \tilde{p}$ we can write

$$\tilde{A} = S^{-1}\,A\,S \;-\; x^p\,Diag\,(\alpha_1, \ldots, \alpha_d).$$

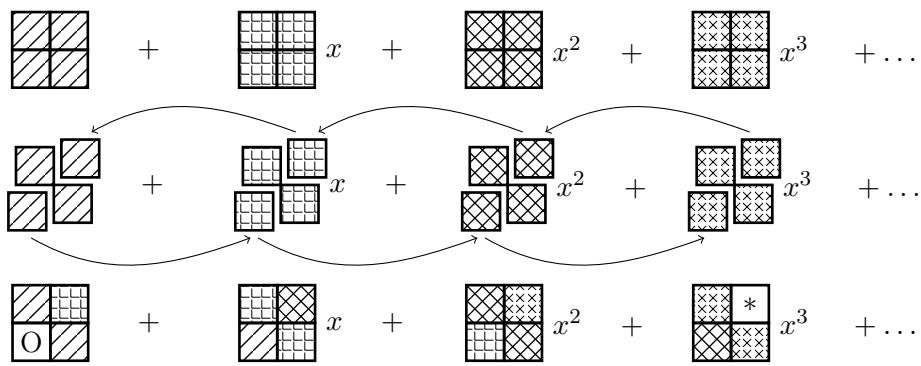
where given $A \;=\; [a_{ij}]_{1\le i,j\le n}$, we have

$$S^{-1}\,A\,S \;=\; [a_{ij}\,x^{\alpha_j-\alpha_i}]_{1\le i,j\le n}, \quad \text{or more explicitly,}$$

$$\begin{aligned}\tilde{A} \;&=\; S^{-1}\,A\,S \;-\; x^p\,Diag\,(\alpha_1, \ldots, \alpha_d)\\ &= \begin{bmatrix} a_{11} - \alpha_1 x^p & a_{12}\,x^{\alpha_2-\alpha_1} & \ldots & a_{1n}\,x^{\alpha_n-\alpha_1} \\ a_{21}\,x^{\alpha_1-\alpha_2} & a_{22} - \alpha_2 x^p & \ldots & a_{2n}\,x^{\alpha_n-\alpha_2} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}\,x^{\alpha_1-\alpha_n} & a_{n2}\,x^{\alpha_2-\alpha_n} & \ldots & a_{nn} - \alpha_n x^p \end{bmatrix}.\end{aligned}$$

| Example 3.1

Consider $S = Diag\,(x, \ldots, x, 1, \ldots, 1)$. The following diagram exhibits the shearing ("cropping") effect of S on $[A]$.



□

4 Equivalence between a system and an equation

In the introduction, we saw that an n^{th}-order differential equation can be rewritten equivalently as a first-order linear differential system. The other direction holds as well although it is nontrivial. The formal solutions of the latter can then be computed from the former (see, e.g. [5] and references therein). However, it is debatable whether such a treatment is satisfactory. Some experimental results show that the size of the coefficients grows dramatically. In any case, its major drawback is that it overlooks the information that can be derived directly from the system itself. Nevertheless, it plays a key role in the theoretical basis of algorithms treating system $[A]$. And so, it is convenient to recall it in this subsection.

Given system $[A]$, a standard procedure is to consider a nonzero row vector $U(x) = (u_1, \ldots, u_n)$ with entries in $\bar{\mathcal{K}}((x))$ and define inductively the row-vector sequence $\{U_i(x)\}_{0 \le i \le n}$ as follows:

$$U_0 = U, \quad U_i = \partial U_{i-1} \; + \; U_{i-1} \, \frac{A}{x^{p+1}}, \quad 1 \le i \le n.$$

Let

$$f \; = \; U_0 \, F \; = \; u_1 \, f_1 + \cdots + u_n f_n \, .$$

By substituting $\partial \, F \; = \; \frac{A}{x^{p+1}} F$ in the successive computations of $\partial f, \ldots, \partial^n f$, we obtain:

$$\partial^i f = U_i F, \quad 0 \le i \le n. \tag{5}$$

Now let T (resp. $\tilde{A}$) be the n-square matrices whose i^{th} rows are formed by U_{i-1} (resp. U_i) for $1 \le i \le n$; and let G be the column vector whose i^{th} component is $\partial^{i-1} f$. Then

the set of equations in (5) can be rewritten as

$$G \;=\; T\,F. \tag{6}$$

Noting that $\tilde{A} = \partial T + T\frac{A}{x^{p+1}}$, (6) can be rewritten as

$$\partial G \;=\; \tilde{A}\,F. \tag{7}$$

We say that $U(x)$ is a cyclic vector if the matrix T is invertible. Cyclic vectors always exist (see, e.g. [14] and references therein). We can thus rewrite (7) and (6) as

$$F \;=\; T^{-1}\,G \quad \text{and} \quad \partial G \;=\; \tilde{\tilde{A}}\,G, \quad \text{where} \quad \tilde{\tilde{A}}(x) = \tilde{A}T^{-1} = T\frac{A}{x^{p+1}}T^{-1} + \partial T T^{-1}$$

is a companion matrix. Denoting the entries in its last row by $(\tilde{\tilde{a}}_i)_{0\le i\le n-1}(x) \in \bar{\mathcal{K}}((x))$, the system is obviously equivalent to the n^{th}-order scalar differential equation:

$$\partial^n f \;-\; \tilde{\tilde{a}}_{n-1}(x)\,\partial^{n-1} f \;-\; \ldots \tilde{\tilde{a}}_1(x)\,\partial f - \tilde{\tilde{a}}_0(x)\,f = 0.$$

Another algorithm is that of [6] which computes a companion block diagonal form for system $[A]$.

In the sequel, we offer a direct treatment of the system $[A]$, i.e. without resorting to an equivalent n^{th}-order scalar equation. The classical direct treatment of system $[A]$ depends on the nature of the eigenvalues of the leading matrix coefficient A_0. For the clarity of the presentation, we can assume without loss of generality that A_0 is in Jordan canonical form (JCF). This can always be achieved by a constant transofrmation and, as we will see later, weaker forms suffice.

5 A_0 has at least two distinct eigenvalues

Whenever A_0 has at least two distinct eigenvalues, system $[A]$ can be uncoupled into systems of lower dimensions via the classical Splitting lemma which we recall here with its constructive proof (see, e.g. [31, Section 12, pg 52-54] or [2, Lemma 3, pg 42-43]).

| Theorem 5.1

Given system $[A]$ with $A(x) \in \mathcal{M}_n(\mathcal{K}[[x]])$. If the leading matrix coefficient is of the form

$$A_0 = \begin{bmatrix} A_0^{11} & O \\ O & A_0^{22} \end{bmatrix} \tag{8}$$

where A_0^{11} and A_0^{22} have no eigenvalues in common, then there exists a unique transformation $T(x) \in GL_n(\mathcal{K}[[x]])$ given by

$$T(x) = \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \sum_{k=1}^{\infty} \begin{bmatrix} O & T_k^{12} \\ T_k^{21} & O \end{bmatrix} x^i,$$

such that the transformation $F = TG$ *gives*

$$x^{p+1}\frac{d}{dx}\,G \;=\; \tilde{A}(x)\;Z \;=\; \begin{bmatrix}\tilde{A}^{11}(x) & O\\ O & \tilde{A}^{22}(x)\end{bmatrix} G$$

where $\tilde{A}_0 = A_0$ *and* $\tilde{A}(x) \in \mathcal{M}_n(\mathcal{K}[[x]])$.

Proof. We have from (4):

$$x^{p+1}\partial T(x) = A(x)T(x) - T(x)\tilde{A}(x). \tag{9}$$

Assume the above form for $T(x)$, then we have for $1 \le \varrho \ne \varsigma \le 2$:

$$\begin{cases} A^{\varrho\varrho}(x) - \tilde{A}^{\varrho\varrho}(x) + A^{\varrho\varsigma}(x)\;T^{\varsigma\varrho}(x) = O\\ A^{\varsigma\varrho}(x) + A^{\varsigma\varsigma}(x)\;T^{\varsigma\varrho}(x) - T^{\varsigma\varrho}(x)\;\tilde{A}^{\varrho\varrho}(x) = x^{p+1}\partial T^{\varsigma\varrho}(x)\end{cases} \tag{10}$$

Inserting the series expansions $A(x) = \sum_{k=0}^{\infty} A_k x^k$ and $\tilde{A}(x) = \sum_{k=0}^{\infty} \tilde{A}_k x^k$ in (10), and equating the power-like coefficients, we get recursion formulas of the form:
For $k = 0$ we have

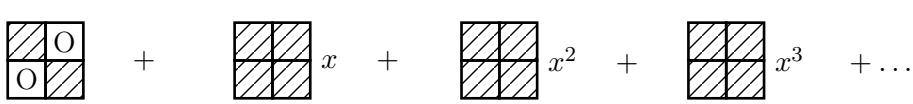
$$\begin{cases} A_0^{\varrho\varrho} = \tilde{A}_0^{\varrho\varrho}\\ A_0^{\varsigma\varsigma}\;T_0^{\varsigma\varrho} - T_0^{\varsigma\varrho}\;\tilde{A}_0^{\varrho\varrho} = O\end{cases}$$

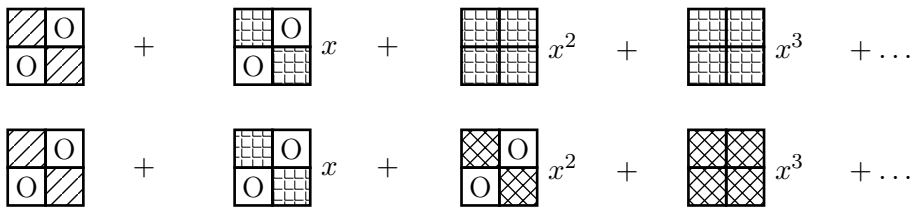
which are satisfied by setting$A_0^{\varsigma\varsigma} = \tilde{A}_0^{\varsigma\varsigma}$ and $T_0^{\varsigma\varsigma} = I,\; T_0^{\varsigma\varrho} = O$.
For $k \ge 1$ we have

$$A_0^{\varsigma\varsigma}T_k^{\varsigma\varrho} - T_k^{\varsigma\varrho}A_0^{\varrho\varrho} \;=\; -A_k^{\varsigma\varrho} - \sum_{j=1}^{k-1}(A_{k-j}^{\varsigma\varsigma}T_j^{\varsigma\varrho} - T_j^{\varsigma\varrho}\tilde{A}_{k-j}^{\varrho\varrho}) + (k-p)\;T_{k-p}^{\varsigma\varrho} \tag{11}$$

$$\tilde{A}_k^{\varrho\varrho} \;=\; A_k^{\varrho\varrho} + \sum_{j=1}^{k} A_{k-j}^{\varrho\varsigma}T_j^{\varsigma\varrho} \tag{12}$$

where $T_{k-p}^{\varsigma\varrho} = O$ for $k \le p$.
It's clear that (11) is a Sylvester matrix equation that possesses a unique solution due the assumption on the disjoint spectra of $M_0^{\varsigma\varsigma}$ and $M_0^{\varrho\varrho}$ (Lemma 5.1). Remarking that the right hand sides depend solely on the T_j, $\tilde{A}_j$ with $j < k$, the system of equations (11) and (12) are successively soluble as illustrated by the following diagram:





□

| Lemma 5.1 (see, e.g., [2], Appendix A.1, pg 212-213)
Given $M \in \mathcal{M}_m(\mathcal{K})$ and $N \in \mathcal{M}_n(\mathcal{K})$. If M and N have disjoint spectra, i.e. do not have an eigenvalue in common, then for every $P \in \mathcal{M}_{m\times n}(\mathcal{K})$ the matrix equation

$$M\,X\;-\;X\,N\;=\;P$$

has a unique solution $P \in \mathcal{M}_{m\times n}(\mathcal{K})$.

Thus the system can be split into two subsystems of lower dimensions and formal reduction proceeds on each of the subsystems in parallel.

6 A_0 has a unique nonzero eigenvalue

Supposing that A_0 has a single nonzero eigenvalue $\gamma \in \bar{\mathcal{K}}$, it is easy to verify that the so-called eigenvalue shifting, which is a special type of exponential transformations,

$$F = exp(\int^x \gamma z^{-p-1} dz)G, \tag{13}$$

results in a system with a nilpotent leading matrix coefficient (in fact, in (4), $\tilde{A}(x) = A(x) - \gamma I_n$ and $\tilde{p} = p$).

7 A_0 is nilpotent

This case is the most interesting and requires at least one of the following steps, the first of which is rank reduction. In the following two parts, we will see that the main difficulties arise in the case of nilpotency as well.

7.1 Rank reduction

Given a system $[A]$, one would like to determine whether $x = 0$ is a regular or an irregular singularity. It is well known that a singularity of the first kind of system $[A]$, i.e. for

which $p = 0$, is a regular singularity (see, e.g. [17, Proposition 3.13, pp 258] and references therein). However the converse is not true: Even when $x = 0$ is a multiple pole (a singularity of second kind), it is still possible for $x = 0$ to be a regular singularity. Horn's theorem states that $[A]$ has a regular singularity if and only if it is equivalent to a system whose singularity is of first kind [15]. Rank reduction is the procedure which constructs a transformation that yields such an equivalent system. More generally, whether the system is regular singular or irregular singular, rank reduction reduces the Poincaré rank to its minimal integer value, called the true Poincaré rank p_{true}. In the case of irregular singularity, p_{true} is the minimal integer which gives an upper bound to the growth order of the solutions in a neighborhood of $x = 0$. There exists several reduction criteria to determine p_{true} and several algorithms to construct an equivalent system whose Poincaré rank is the true Poincaré rank [26, 30, 20, 9]. In this section we are interested in the Moser-based ones.

Moser-based rank reduction of system $[A]$ is a reduction based on the criterion defined by Moser in [24]. Consider the following two rational numbers called the Moser rank and Moser invariant respectively:

$$\begin{cases} m(A) = \max\,(0, p + \frac{rank(A_0)}{n}) \\ \mu(A) = \min\,\{\ m(T[A]) \text{ for all possible choices of } T \text{ in } GL_n(\bar{\mathcal{K}}((x)))\}, \end{cases} \tag{14}$$

If $\mu(A) \leq 1$ then system $[A]$ is regular. For $m(A) > 1$, it is proved in [24, Theorems 1 and 2, pg 381] that $m(A) > \mu(A)$ if and only if the polynomial

$$\theta_A(\lambda) := x^{rank(A_0)}\ det(\lambda I + \frac{A_0}{x} + A_1)|_{x=0}$$

vanishes identically in λ. If it is the case, we say that system $[A]$ (resp. $A(x)$) is reducible. Constant transformations cannot change the rank of the system unless combined with shearings : $m(A)$ can be diminished by applying the transformation $Y = TZ$ where $T \in GL_n(\mathcal{K}((x)))$ is a product of transformations of the form [4]

$$P\ Diag(x, \ldots, x, 1, \ldots, 1) \quad \text{where} \quad P \in GL_n(\mathcal{K}).$$

Otherwise, the system (resp. $A(x)$) is said to be irreducible[4].

Thus, a system has a regular singularity at a point $x = 0$ if and only if it is equivalent to a system $[\tilde{A}]$ with a first-kind singularity at $x = 0$. Such a $T(x) \in GL_n(\mathcal{K}((x)))$ can be constructed via the rank reduction algorithms of [9] and references therein. Moser-based rank reduction algorithms result in a system equivalent to $[A]$ which holds not only the true Poincaré rank but also a minimal algebraic rank for the leading matrix coefficient A_0. A minimal algebraic rank of A_0 is a prerequisite for later computations (Subsection 7.2) and from here stems our interest in such algorithms.

[4]In literature, the terminology of *Moser-reducible* and *Moser-irreducible* is also used.

We can now suppose without any loss of generality that $[A]$ is an irreducible system. Three possibilities arise:

- System $[A]$ is regular singular and so it is transformed into an equivalent system whose Poincaré rank is zero. We proceed in the formal reduction as in Section 8.
- System $[A]$ is irregular singular and the leading matrix coefficient has at least two distinct eigenvalues (resp. unique nonzero eigenvalue). We thus retreat to Section 5 (resp. Section 6) .
- System $[A]$ is irregular singular and the leading matrix coefficient is nilpotent. This case demands introducing a ramification in x. This ramification (re-adjustment of the independent variable) is not known from the outset but can be computed as in Subsection 7.2.

We remark that more recent rank reduction algorithms were given in [9] and references therein. The transformations considered therein were optimal in the following sense: If p can be dropped by one then the similarity transformation computed acheives this goal in one single step. Moreover, $\theta_A(\lambda)$ gives other valuable information about the system leading to a generalized splitting lemma [25]. Roughly speaking, the latter uncouples the system into two subsystems, one of which does not demand a ramification for the retrieval of the leading term of the exponential parts. The former are however out of the scope of this brief description.

7.2 Formal exponential order $\omega(A)$

Definition 7.1

[3, Theorem 1] Given system $[A]$ and its *exponential part* in (2):

$$Q(x^{-1/s}) = Diag(q_1(x^{-1/s}), q_2(x^{-1/s}), \ldots, q_n(x^{-1/s})).$$

Then, the *formal exponential order*, exponential order in short, of $[A]$ (resp. $A(x)$) is the rational number

$$\omega(A) \;=\; -min_{1\leq i\leq n}\, val_x(q_i).$$

In literature, $\omega(A)$ is also referred to as Katz invariant. As mentioned, one cannot retrieve $\omega(A)$ from the outset. However, it is proved in [3, Theorem 1], which we recall here, that $\omega(A)$ can be computed from the characteristic polynomial of $A(x)/x^p$ whenever $A(x)$ is irreducible. Supposing that $\omega(A) = \frac{\ell}{d}$ with ℓ, d relatively prime positive integers, one can then set $t = x^{1/d}$, and perform again rank reduction. The resulting equivalent system has Poincaré rank equal to ℓ and leading matrix coefficient with at least d distinct eigenvalues. Consequently, the Splitting lemma can be reapplied to uncouple the system.

| Theorem 7.1 ([3], Thm 1)

Given system $[A]$ *with* $p > 1$. *Let*

$$Det\ (\lambda I_n - \frac{A(x)}{x^p}) = \lambda^n + \alpha_{n-1}\lambda_{n-1} + \cdots + \alpha_0.$$

such that $\alpha_n = 1$ *and* $\alpha_i = \sum_{j=val_x(\alpha_i)}^{\infty} \alpha_{i,j}\ x^j$ *for* $0 \leq i < n$. *If* $p > n -$ *rank* (A_0), *then we have*

$$\omega(A) = \max_{0}^{n-1} (0,\ \frac{-val_x(\alpha_i)}{n-i}).$$

One can analyze $Det(\lambda I_n - \frac{A(x)}{x^p})$ by associating a Newton polygon (see, e.g. [16, Section 2.1]), and $\omega(A)$ would then be the steepest slope of this polygon. This theorem establishes a relationship between the algebraic Newton polygon, that is the Newton polyon of $A(x)$ on one hand and the differential Newton polygon, that is the Newton polygon of the equivalent scalar n^{th}-order differential equation on the other hand (see also [12, 32, 5]).

/ Remark 7.1

The condition $p > n -$ *rank* (A_0) *is non-restrictive as it can be always attained by a suitable choice of ramification and a computable transformation [3, Lemma 5].*

The leading term of $Q(x^{-1/s})$ is then given by

$$-\frac{1}{\omega(A)\ x^{\omega(A)}}\ Diag\ (a_1, \ldots a_{deg\ (P)}, 0, \ldots, 0),$$

where the a_k's denote the roots of the (Newton) polynomial given by the algebraic equation

$$P(X) = \sum_{k=0}^{\ell} \alpha_{(i_k, val_x(\alpha_{i_k}))}\ X^{(i_k - i_0)} \tag{15}$$

where $0 \leq i_0 < i_1 < \cdots < i_\ell = n$ denote the integers i for which $k(n-i) = -val(\alpha_i)$.
Or equivalently $Q(x^{-1/s})$ is given by,

$$-\int^x \frac{1}{(p+1)\ z^{\,p+1}}\ Diag\ (\gamma_1, \ldots, \gamma_n)\ dz,$$

where the γ_i's are the eigenvalues of A_0 in $\bar{\mathcal{K}}$.

8 Regular systems

This section is devoted to systems which are regular singular $x = 0$, i.e. system$[A]$ with $p_{true} = 0$. Without loss of generality, we suppose that $[A]$ is irreducible and consequently $p = p_{true} = 0$. Relevant methods of resolution are discussed in [31, Chapter 1], [10, 7] for more general contexts, and references therein. The discussion is, again, based on the nature of the eigenvalues of A_0:

| Theorem 8.1

Given system $[A]$ with $p = 0$. If the eigenvalues of A_0 do not differ by nonzero integers, then there exists

$$T(x) \;=\; \sum_{i=0}^{\infty} T_k\, x^k \;\in GL_n(\mathcal{K}[[x]]), \quad \textit{where} \quad T_0 = I_n,$$

which yields an equivalent system $[\tilde{A}]$ for which $\tilde{A} = A_0$. Consequently, a formal fundamental matrix of formal solutions of $[A]$ is given by $T(x)\, x^{A_0}$.

Proof. If follows from (4) that

$$x\partial T(x) \;=\; A(x)\, T(x) \;-\; T(x)\, A_0.$$

Inserting the series representation of $A(x)$ and $T(x)$ yields:

$$\begin{cases} A_0\, T_0 \;- T_0\, A_0 \;=\; O_n \\ (A_0 \;-\; k\, I_n)\, T_k \;-\; T_k\, A_0 \;=\; -\sum_{i=0}^{k-1}\; A_{k-i}\, T_i\;,\; k \geq 1. \end{cases}$$

Choosing $T_0 = I_n$, the T_k's can be successively computed since $A_0 - kI_n$ and A_0 have disjoint spectra. □

| Proposition 8.1

If the eigenvalues of A_0 differ by a nonzero integer then there exists $T \in GL_n(\mathcal{K}((x)))$, product of shearing (polynomial) transformations and constant transformations such that the eigenvalues of the leading matrix coefficient of the equivalent system do not differ by nonzero integers.

The proof is constructive and is skipped here. An efficient version of the resulting algorithm is proposed in [18, pp 67 - 68].

| Example 8.1

Given the famous modified Bessel equation of order ν:

$$\partial^2 f + \frac{1}{x}\partial f - (1 + \frac{\nu^2}{x^2})f = 0.$$

We discuss the construction of a fundamental matrix of formal solutions at $x = 0$ according to the values of ν. Let $F = (f, \partial f)^T$. Then the given equation can be rewritten as the following linear first-order differential system of dimension 2:

$$x^2\ \partial F \ = \ A(x)\ F = \begin{bmatrix} 0 & x^2 \\ x^2 + \nu^2 & -x \end{bmatrix}\ F. \tag{16}$$

The leading term $A_0 := A(0)$ is nilpotent and the system is reducible according to Moser's criterion ($\theta(\lambda) = 0$). The transformation $F \ = \ T\ G$ where $T \ = \ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$ yields the equivalent system:

$$x\ \partial G \ = \ \tilde{A}(x)G \ = \ \begin{bmatrix} -1 & 1 \\ x^2 + \nu^2 & -x \end{bmatrix}\ G.$$

Thus, $p_{true} = 0$ and the system is regular singular at $x = 0$. The new leading matrix $\tilde{A}_0 = \tilde{A}(0) = \begin{bmatrix} -1 & 1 \\ \nu^2 & 0 \end{bmatrix}$ is not nilpotent. The difference between its two eigenvalues is given by $\sqrt{4\nu^2 + 1}$. We thus distinguish two cases:

- **Case 1** : If $\sqrt{4\nu^2 + 1}\ \in \mathbb{N}^*$ then Proposition 8.1 is applied.
- **Case 2** : Otherwise, we proceed to constructing $T(x)$ of Theorem 8.1 and hence a fundamental matrix of formal solutions is given by $T(x)\ x^{\tilde{A}_0}$.

□

9 Formal reduction algorithm

Based on the above, the terms of $Q(x^{-1/s})$ of largest degrees in $x^{-1/s}$ can be found recursively. This process is exhibited in the Algorithm 9[5], which computes the exponential part of a FMFS (2), and eventually, a full fundamental matrix of formal solutions of system $[A]$. In fact, the transformations performed in the process of computing the former, are endowed in the latter.

Algorithm 9 results in a set of decoupled systems with dimension $n = 1$ (case of first-order linear scalar equations) and/or a set of system(s) whose Poincaré ranks are zeros, and consequently, has(ve) zero exponential part(s) and can be treated as in Section 8.

[5]The pseudocode is given only to serve the illustration of the procedure. The implementation however does not follow this pseudocode.

Algorithm 1 FMFS_ODS $(p, A(x))$: Computes a fundamental matrix of formal solutions (2) of $[A]$

Input: p, $A(x)$ of (1)
Output: A fundamental matrix of formal solutions FMFS (2)

$Q \leftarrow O_n$, $C \leftarrow O_n$, $\Phi \leftarrow I_n$;
while $p > 1$ and $n \neq 1$ **do**
  if A_0 has at least two distinct eigenvalues **then**
    Split system as in Subsection 5; Update Φ;
    FMFS_ODS $(p, \tilde{A}_{11}(x))$; Update Φ, C, Q;
    FMFS_ODS $(p, \tilde{A}_{22}(x))$; Update Φ, C, Q;
  else if A_0 has one non-zero eigenvalue **then**
    Update Q from the eigenvalues of A_0;
    $A(x) \leftarrow$ Follow Subsection 6; (A_0 is now nilpotent);
    FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
  else
    Apply rank reduction of Subsection 7.1; Update Φ;Update p; Update A;
    if $p > 1$ and A_0 has at least two distinct eigenvalues **then**
      Split system as in Subsection 5;
      FMFS_ODS $(p, \tilde{A}_{11}(x))$; Update Φ, C, Q;
      FMFS_ODS $(p, \tilde{A}_{22}(x))$; Update Φ, C, Q;
    else if A_0 has one non-zero eigenvalue **then**
      Update Q from the eigenvalues of A_0;
      $A(x) \leftarrow$ Follow Subsection 6; (A_0 is now nilpotent)
      FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
    else
      Follow Subsection 7.2;
      $\omega(A) = \frac{\ell}{d}$; $x \leftarrow x^d$;
      $A(x) \leftarrow$ Follow Subsection 7.1; Update Φ; $p \leftarrow \ell$;
      Update Q from eigenvalues of A_0;
      $A(x) \leftarrow$ Follow Subsection 6; (A_0 is now nilpotent)
      FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
    end if
  end if
end while
if $n = 1$ **then**
  Proceed by integration up to the first $p - 1$ terms; Update Φ, C, Q;
else if $p = 0$ **then**
  Update Φ, C from Subsection 8;
end if
return Φ, C, Q.

10 Remarks about the implementation

In this section, we point out some considerations in the implementation of the packages ISOLDE [8], miniISOLDE, and LINDALG[6]. The implementation in miniISOLDE and LINDALG is iterative although the algorithm itself is recursive.

The polynomial $Q(x^{-1/s})$ of a FMFS given by (2) is determined by at most the first np terms in the Taylor expansion of $A(x)$ in (1) [1, 21]. Thus, finitely many terms are needed for the construction of the exponential part and this number is reduced as the system splits into two or several systems and as the Poincaré rank drops. Moreover, any additional number of terms can be taken into account to increase the precision of $\Phi(x)$ of a FMFS . The base field can be any commutative field $\mathcal{K}$ of characteristic zero. Algorithm 9 and its underlying sub-algorithms can be refined to handle efficiently algebraic extensions of the constant base field, as explained below.
The JCF can be avoided in Theorem 5.1, Proposition 8.1, and whenever appropriate, by making use of the following lemma.

Lemma 10.1 (Lemma A.1, [9])
Given a rank deficient matrix $M \in \mathcal{M}_n(\mathcal{K})$. Then there exists a constant matrix $T \in GL_n(\mathcal{K})$ such that

$$\tilde{M} \;=\; T^{-1}\, M\, T \;=\; \begin{bmatrix} D & O_{r\times(n-r)} \\ O_{(n-r)\times r} & N \end{bmatrix} \quad \in \mathcal{M}_n(\mathcal{K}),$$

where $r := \; rank\;(M^n)$, $D \in \mathcal{M}_r(\mathcal{K})$ is nonsingular, and N is a nilpotent matrix.

The proof follows from the rank-nullity theorem. In practice we proceed as in Algorithm 10.

Variations of Lemma 10.1 can also be applied to isolate distinct eigenvalues, e.g. to put A_0 in the form 8 to prepare $[A]$ for the Splitting lemma or Proposition 8.1. However, despite avoiding JCF, some algebraic extensions might be introduced with the roots of (15) and eigenvalue shifting. In fact, let $\omega(A) = \frac{\ell}{d}$ with $gcd(\ell, d) = 1$. If a is a root of the polynomial (15) of multiplicity ν then there exists ν polynomial entries on the diagonal of the exponential part $Q(x^{-1/s})$ that have $-\frac{a}{\omega(A)\; x^{\;\omega(A)}}$ as a leading term, i.e. the leading term of $Q(x^{-1/s})$ is given by

$$-\frac{1}{\omega(A)\; x^{\omega(A)}}\; Diag\;(a, \ldots, a, 0, \ldots, 0).$$

Following Algorithm 9, we apply the ramification $x = t^d$ and eventually carry out the

[6]The source code of LINDALG is accessible with examples of computation within the current release of MATHEMAGIX.

Algorithm 2 Block-diagonalization of M over $\mathcal{K}$

Input: $M \in \mathcal{M}_n(\mathcal{K})$

Output: The transformation of Lemma 10.1

$r = rank(M^n)$;
if $0 < r < n$ **then**
Compute a basis $\{T_1, \ldots, T_r\}$ of the space generated by the columns of M^n;
Compute a basis $\{T_{r+1}, \ldots, T_n\}$ of the kernel of M^n (using GUASSIAN ELIMINATION);
Form the matrix $T(x)$ whose columns are $\{T_1, \ldots, T_r, T_{r+1}, \ldots, T_n\}$;
else
$T = I_n$;
end ifreturn T.

shifting

$$F \ = \ exp\,(\frac{-a}{\omega(A)\ t^\ell})\ G.$$

At this point, we leave the constant field $\mathcal{K}$ and start working in its extension $\mathcal{K}(a)$. These extensions arise naturally but it is possible to restrict their sizes using a trick described in [3, Section 5] which we outline here:
Let u and v be two integers verifying $u\ell \ + \ vd \ = \ 1$. Let $z = a^{-u}\ t$ and $b = a^d$. Then we have:

$$\begin{cases} \frac{-a}{\omega(A)\ t^\ell} \ = \ \frac{-a^{u\ell+vd}}{\omega(A)\ t^\ell} \ = \ \frac{-a^{dv}}{\omega(A)\ (a^{-u}t)^\ell} \ = \ \frac{b^v}{\omega(A)\ z^\ell}, \\ x \ = \ t^d \ = \ a^{du}\ z^d \ = \ b^u\ z^d. \end{cases}$$

One notices that if a is a root of the polynomial (15) then $b = a^d$ is a root of a reduced polynomial given by $P_{reduced}\ (X^d) \ = \ P(X)$. Hence, Algorithm 9 can then be modified as follows:

- Choose a root b of $P_{reduced}$;
- compute u and v satisfying $u\ell \ + \ vd \ = \ 1$;
- substitute x by $b^u\ z^d$;
- apply the shifting

$$F \ = \ exp\,(\frac{-b^v}{\omega(A)\ z^\ell})\ G.$$

The new constant field is then $\mathcal{K}(b)$ where b is a root of a polynomial of degree equal to the degree of the polynomial (15) divided by d. The computations are done up to conjugations.

We illustrate these enhancements with this very simple example.

| Example 10.1 ([3], Example 4)

Let $\mathcal{K} = \mathbb{Q}$ and consider the following system:

$$x^3\,\partial\,F \;=\; A(x)\,F \;=\; \begin{bmatrix} 2x & x \\ x-1 & 1 \end{bmatrix}.$$

Without applying the above trick, Algorithm 9 computes the exponential part to be:

$$Q(x^{-1/2}) \;=\; Diag\,(q_1(x^{-1/2})\,,\; q_2(x^{-1/2})),$$

where

$$\begin{cases} q_1(x^{-1/2}) \;=\; -\frac{2\sqrt{-1}}{3x^{3/2}} - \frac{1}{x} + \frac{2\sqrt{-1}}{\sqrt{x}}, \\ q_2(x^{-1/2}) \;=\; \frac{2\sqrt{-1}}{3x^{3/2}} - \frac{1}{x} - \frac{2\sqrt{-1}}{\sqrt{x}}. \end{cases}$$

However, we proceed instead as follows:

- The leading matrix A_0 is nilpotent and irreducible so we need to compute $\omega(A)$;
- we compute $\omega(A) = 3/2$ and $P(X) = X^2{+}1$ (15). Hence we set $P_{reduced} = U{+}1$;
- we take $u = 1,\ v = -1$, and $b = -1$ (the unique root of $P_{reduced}$);
- we apply the ramification $x = bt^2 = -t^2$ and rank reduction again to obtain the new system

$$t^4\,\partial_t\,G \;=\; \tilde{A}(t)\,G \;=\; \begin{bmatrix} 2t^3 & -2t^2-2 \\ -2 & -4t-t^3 \end{bmatrix};$$

- The shifting

$$G \;=\; H\;exp\,(\frac{2}{3t^3}),$$

shifts the eigenvalues of the leading matrix coefficient by two which yields

$$t^4\,\partial_t\,H \;=\; \tilde{\tilde{A}}(t)\,H \;=\; \begin{bmatrix} 2t^3+2 & -2t^2-2 \\ -2 & -4t-t^3+2 \end{bmatrix};$$

Now, the leading matrix coefficient $\tilde{\tilde{A}}_0$ of the resulting system has zero as a simple eigenvalue and is given by

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

One can then isolate its nilpotent part based on Lemma 10.1 (whose transformation coincides with Jordan tranformation in this case): We compute

$$T \;=\; \begin{bmatrix} 8 & 1 \\ -8 & 1 \end{bmatrix} \quad \text{so that} \quad T^{-1}\,\tilde{\tilde{A}}_0\,T \;=\; \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

- We then split the system to any desired precision by Theorem 5.1. which leads to the scalar equation

$$t^3\,\partial_t f \;=\; n(t)\;f = n(t)\;f = \;(-2-2\,t+\ldots)\;f.$$

  The latter has $exp\;(\frac{1}{t^2}+\frac{2}{t})$ as its exponential part.
- Thus, we obtain a polynomial in $1/t$

$$\mathbf{q}(1/t) \;=\; \frac{2}{3\,t^3} \;+\; \frac{1}{t^2} \;+\; \frac{2}{t}, \quad \text{where} \quad x \;=\; -t^2.$$

Thus, $q_1(x^{-1/2})$ and $q_2(x^{-1/2})$ obtained above by computations in $\mathbb{Q}(i)$ can be both obtained from $\mathbf{q}(t)$ by substituting $t = \pm i x^{1/2}$ in $\mathbf{q}(1/t)$, which is computed in $\mathbb{Q}$.

□

11 Conclusion

LINDALG and MINIISOLDE are currently dedicated to the symbolic computation of local solutions of n^{th}-order linear scalar differential equations and first-order linear differential systems with singularities. Following the track of the MAPLE package ISOLDE, algorithms for computing global solutions will be adjoined as well (e.g. rational solutions) in the future.

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